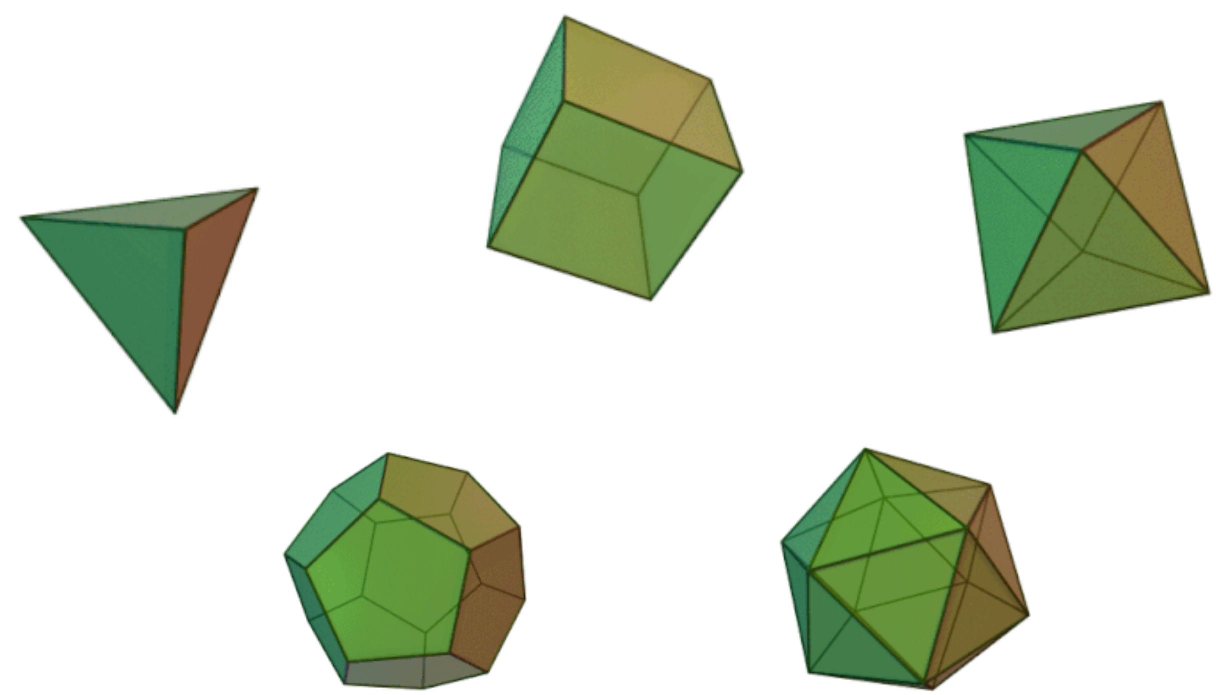
No me amenaces… No me amenaces…

(con que ya son todos)

Seminario de matemáticas
Fismat - UMSNH
16 de Noviembre de 2023



Antonio Montero*

Universidad de Ljubljana

*Financiado por: ARRS Projekt 21 N1-0216

Videos: wikipedia.
Slides y figuras: Isabel Hubard y A.M.

Are Your Polyhedra the Same as My Polyhedra?

Branko Grünbaum

Branko Grünbaum
1929 - 2018

1 Introduction

"Polyhedron" means different things to different people. There is very little in common between the meaning of the word in topology and in geometry. But even if we confine attention to geometry of the 3-dimensional Euclidean space – as we shall do from now on – "polyhedron" can mean either a solid (as in "Platonic solids", convex polyhedron, and other contexts), or a surface (such as the polyhedral models constructed from cardboard using "nets", which were introduced by Albrecht Dürer [17] in 1525, or, in a more modern version, by Aleksandrov [1]), or the 1-dimensional complex consisting of points ("vertices") and line-segments ("edges") organized in a suitable way into polygons ("faces") subject to certain restrictions ("skeletal polyhedra", diagrams of which have been presented first by Luca Pacioli [44] in 1498 and attributed to Leonardo da Vinci). The last alternative is the least usual one – but it is close to what seems to be the most useful approach to the theory

diagrams of which have been presented first by Luca Pacioli [44] in 1498 and attributed to Leonardo da Vinci). The last alternative is the least usual one – but it is close to what seems to be the most useful approach to the theory of *general* polyhedra. Indeed, it does not restrict faces to be planar, and it makes possible to retrieve the other characterizations in circumstances in which they reasonably apply: If the faces of a "surface" polyhedron are simple polygons, in most cases the polyhedron is unambiguously determined by the boundary circuits of the faces. And if the polyhedron itself is without selfintersections, then the "solid" can be found from the faces. These reasons, as well as some others, seem to warrant the choice of our approach.

Before deciding on the particular choice of definition, the following facts – which I often mention at the start of courses or lectures on polyhedra – should be considered. The regular polyhedra were enumerated by the mathematicians of ancient Greece; an account of these five "Platonic solids" is the final topic of Euclid's "Elements" [18]. Although this list was considered to be complete, two millennia later Kepler [38] found two additional regular polyhedra, and in the early 1800's Poinsot [45] found these two as well as two more; Cauchy [7] soon proved that there are no others. But in the 1920's Petrie and Coxeter found (see [8]) three new regular polyhedra, and proved the completeness of that enumeration. However, in 1977 I found [21] a whole

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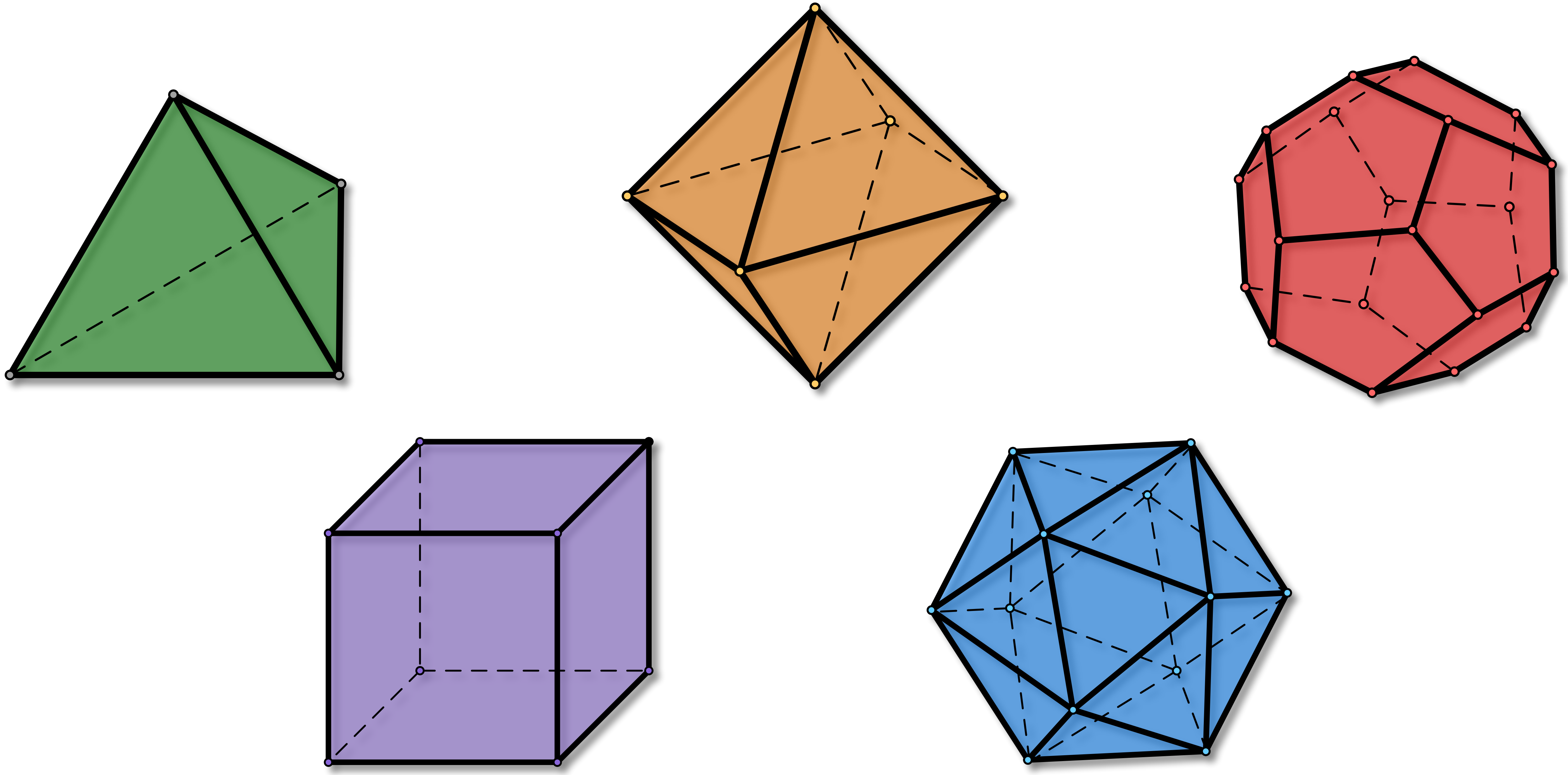
Teorema (Elementos de Euclides)

Existen exactamente 5 sólidos platónicos



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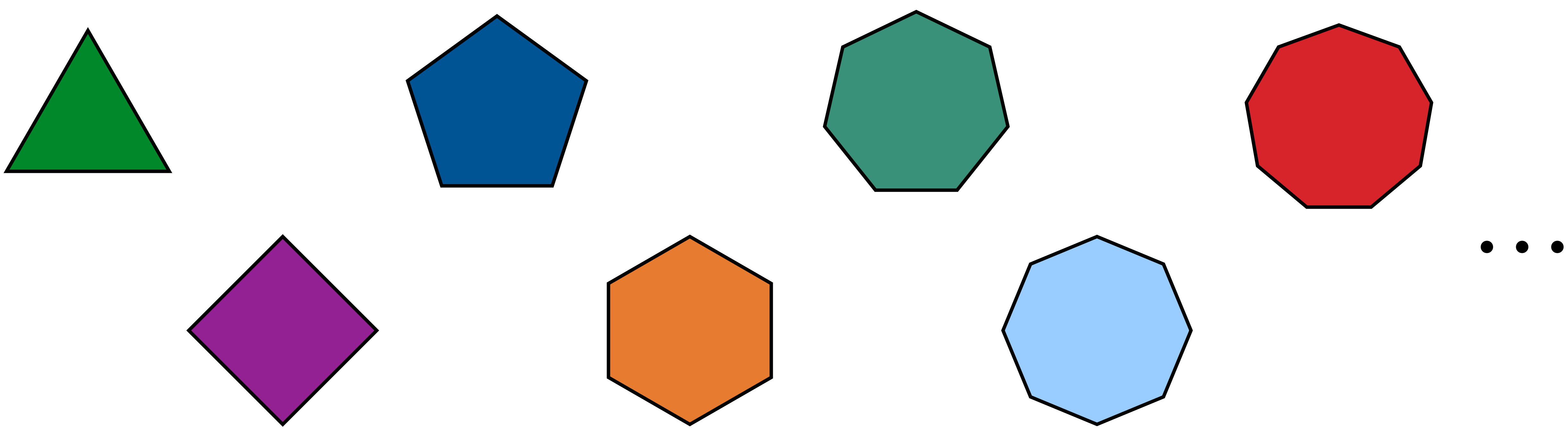
¿Cómo le hacemos para ver que esto es cierto?

- ¿Caras? → Polígonos regulares



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- ¿Caras? $\rightarrow$ Polígonos regulares

- ¿Cuántos hay en cada vértice?

Caras

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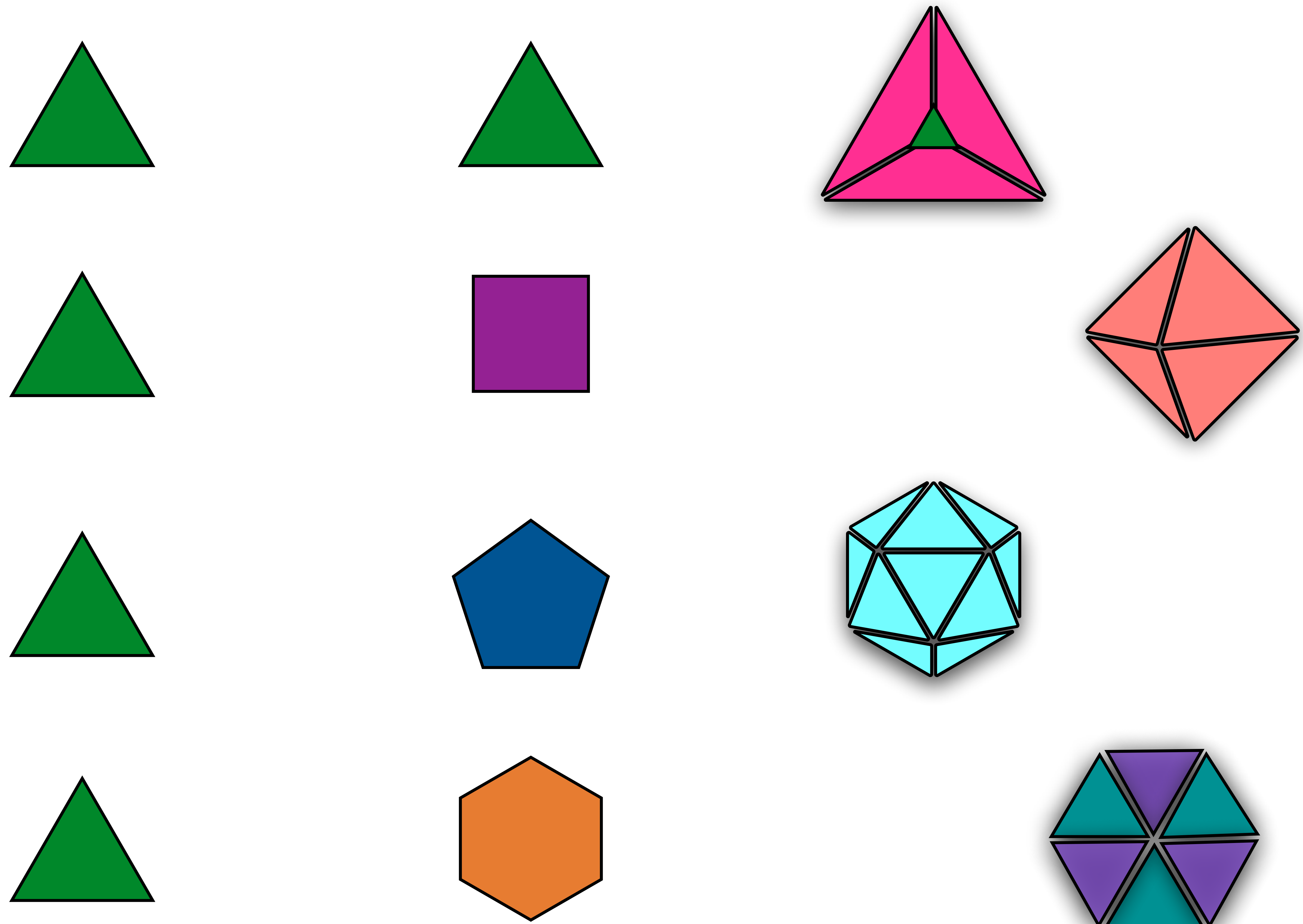
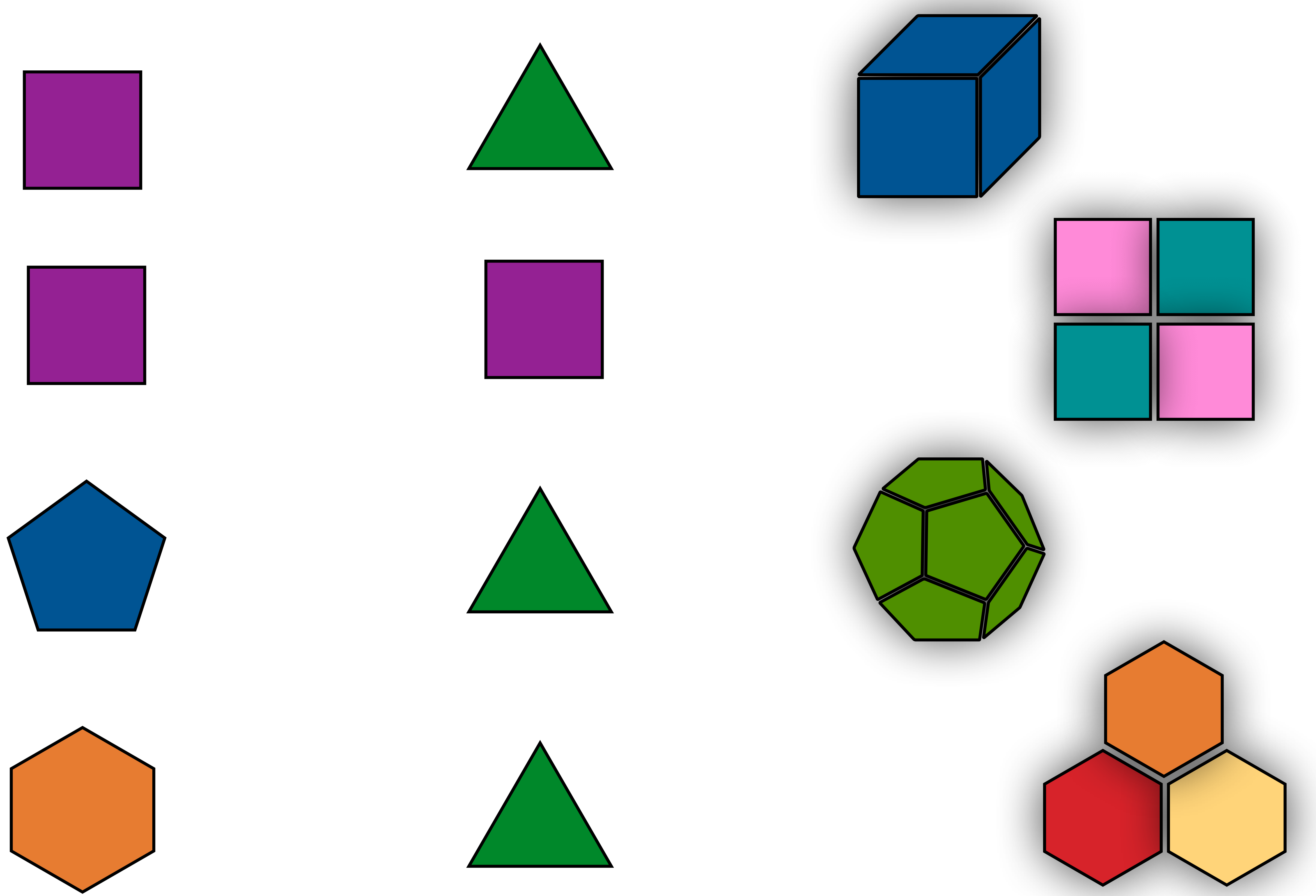


Caras

¿Cuántos hay en cada vértice?



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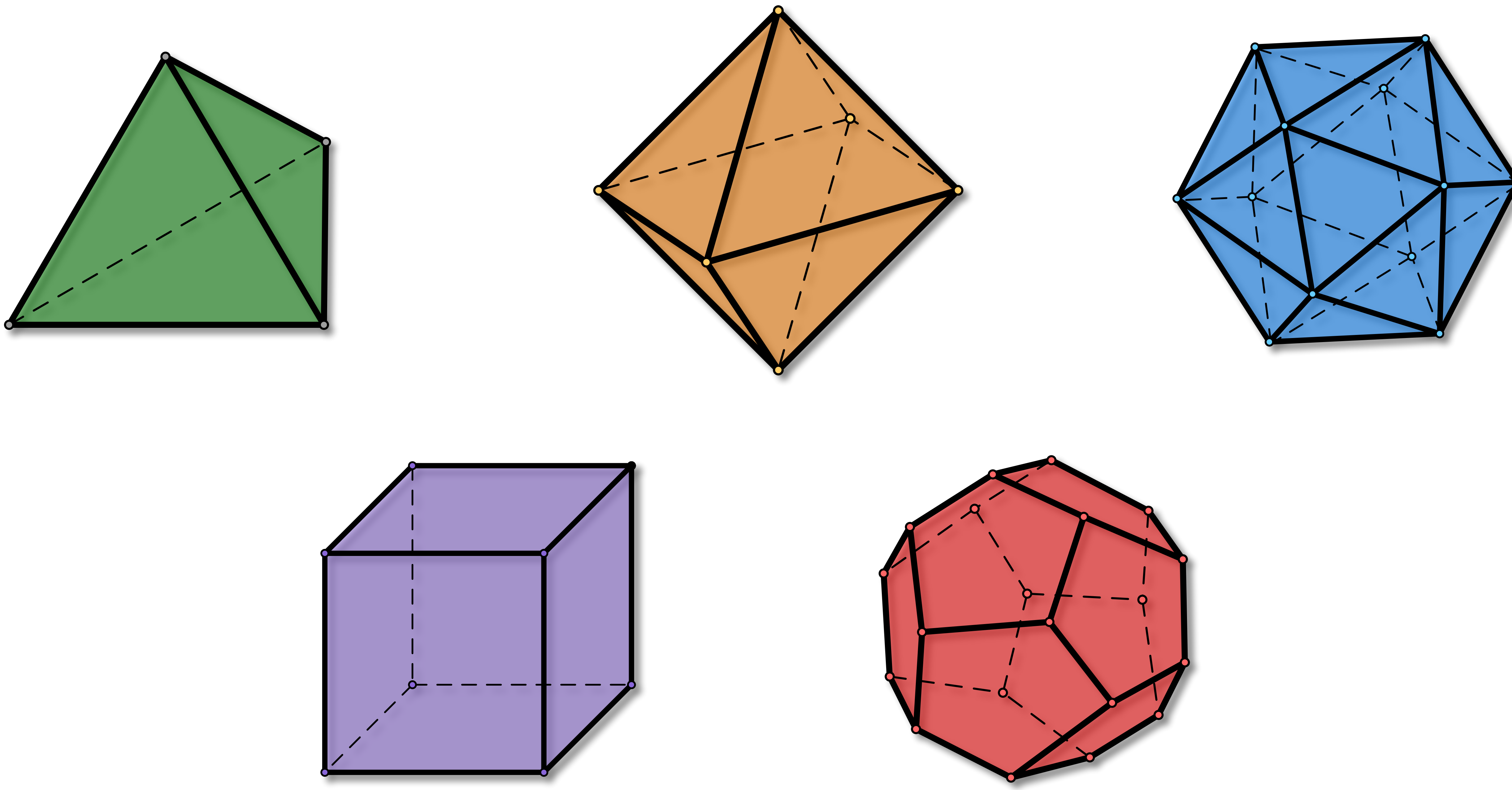

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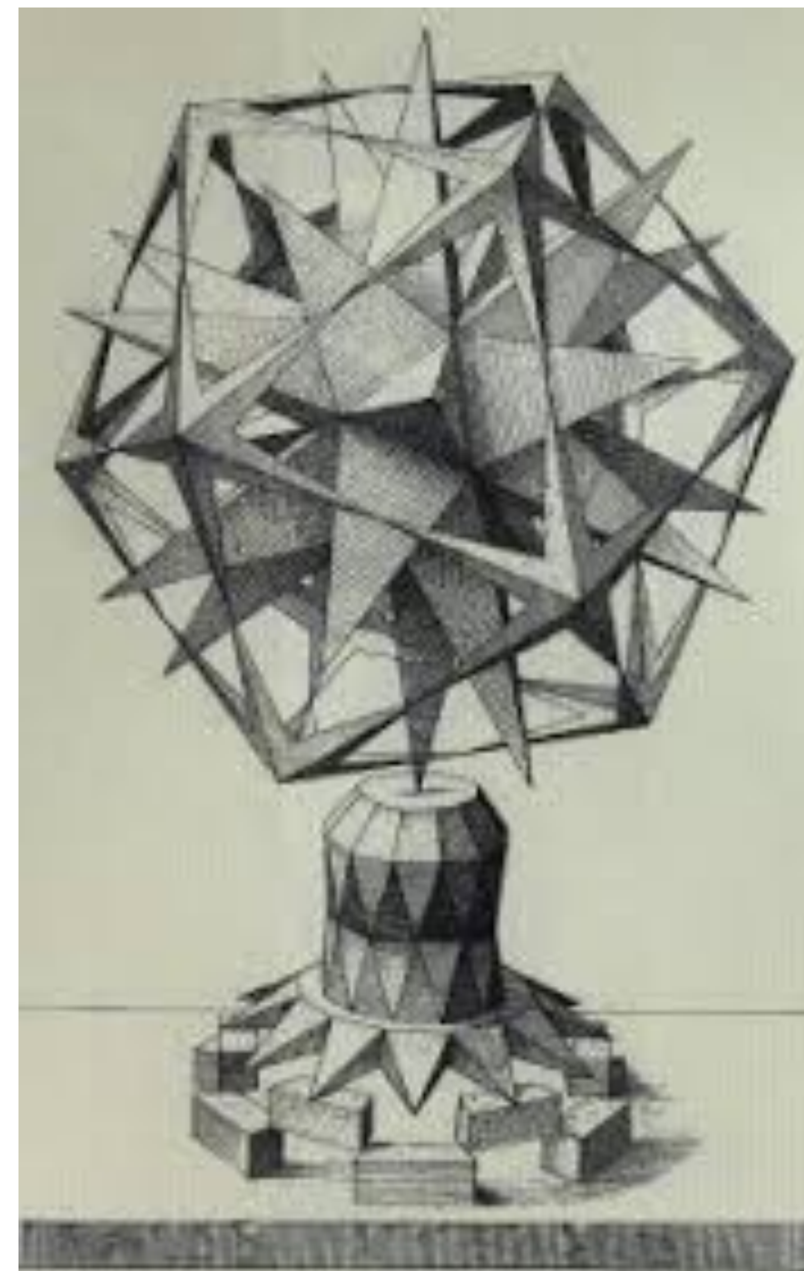
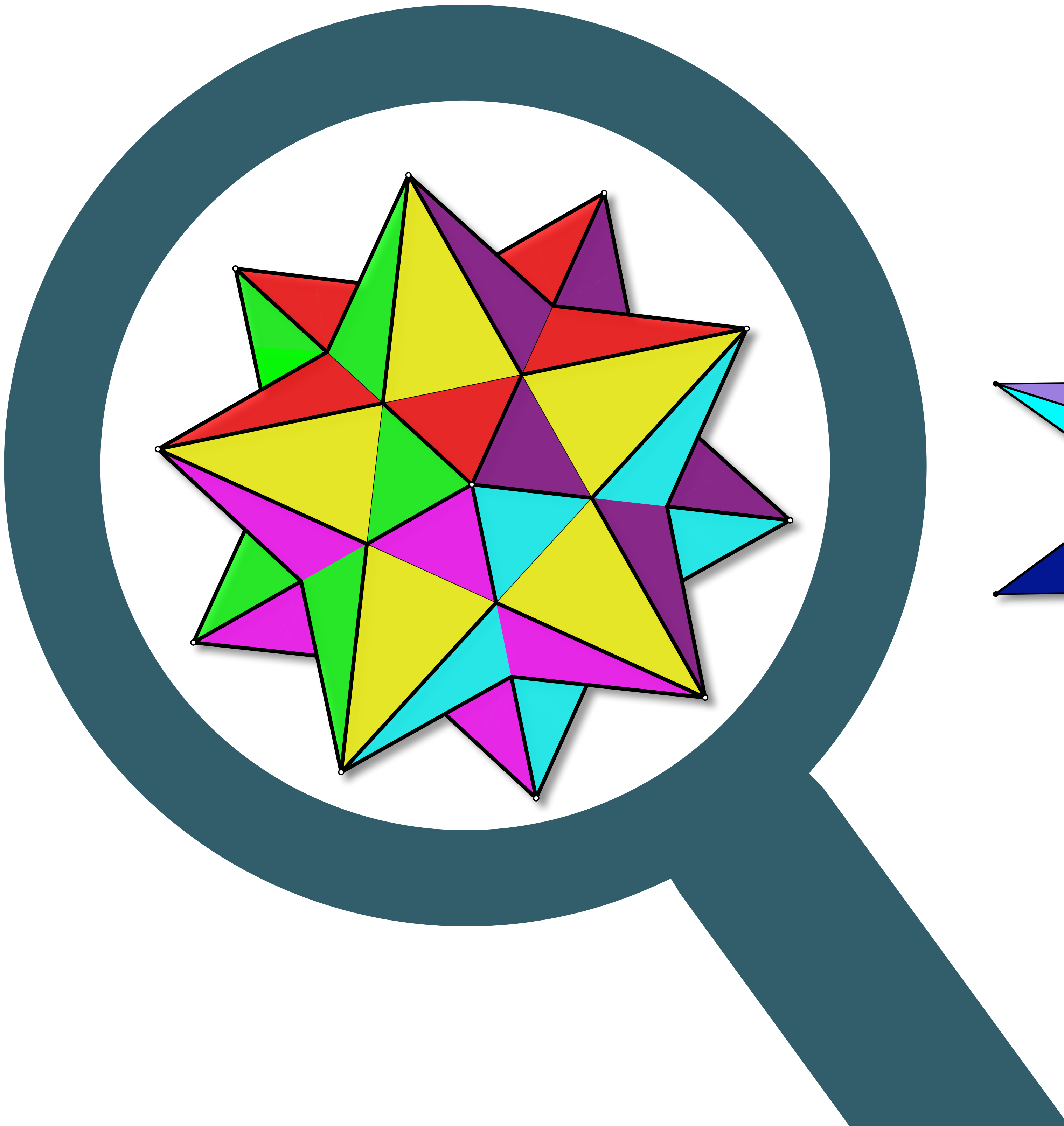
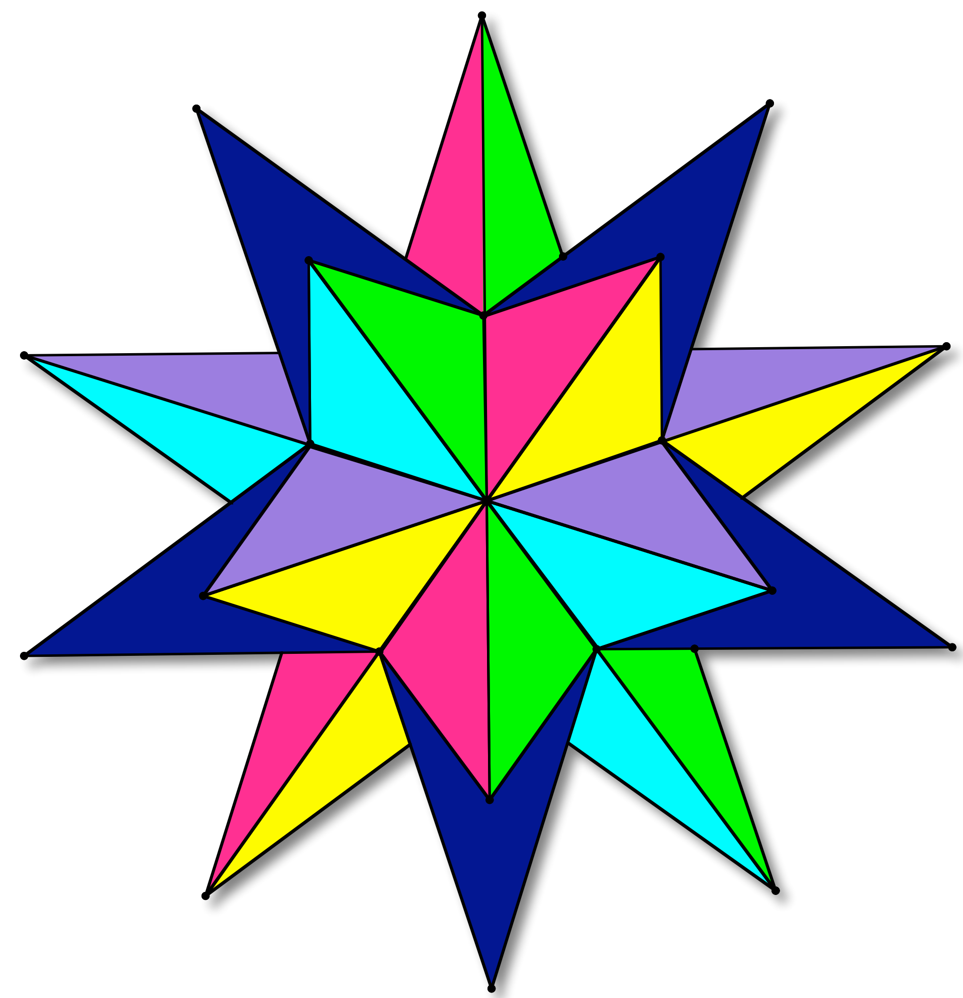
Mosaico en la Catedral de San Marco, Venecia (Paolo Uccello~1425)

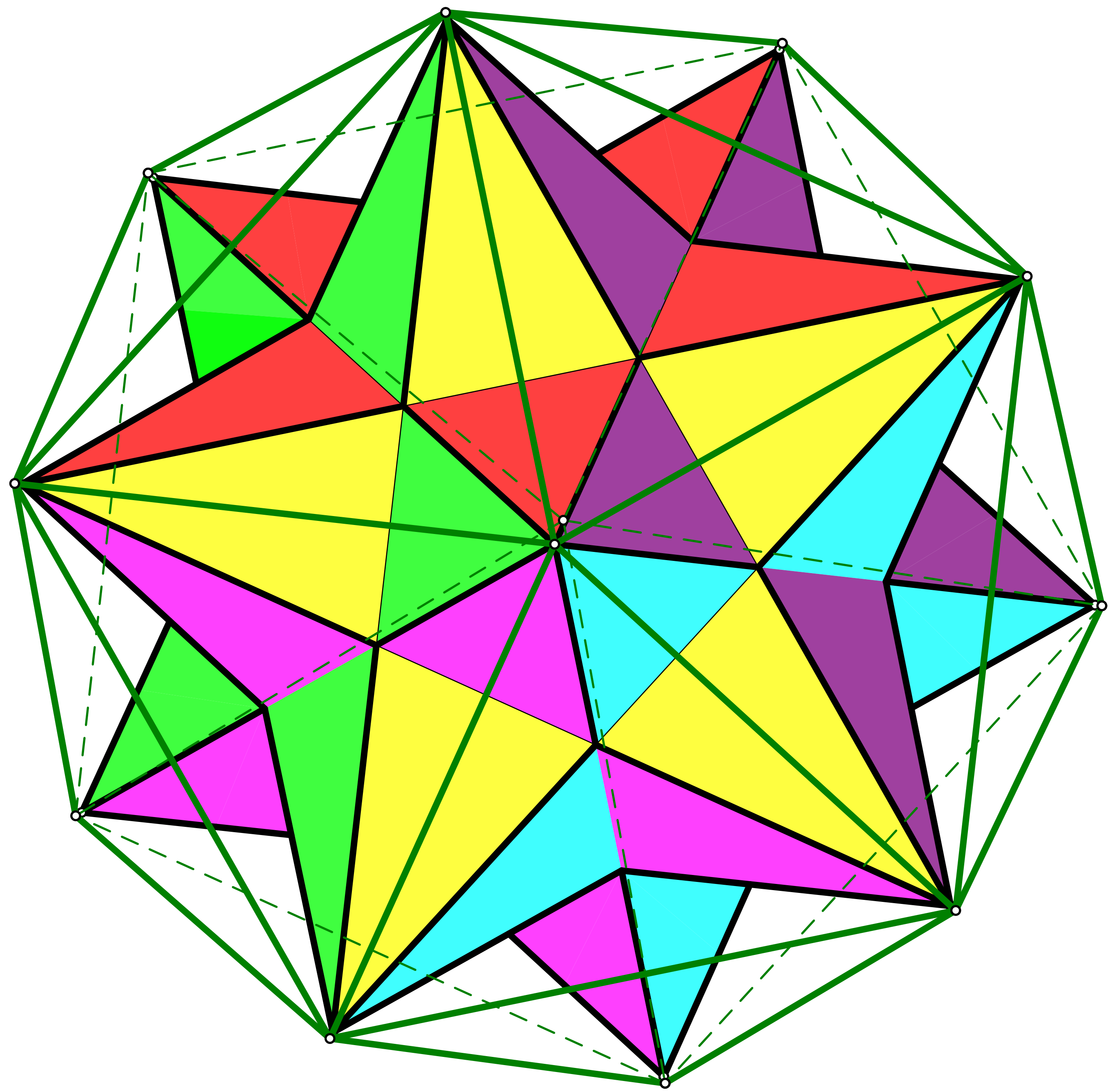
Dibujo de poliedros de Wenzel Jamnitzer (1505)

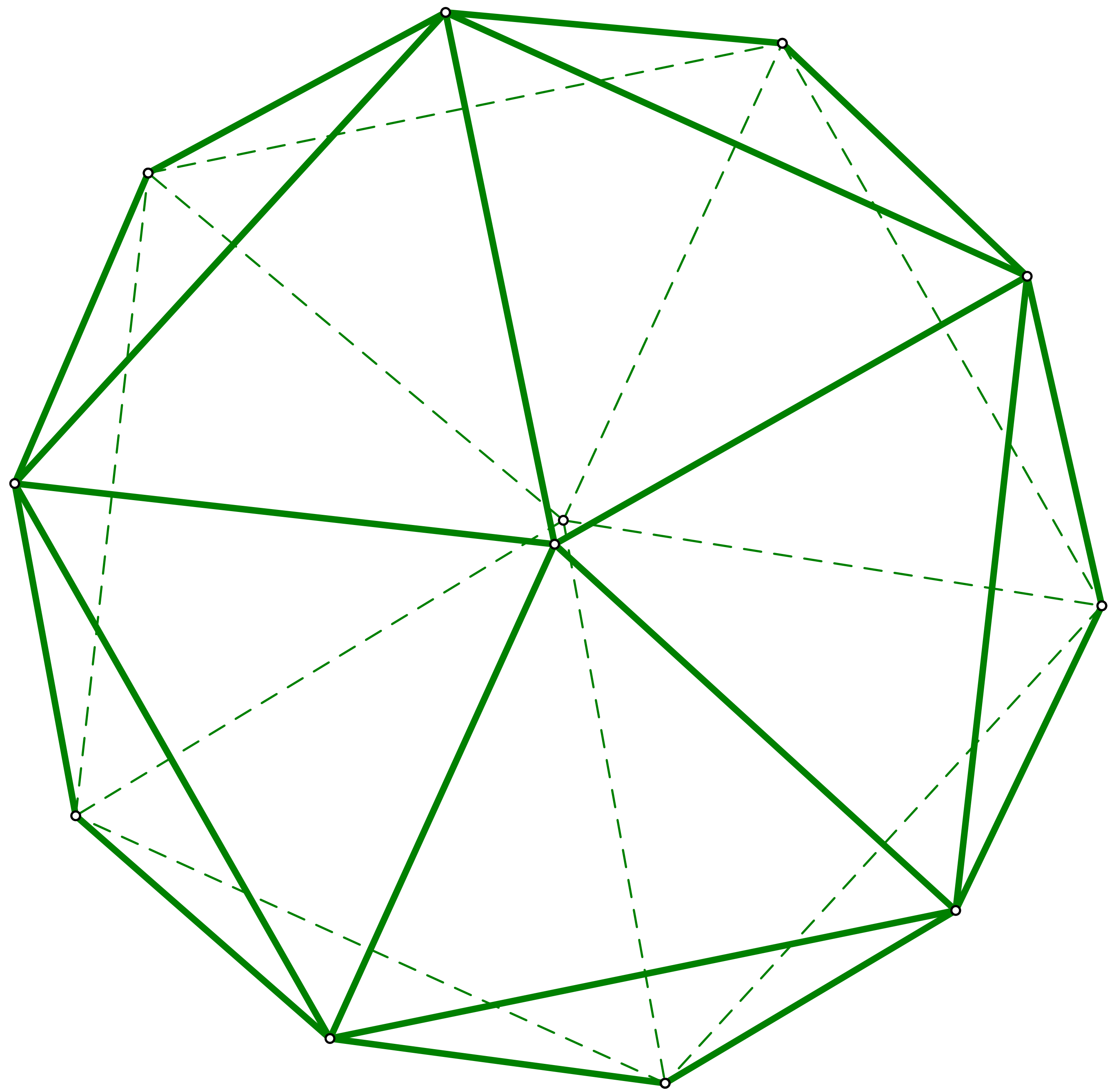


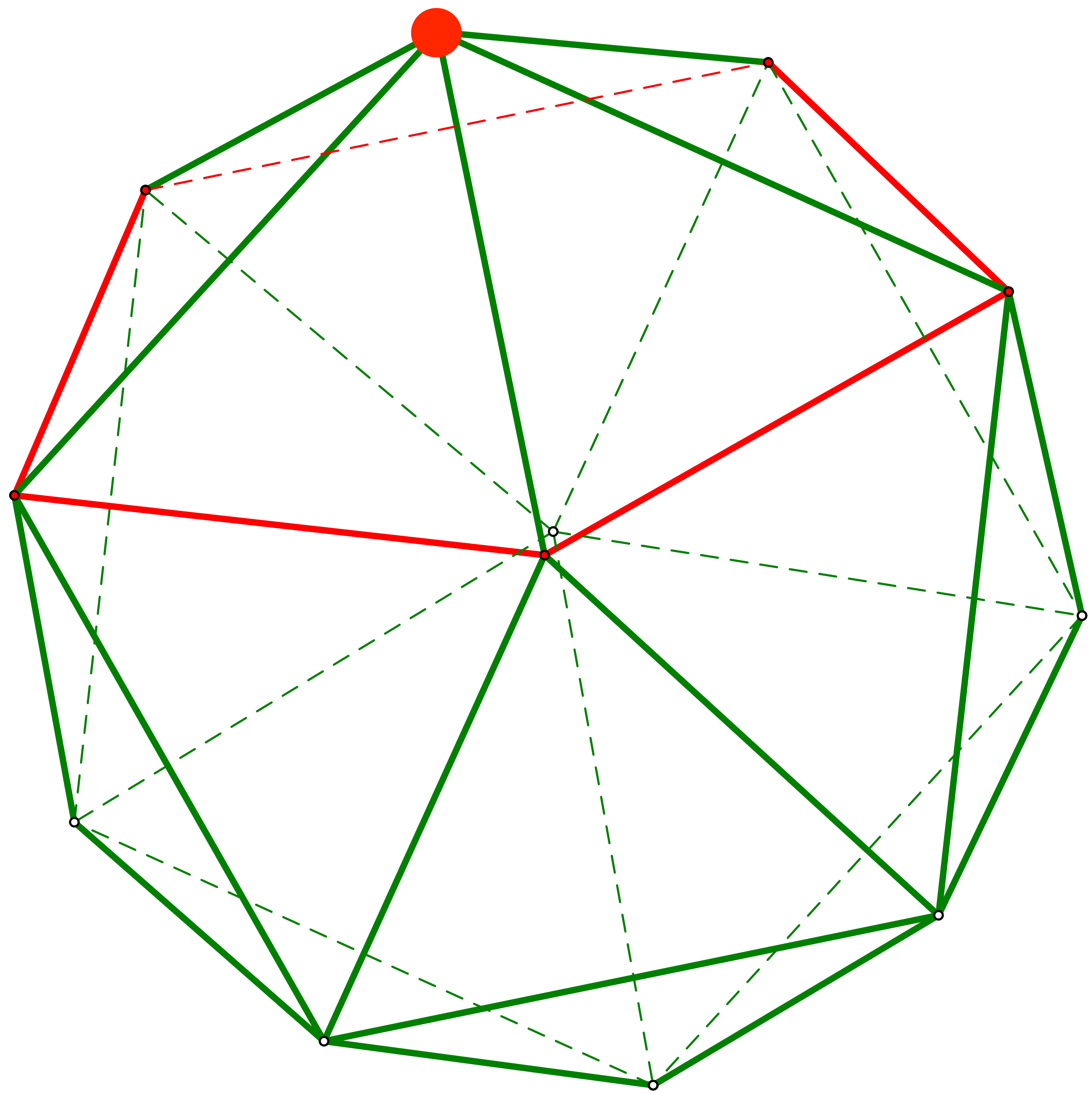
En 1619 Kepler encuentra dos poliedros ¿regulares?:

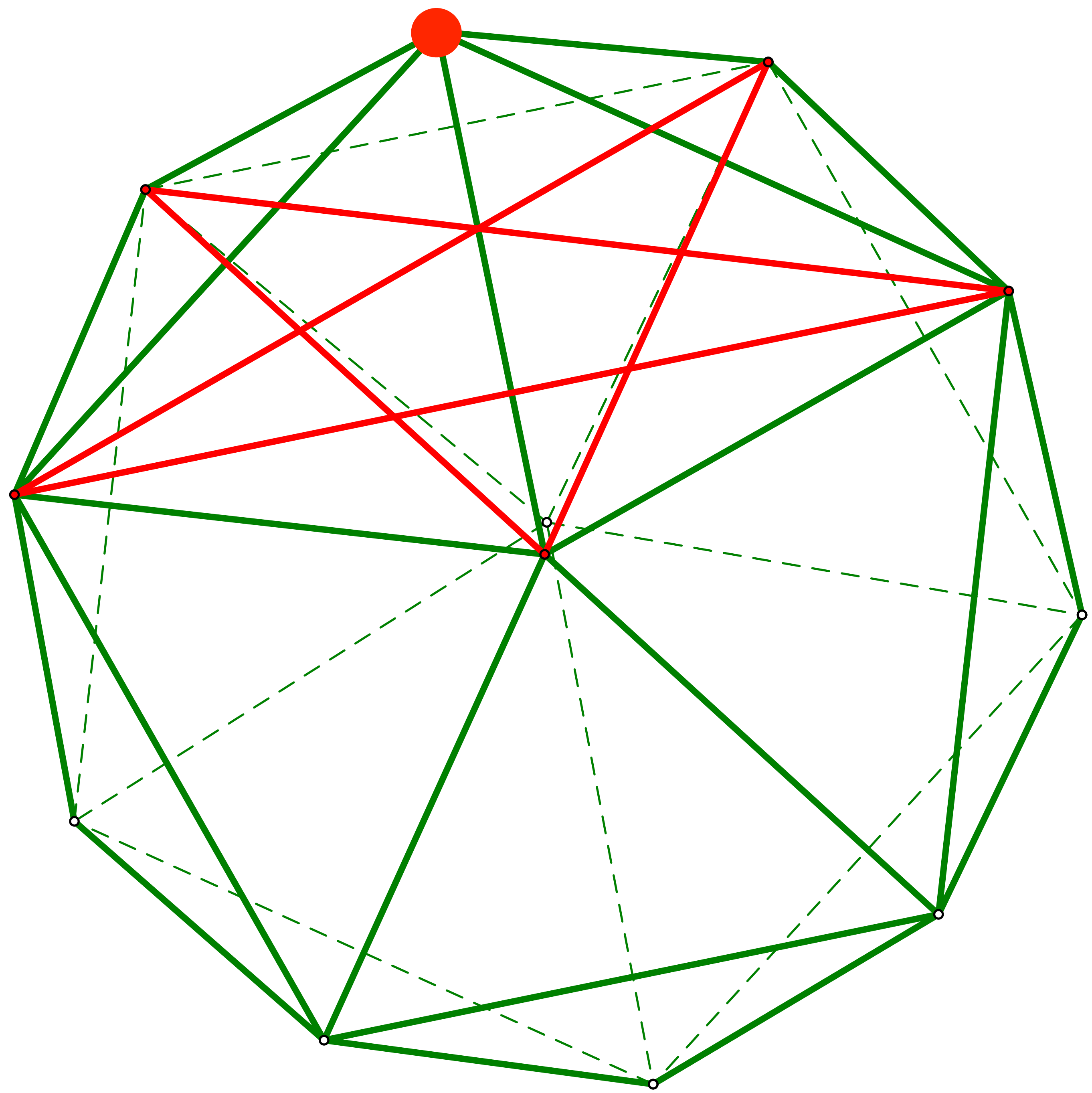


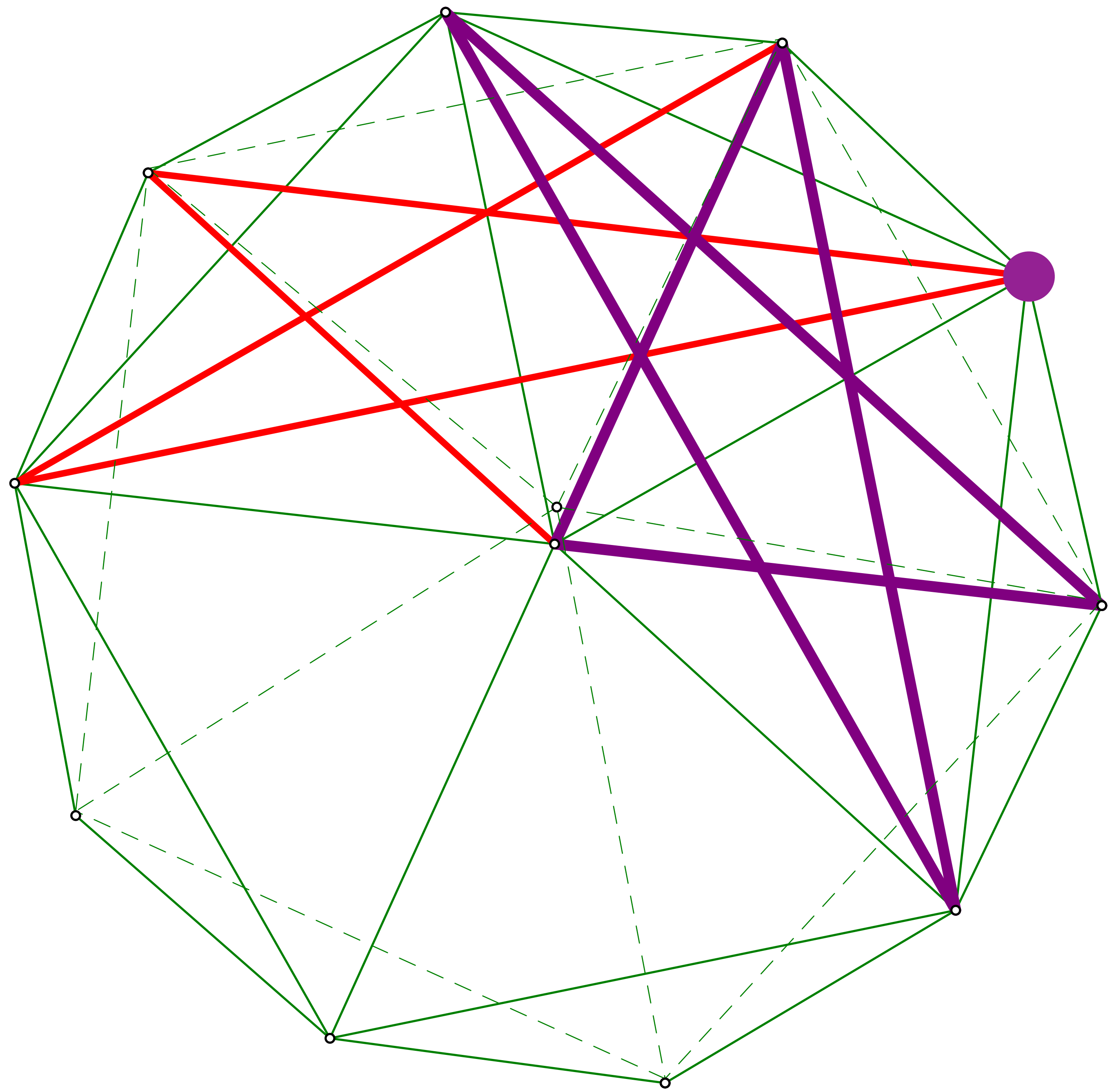


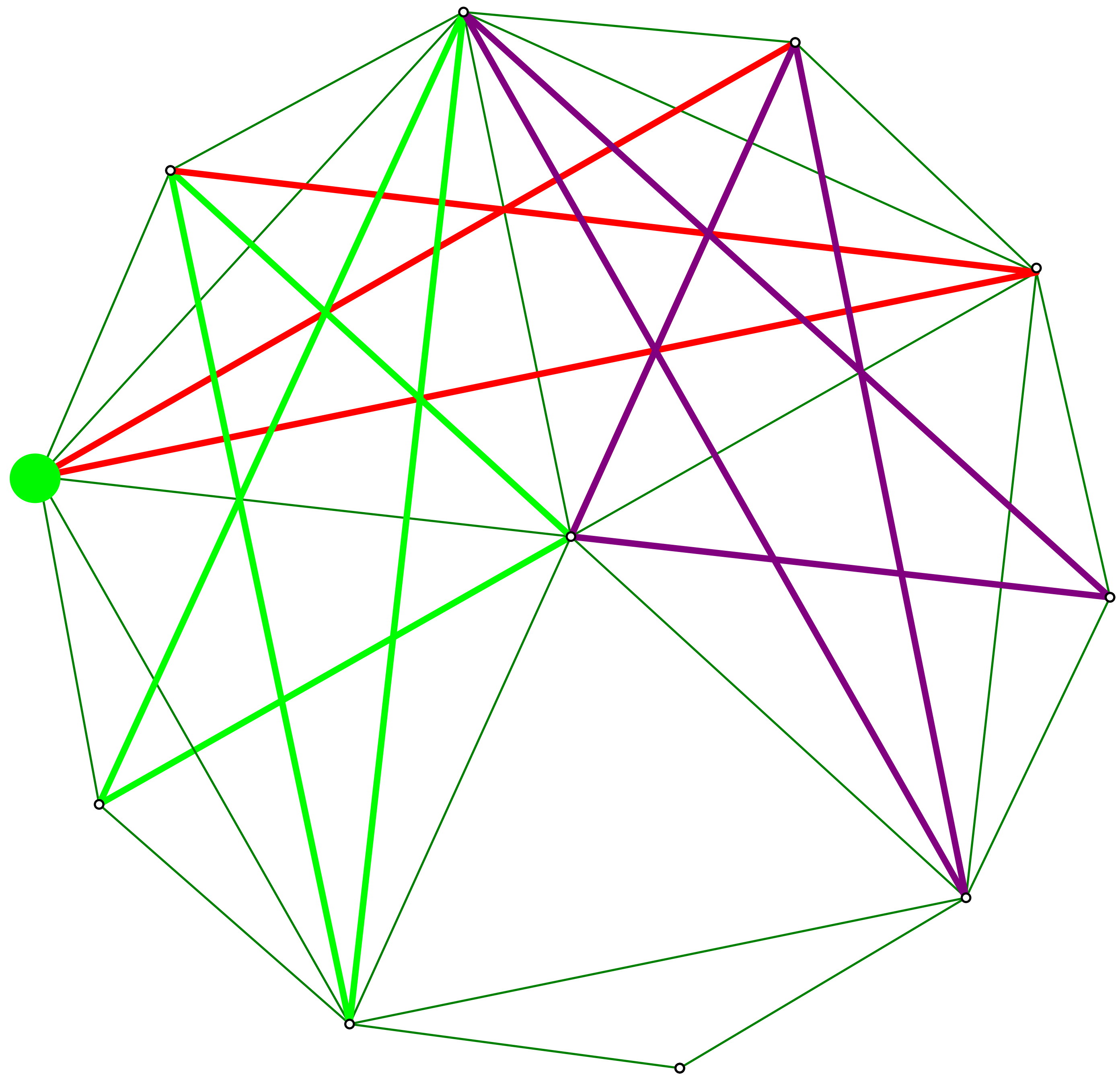


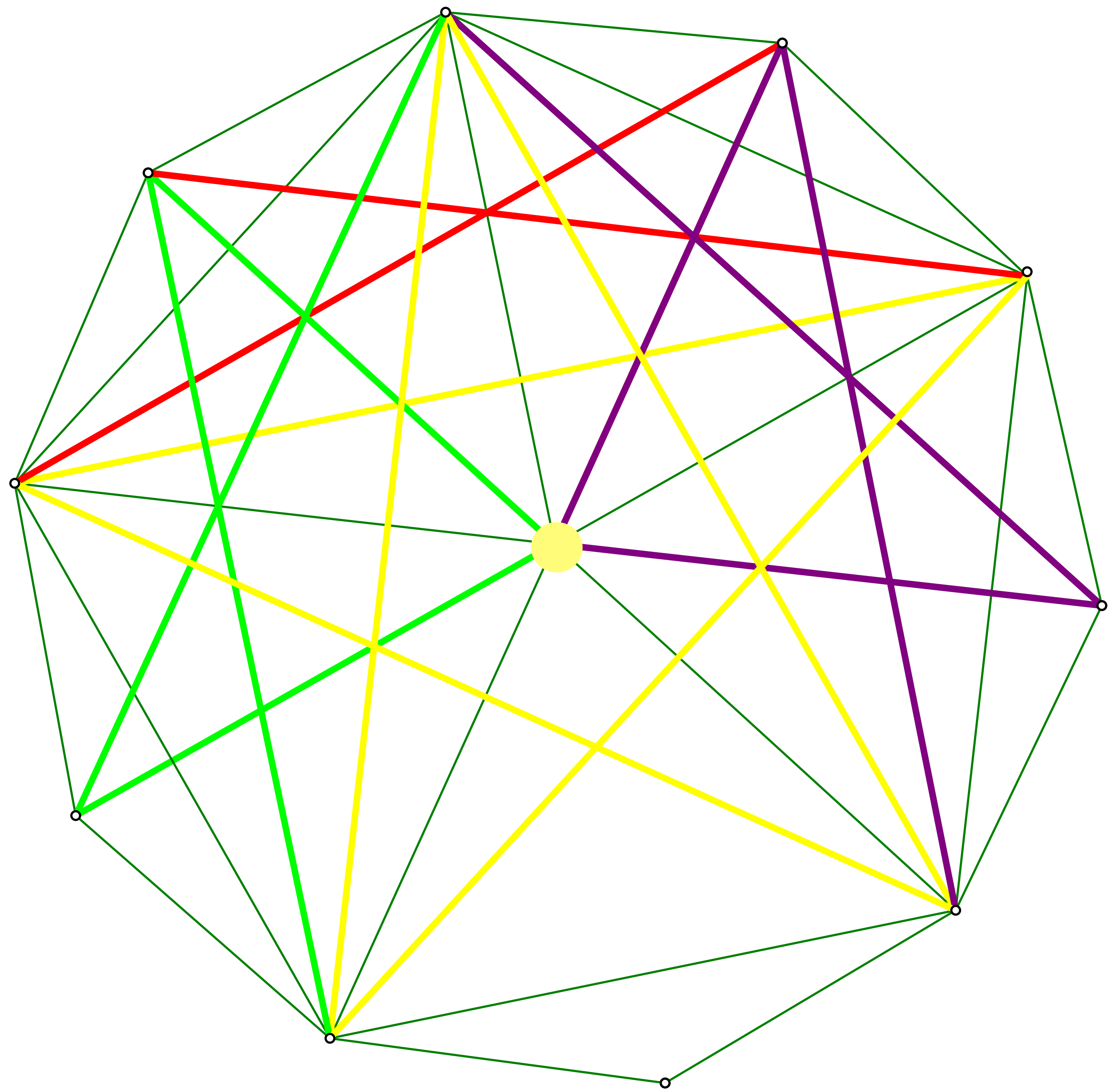


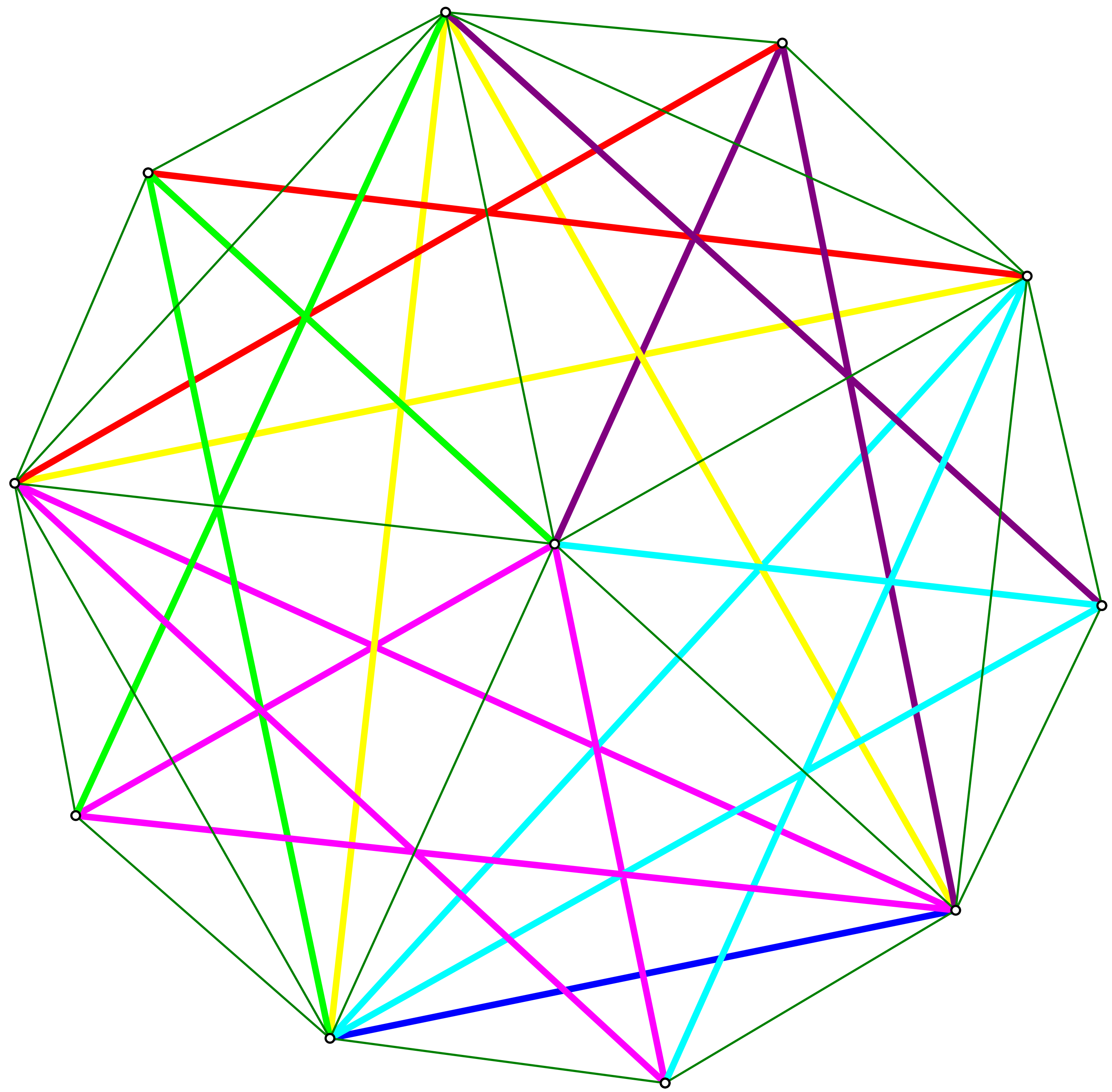


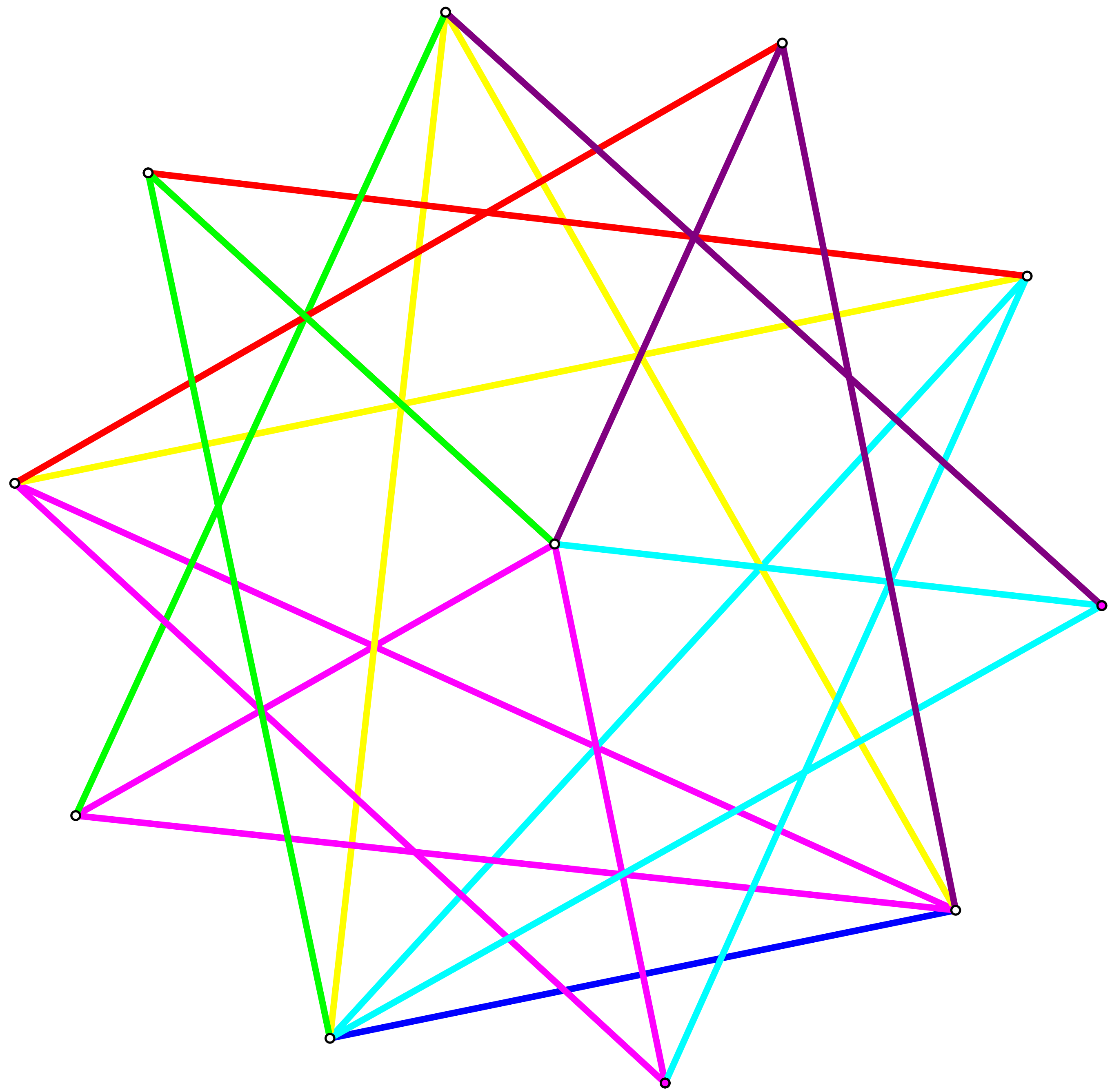


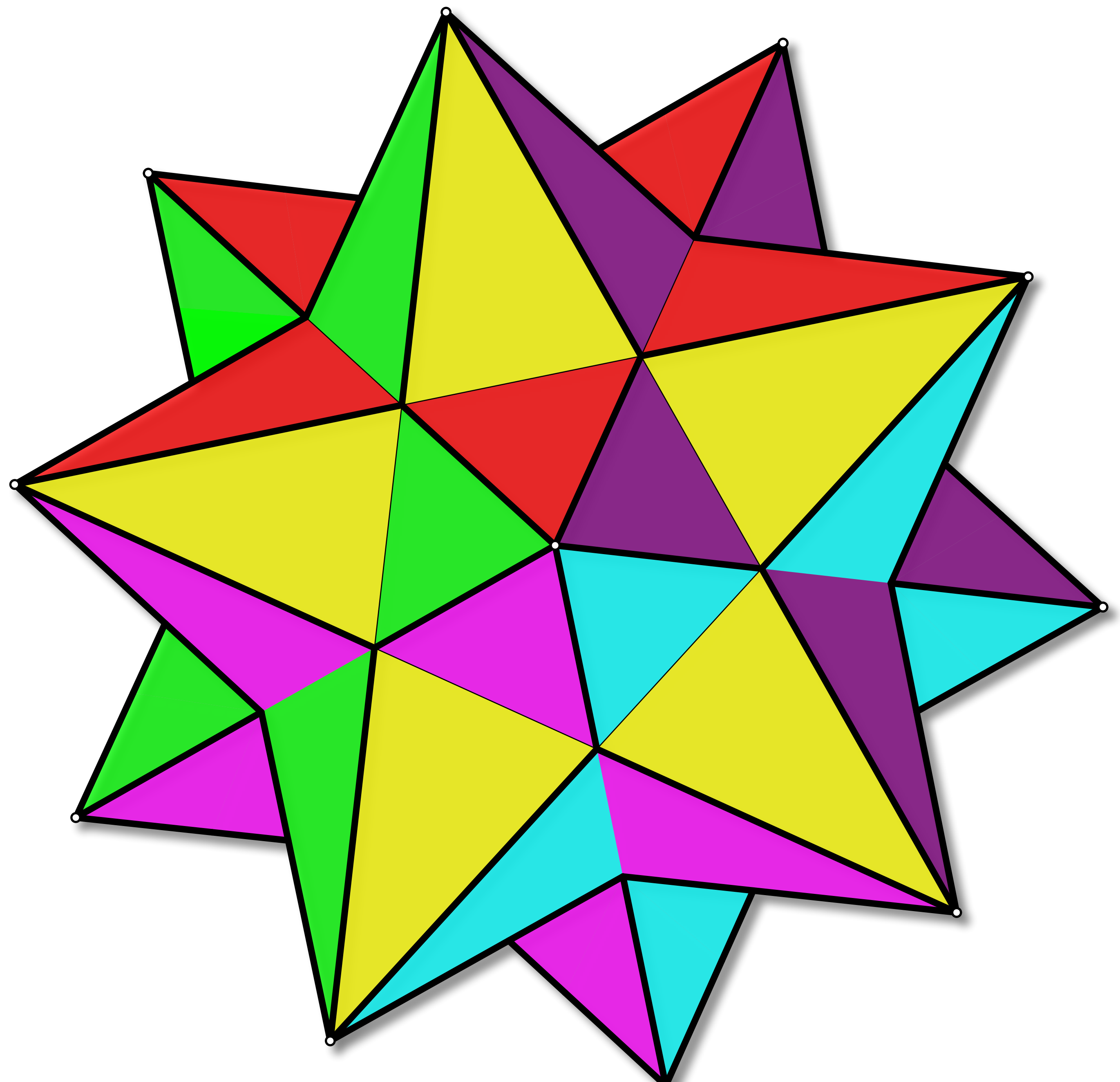






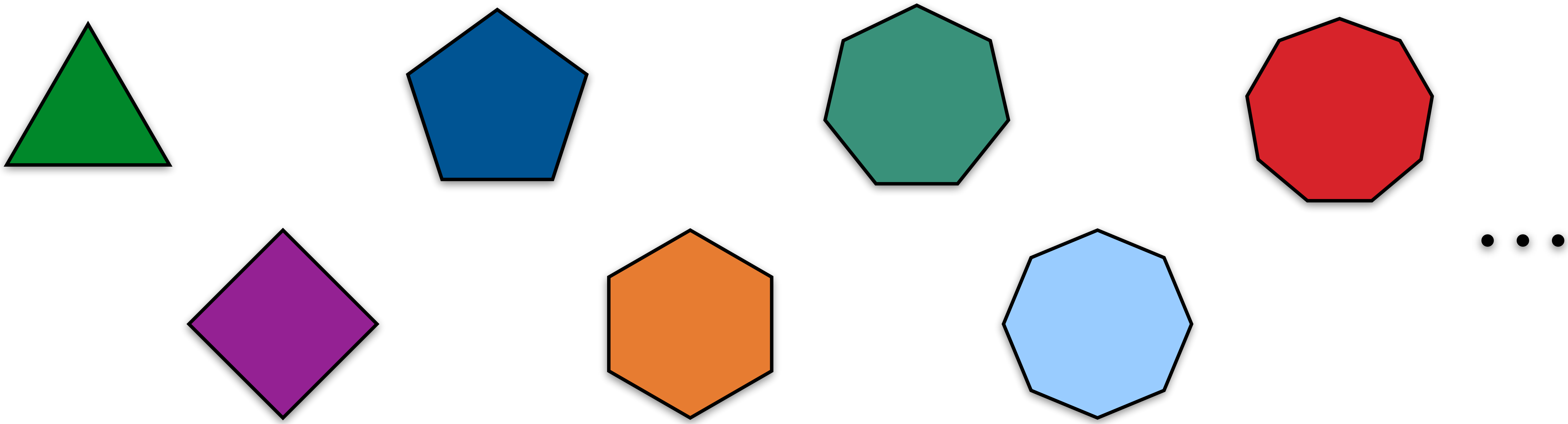


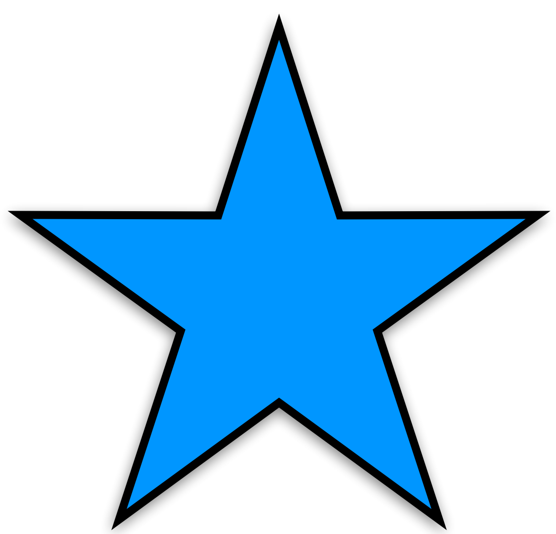




Podríamos pensar que sus caras son triángulos o…

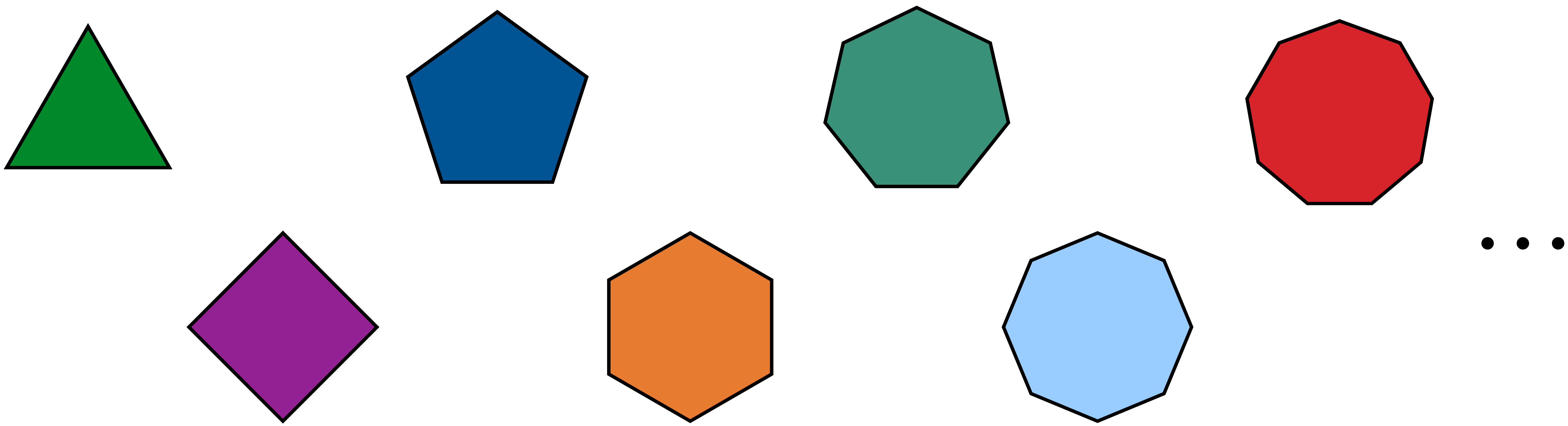
Polígonos regulares





¿Cuántos vértices y aristas tiene?

Polígonos regulares



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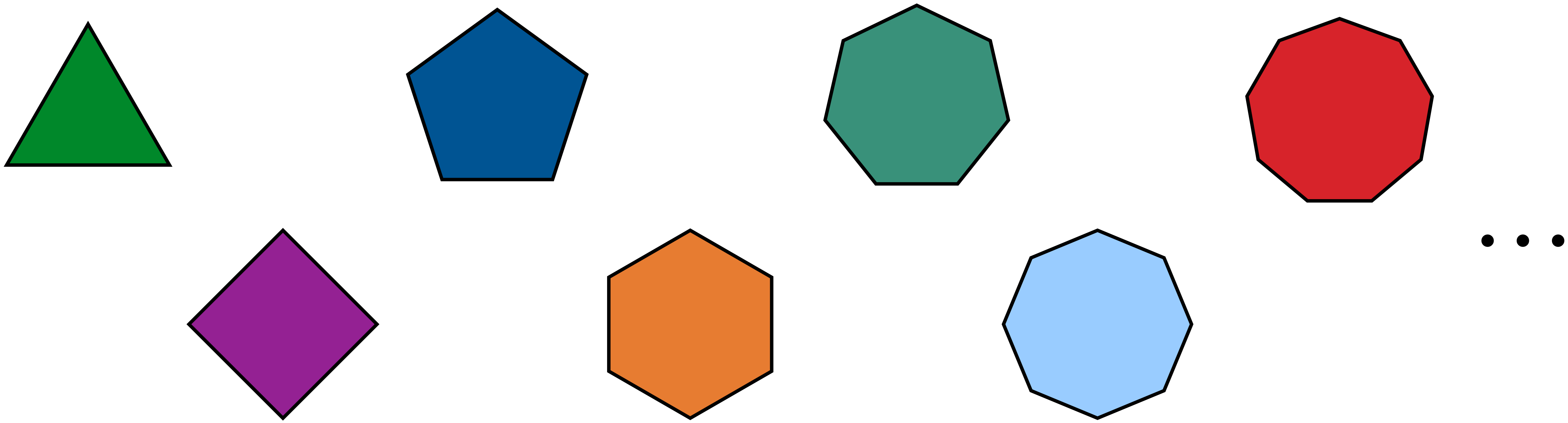

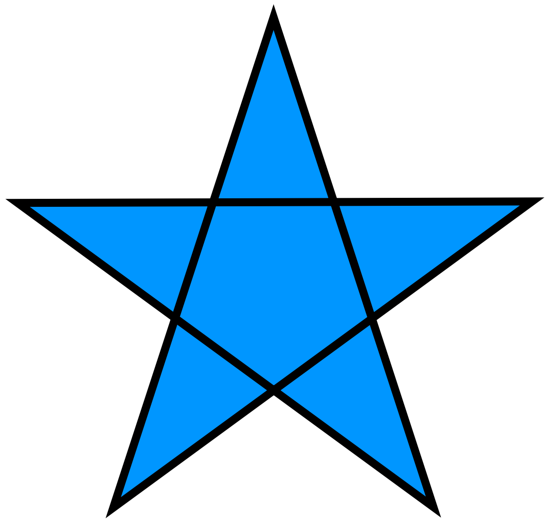

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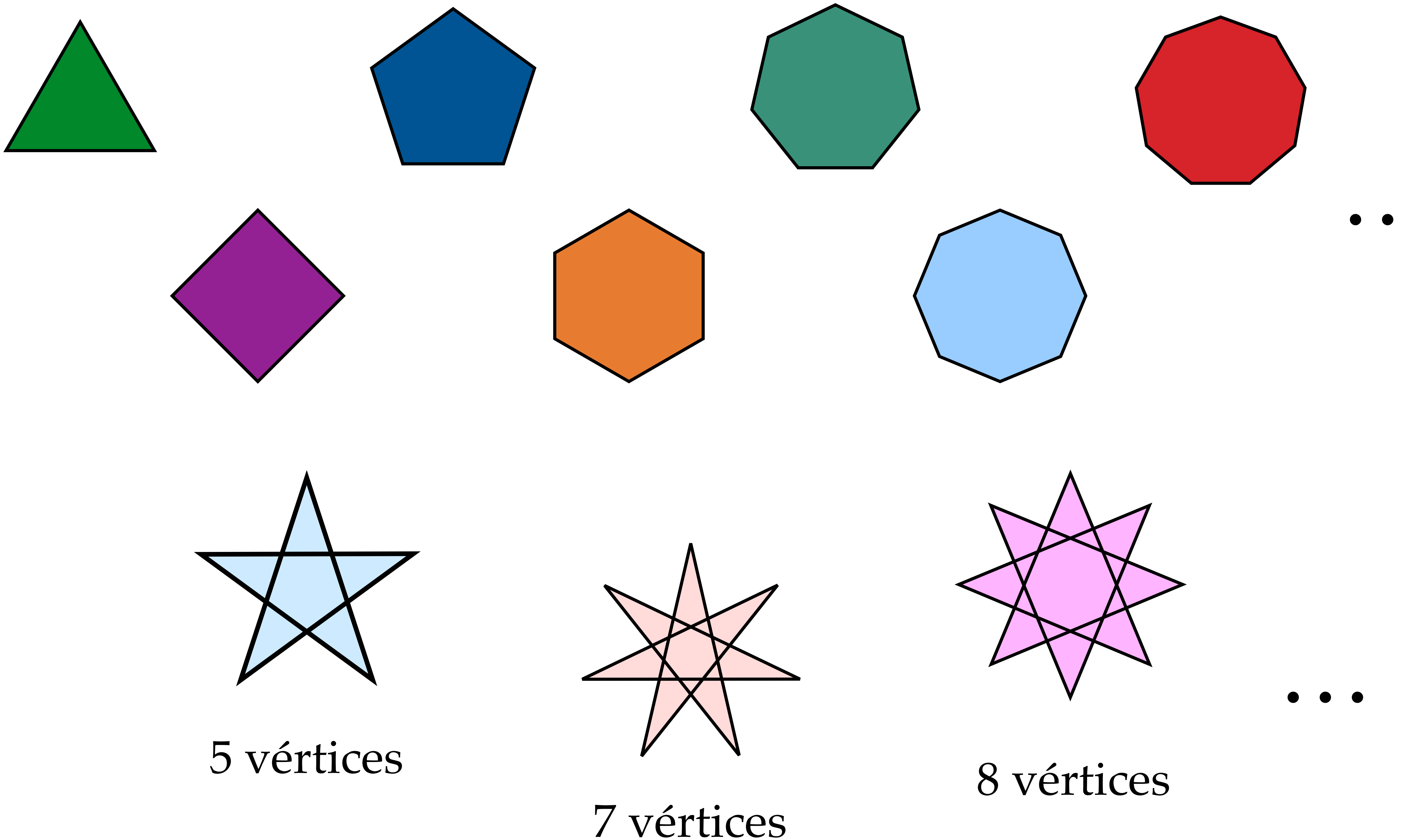
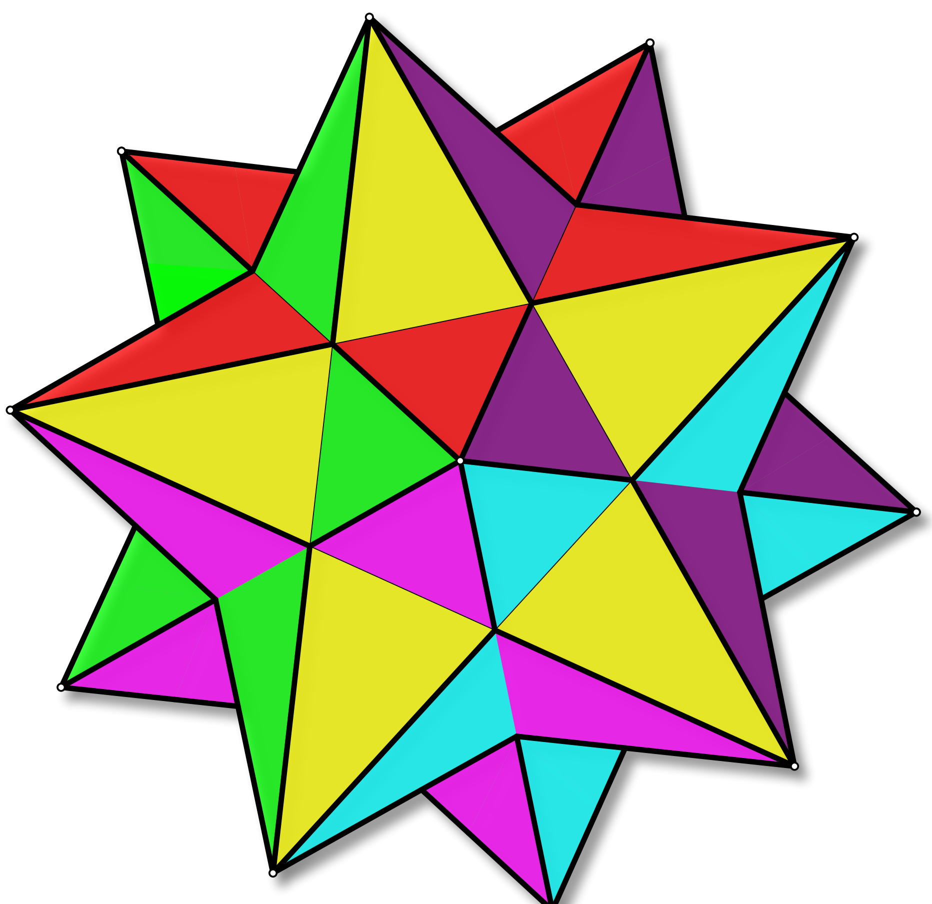
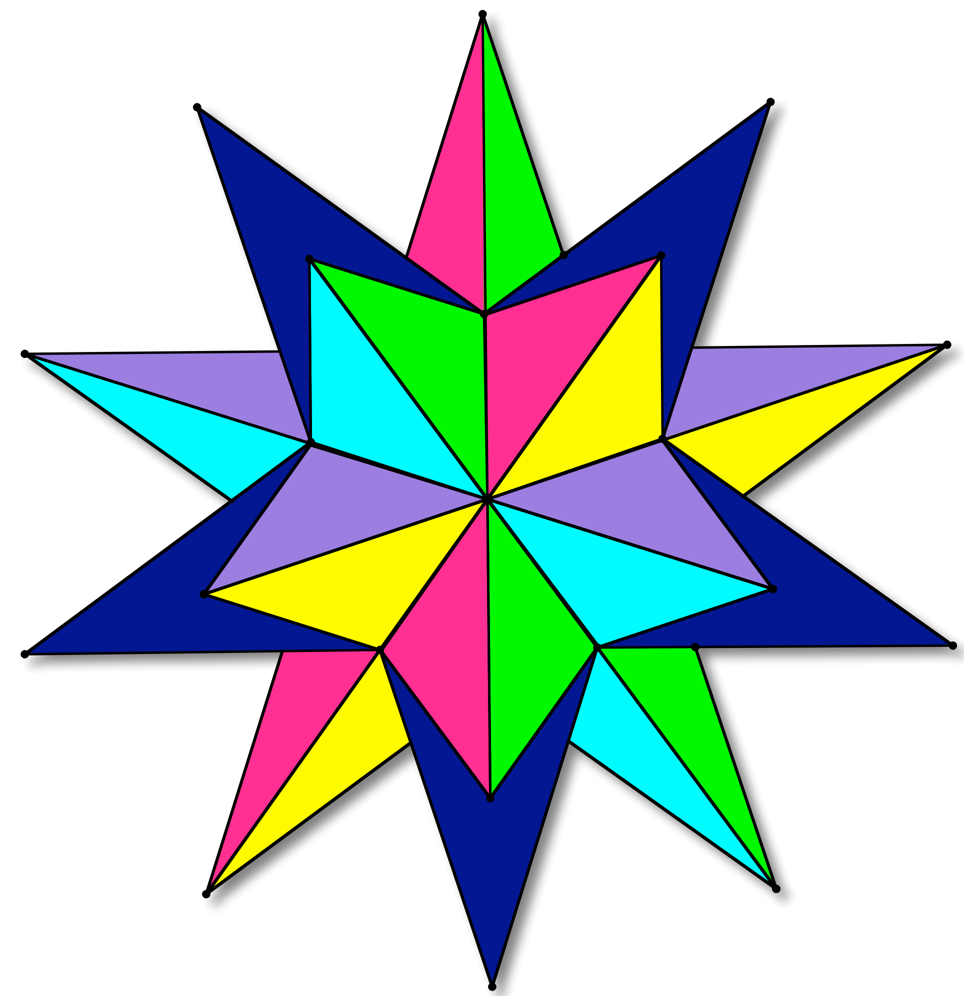


En 1619 Kepler encuentra dos poliedros ¿regulares?:

Las caras de ambos son "pentagramas" o pentágonos estrellados.

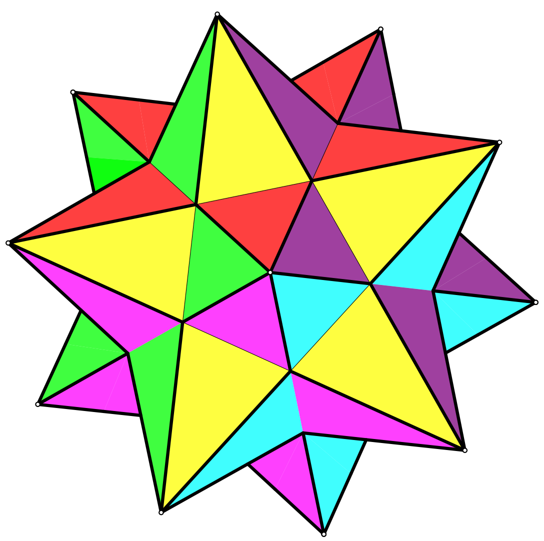


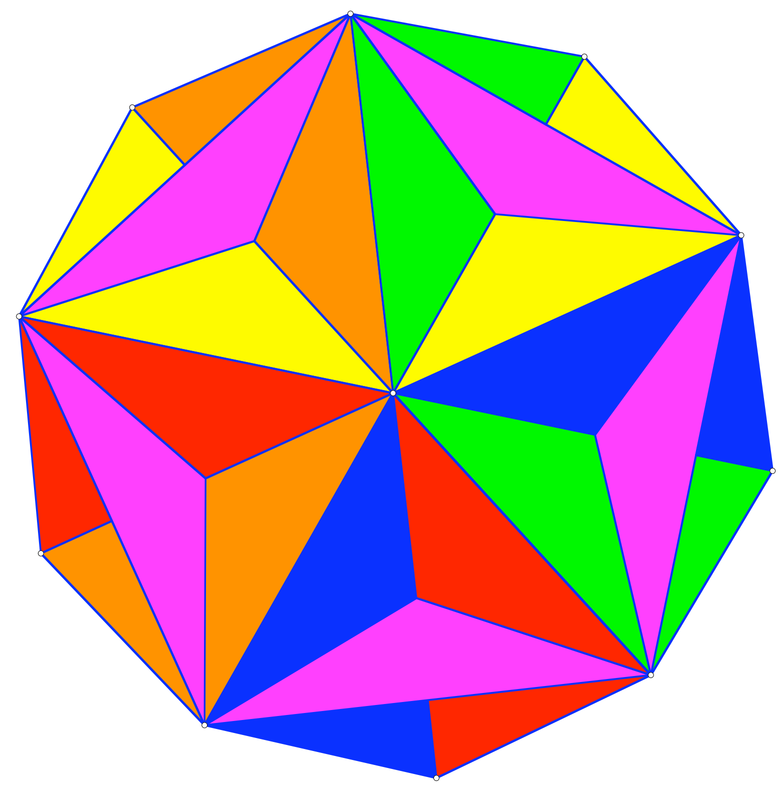
Hay 5 caras alrededor de cada vértice

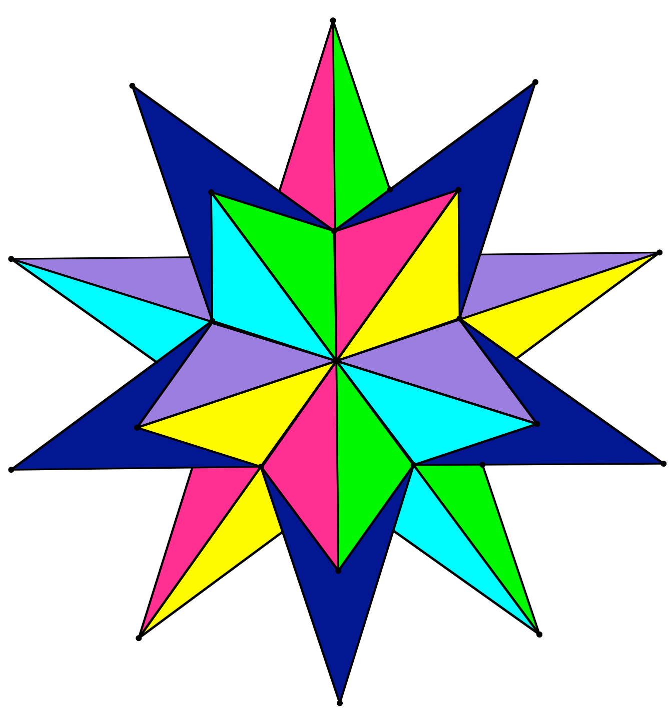


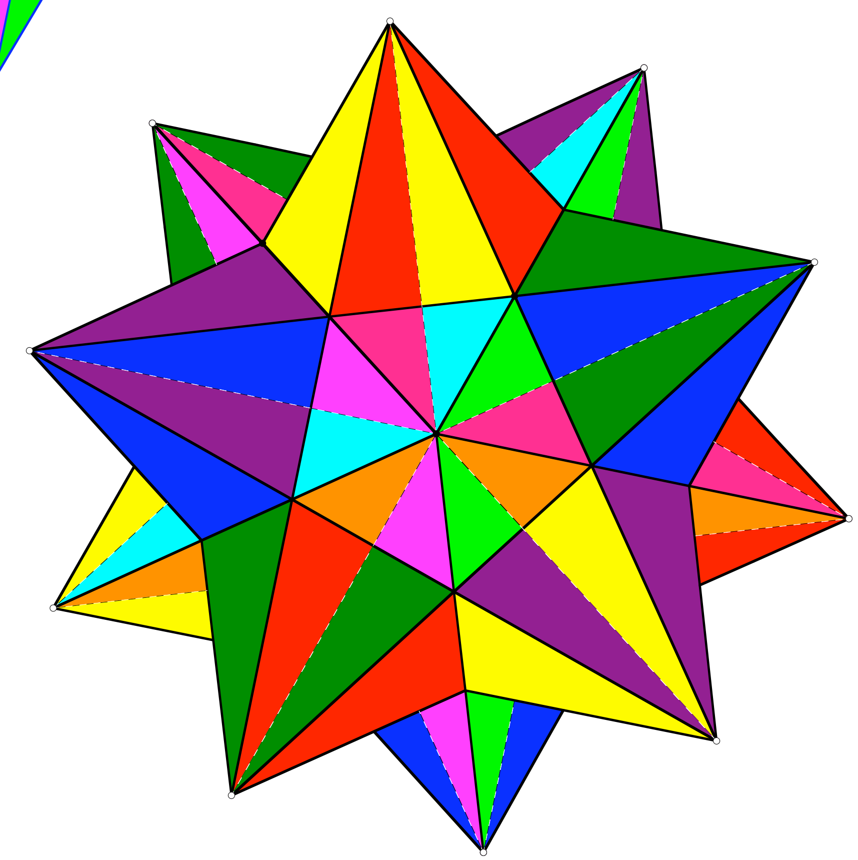
Hay 3 caras alrededor de cada vértice

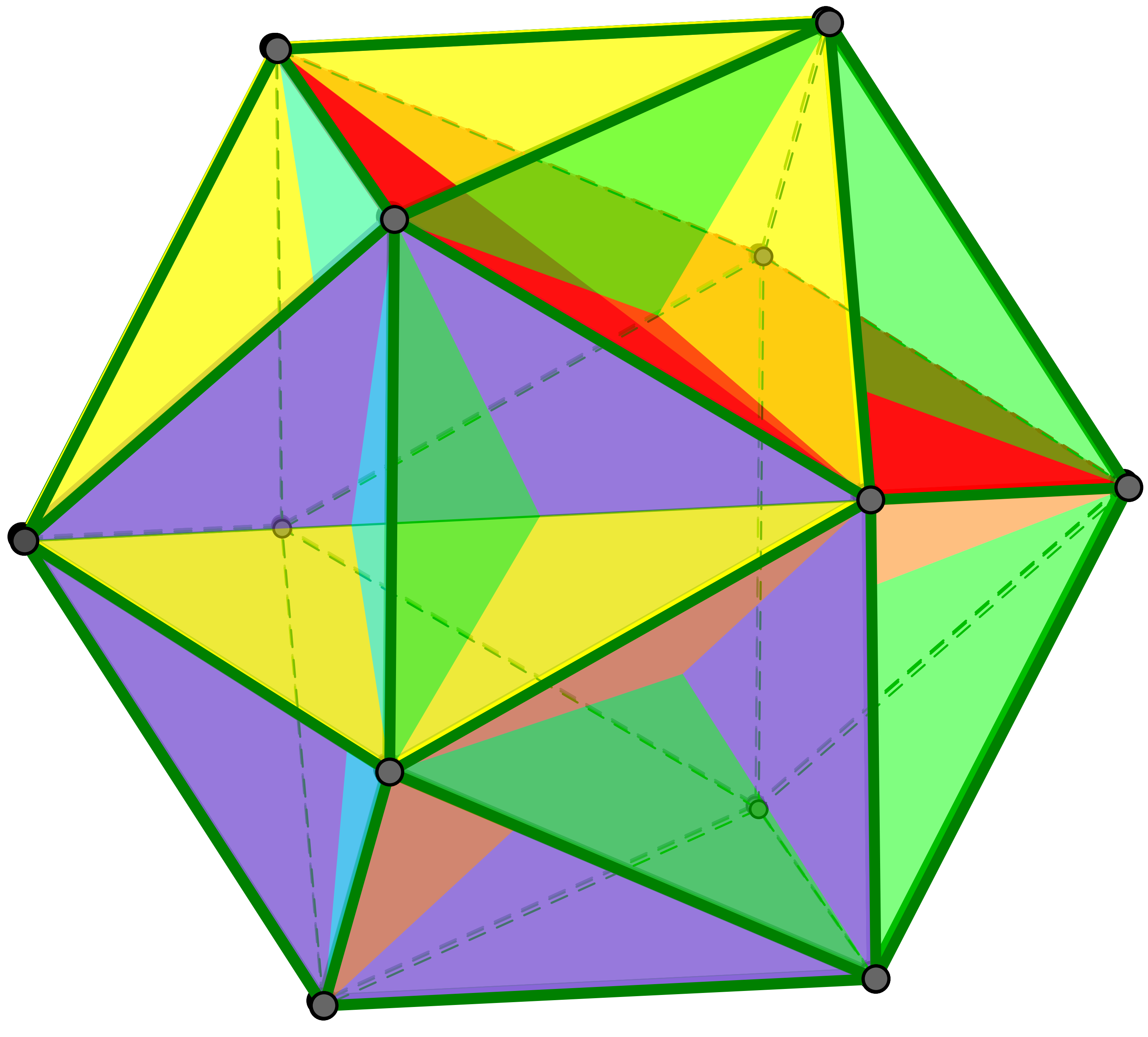
En 1809, Poinsot los redescubre y descubre dos más:



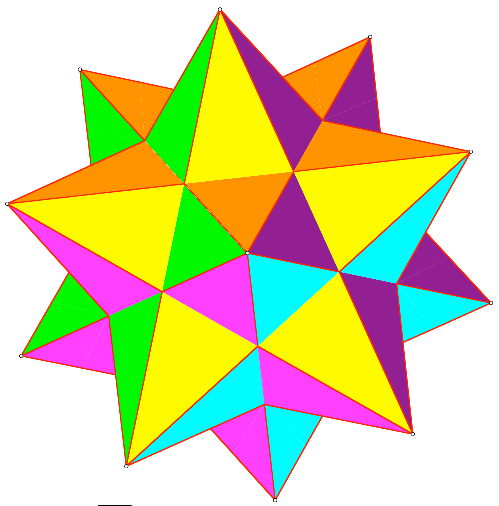




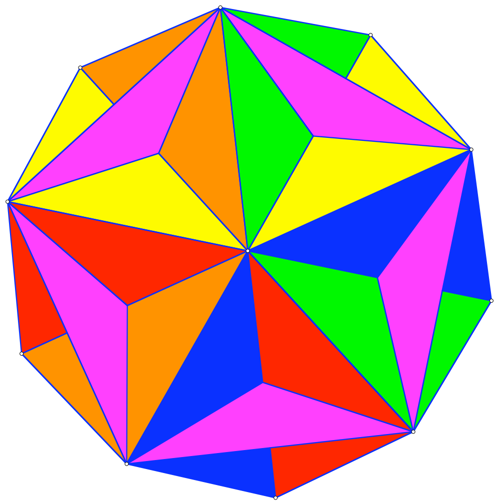




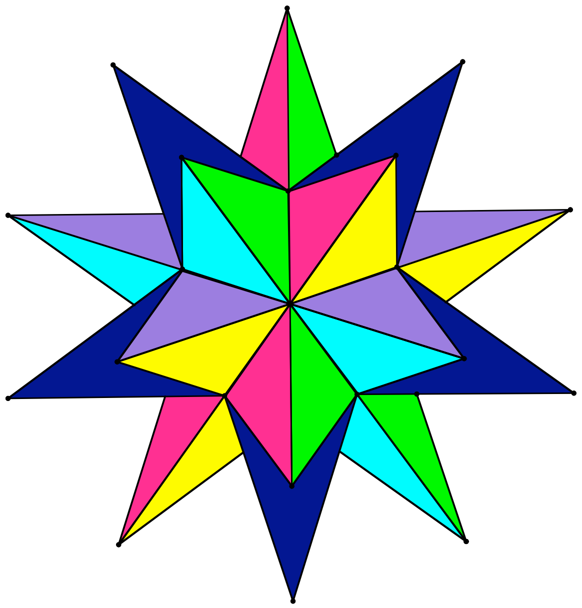
En 1812, Cauchy demuestra que estos son todos los poliedros regulares “estrellados”



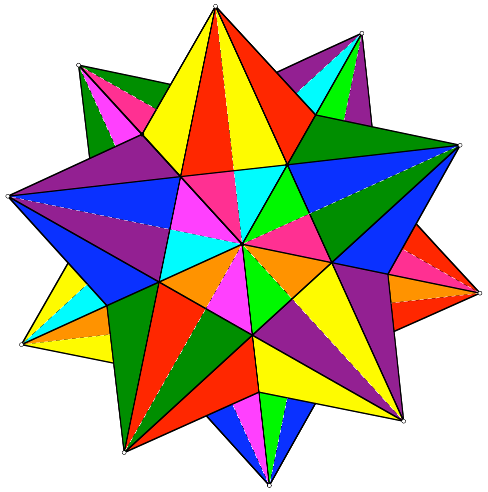
Pequeño dodecaedro estrellado



Gran dodecaedro



Gran dodecaedro estrellado



Gran icosaedro

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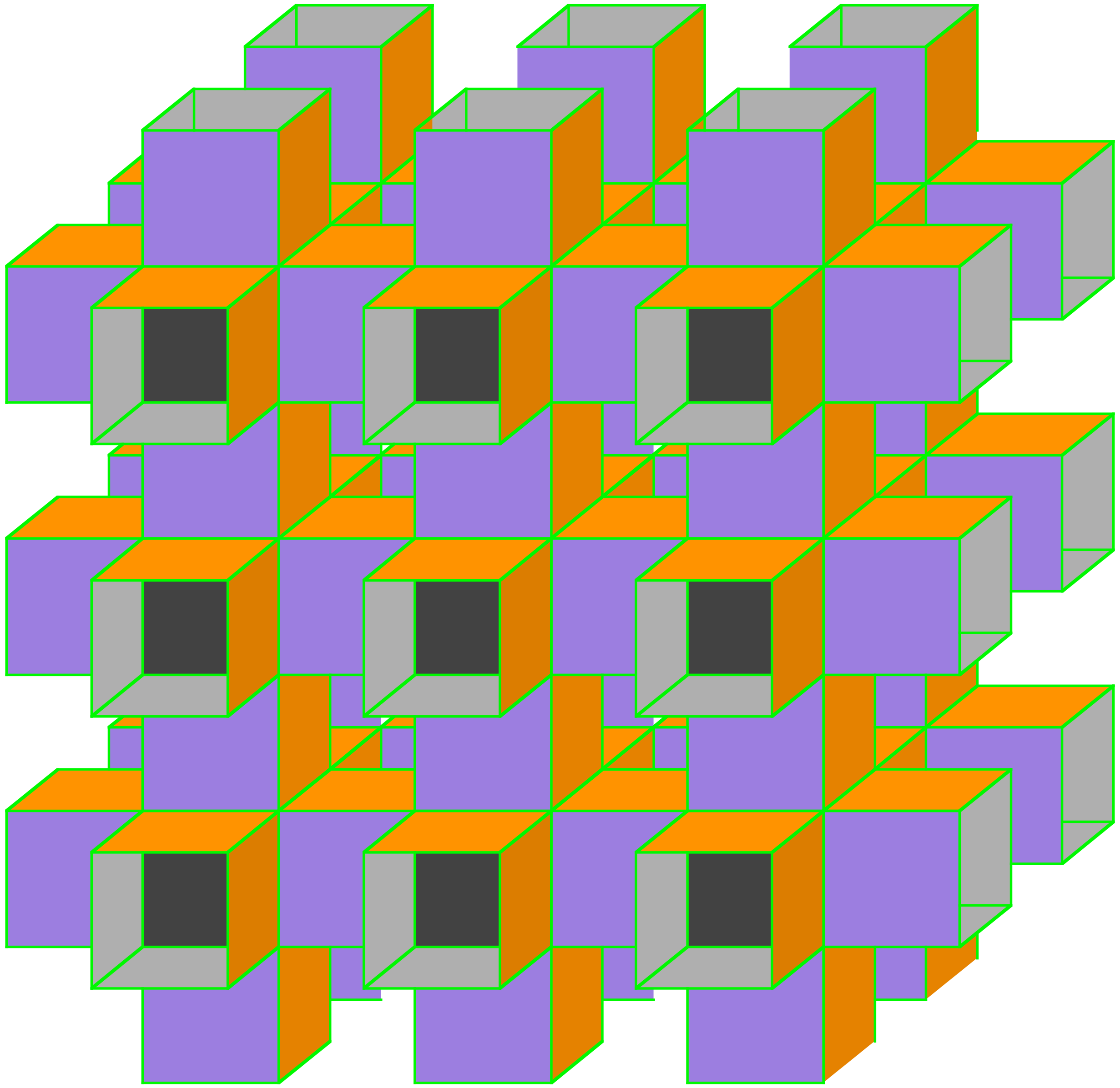
En 1924 Petrie encuentra un nuevo poliedro regular, pero… ¡infinito!



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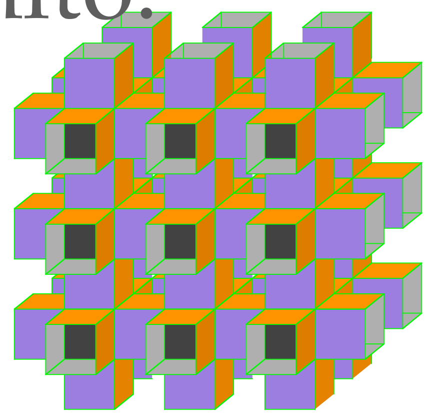

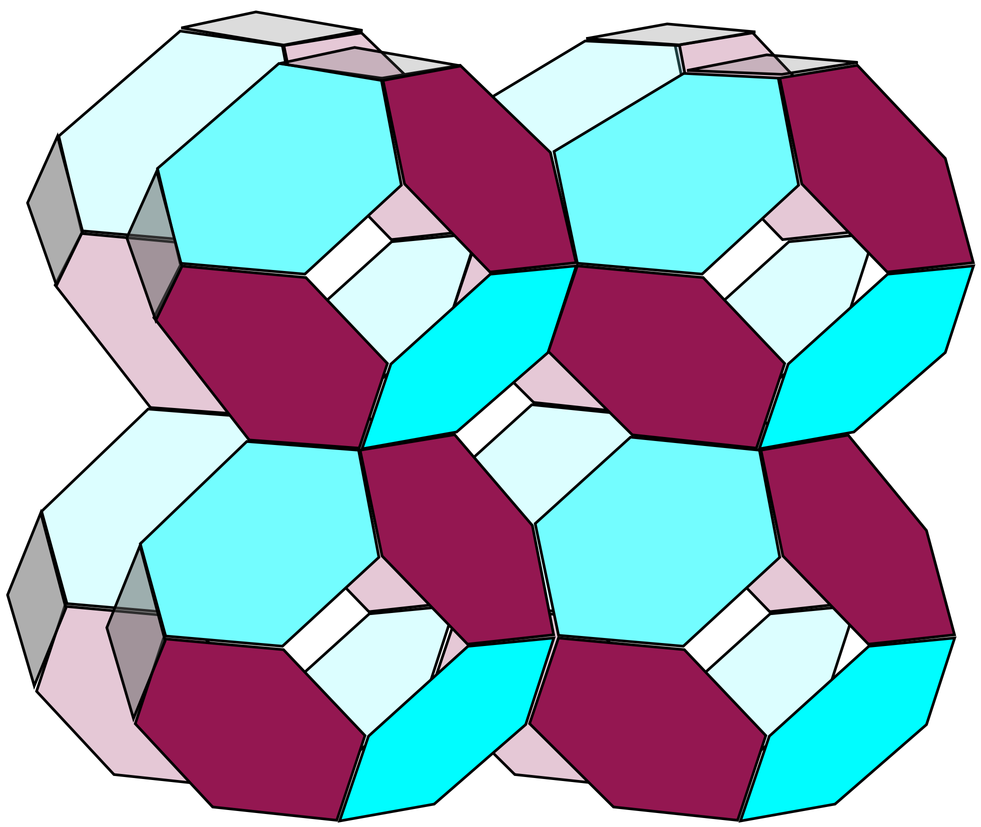

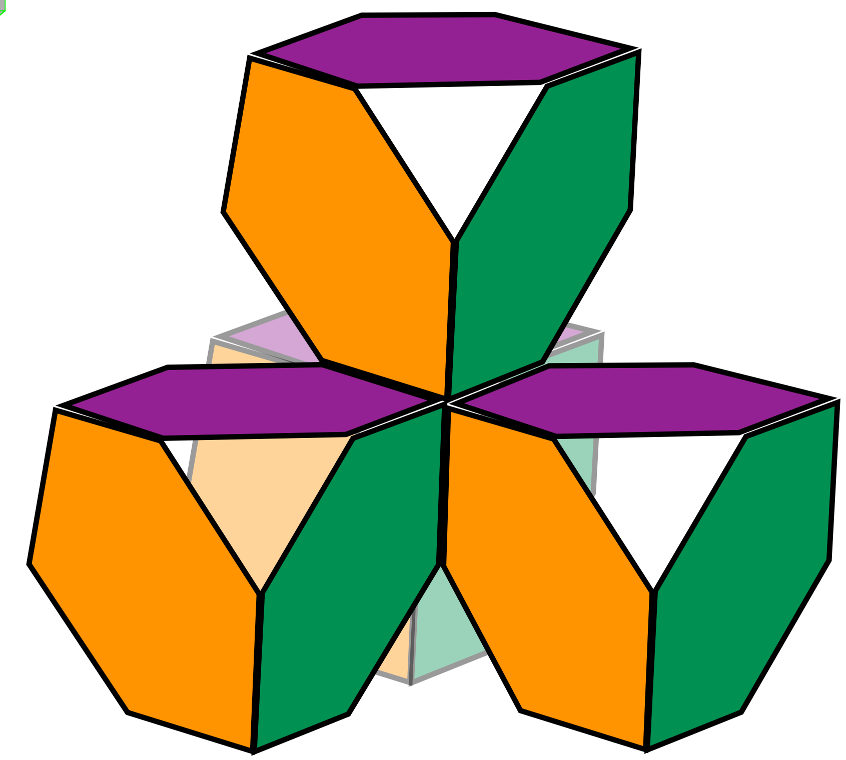

Después de platicarle a Coxeter cómo los construye, él encuentra dos más y demuestra que solo hay tres de este estilo

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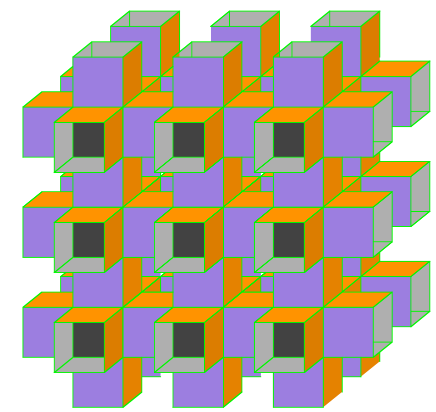
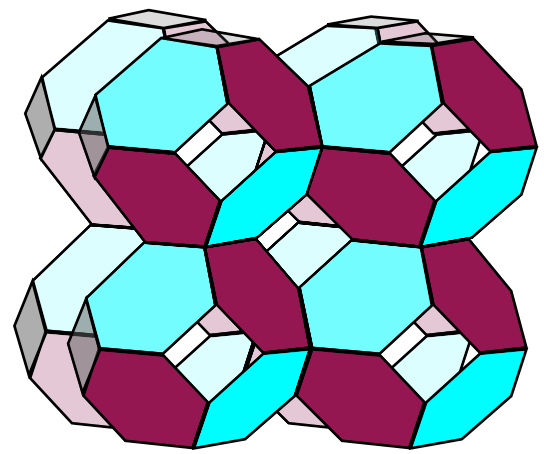
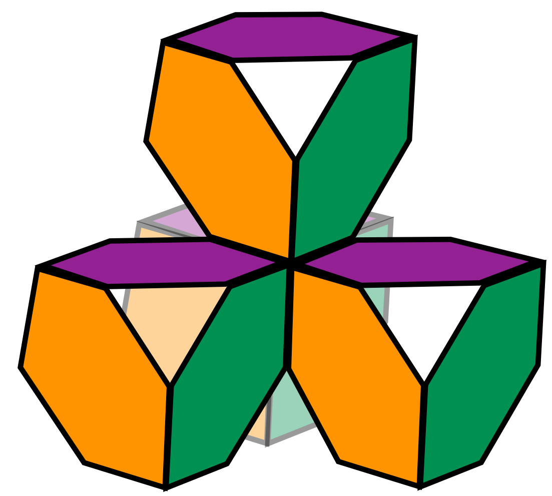


H.S.M. Coxeter
1907 - 2003





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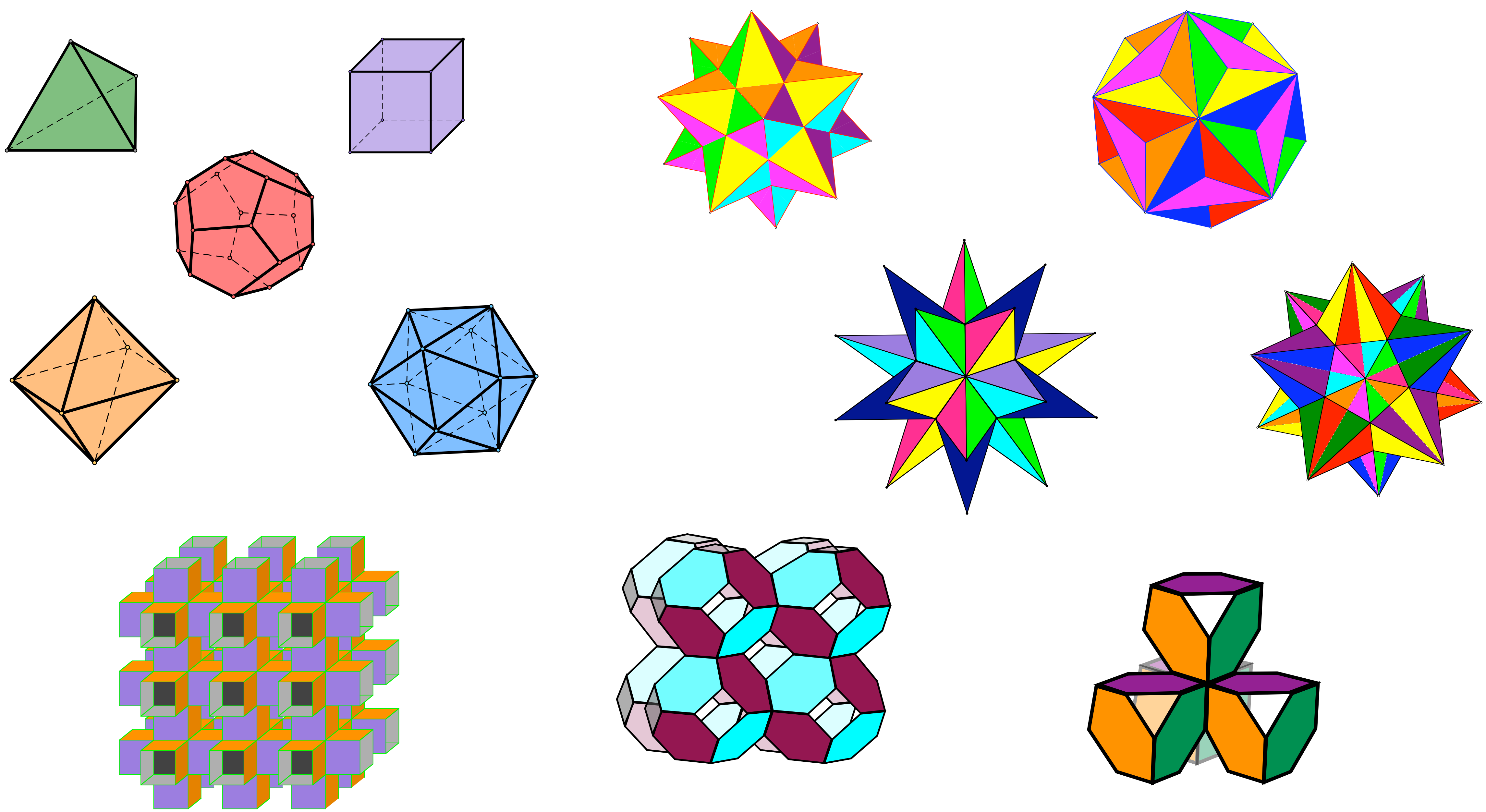
lot of new regular polyhedra, and soon thereafter Dress proved [15], [16] that one needs to add just one more polyhedron to make my list complete. Then, about ten years ago I found [22] a whole slew of new regular polyhedra, and so far nobody claimed to have found them all.

Grünbaum 1977

Encuentra 47 poliedros regulares en $\mathbb{R}^3$



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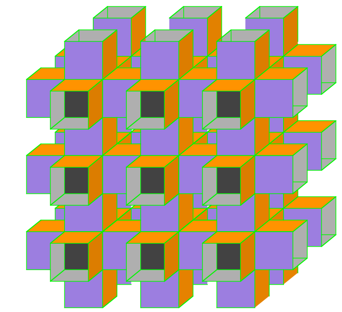
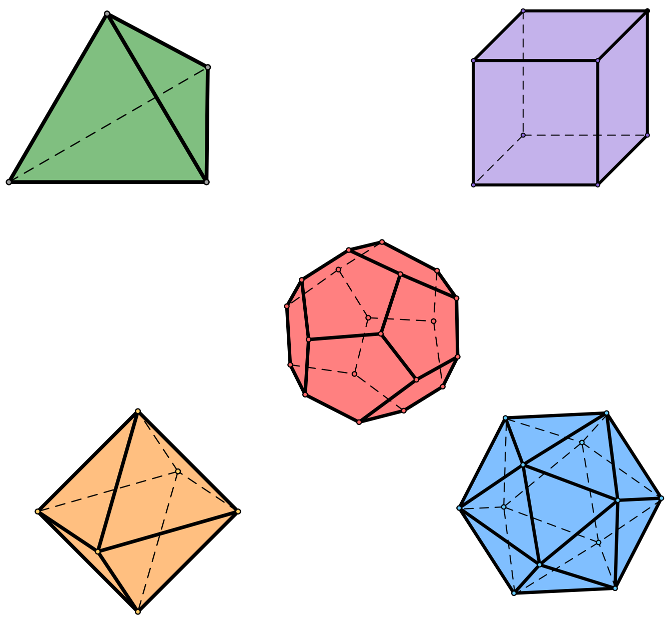
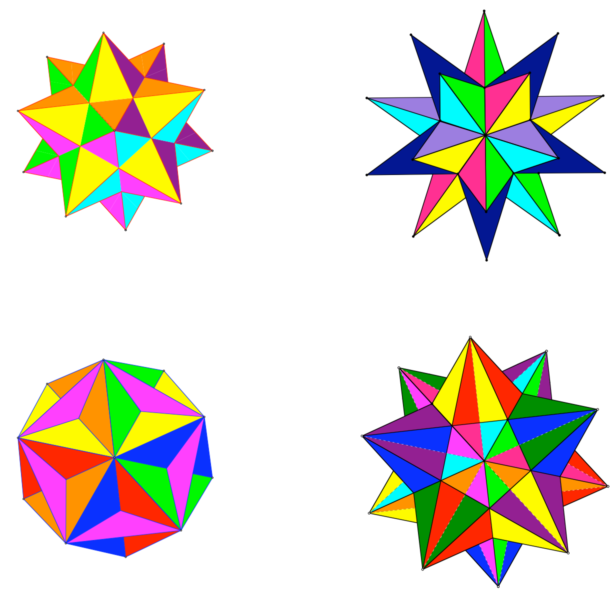
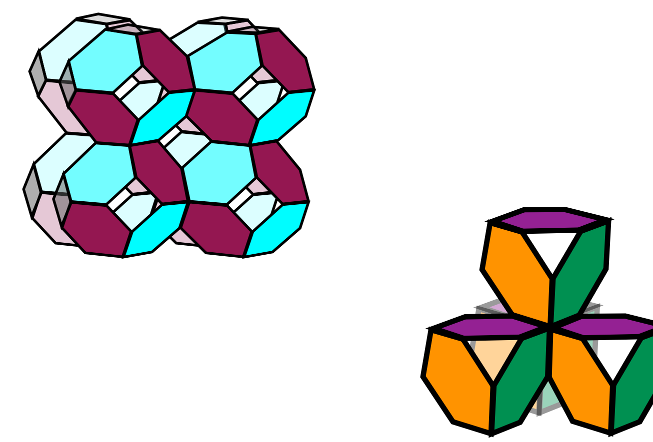






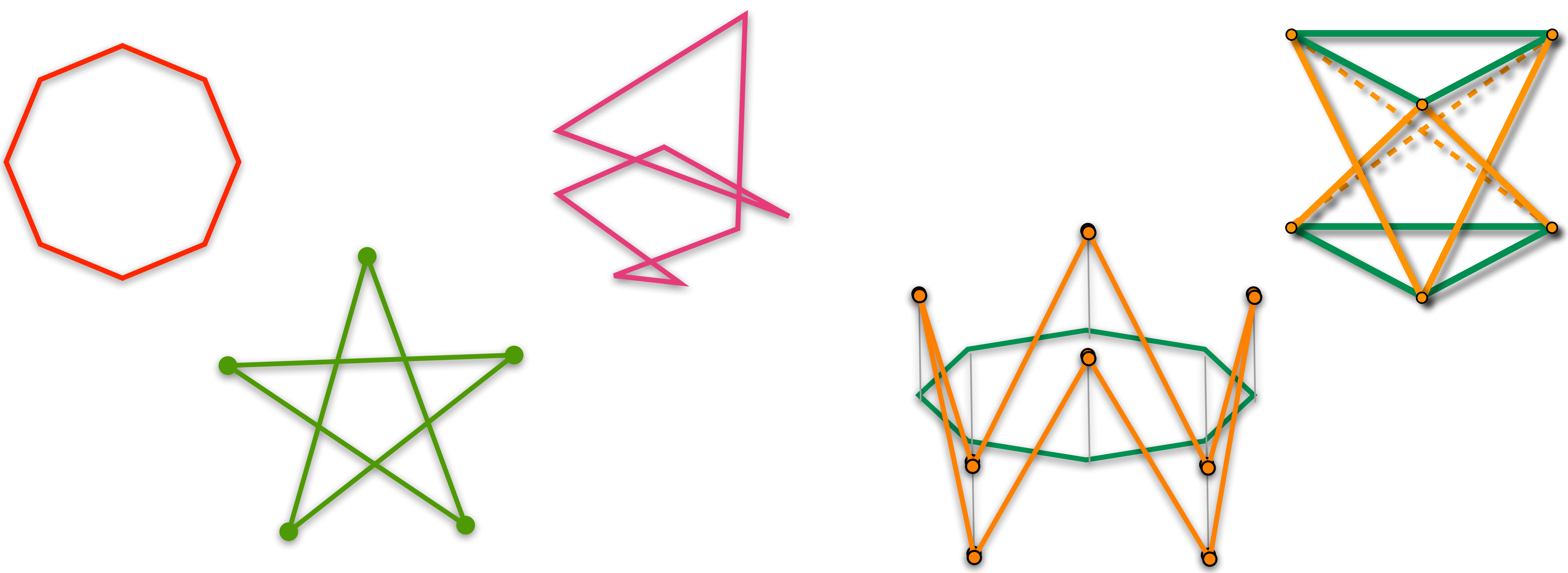


- Es el primero que da una definición de poliedro y de poliedro regular;
- No dice que su numeración está completa

Un poliedro es una colección de polígonos (que llamamos caras) tal que:

polígonos

Un polígono es una colección discreta de puntos en el espacio, llamados vértices, junto con segmentos de recta entre ellos, llamados aristas, tales que cada vértice está en exactamente dos aristas y el objeto es conexo.



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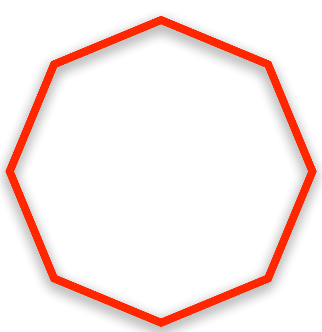
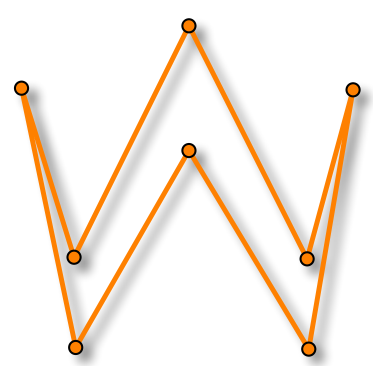
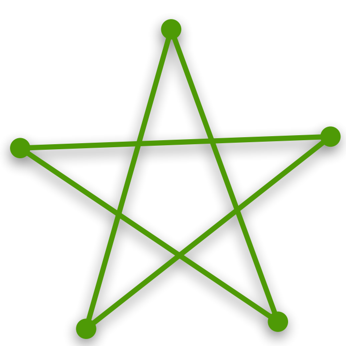
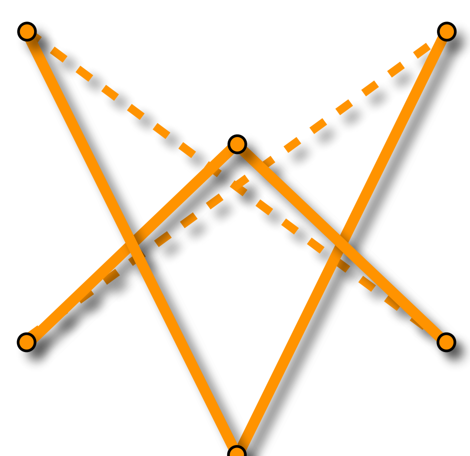
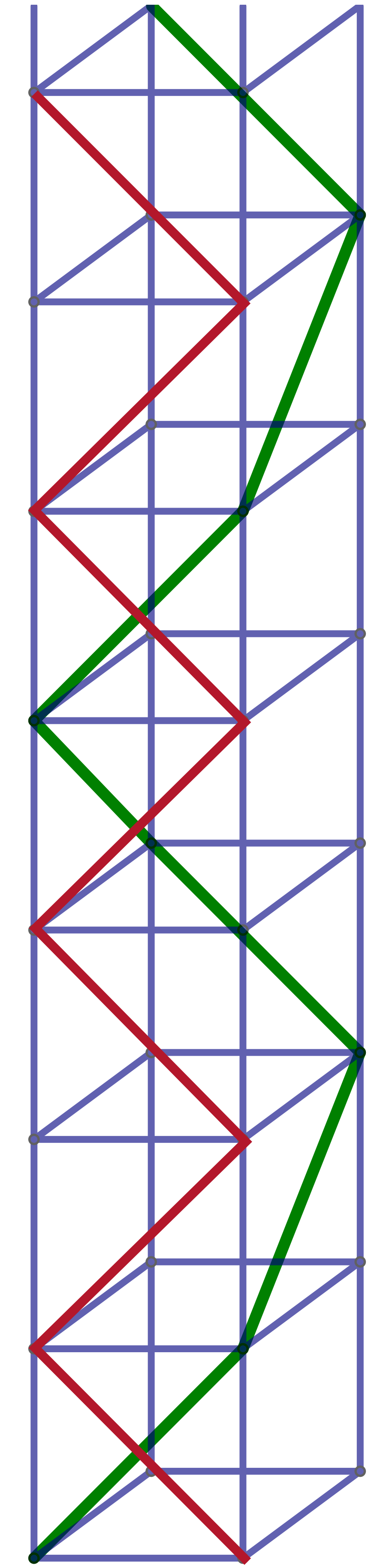




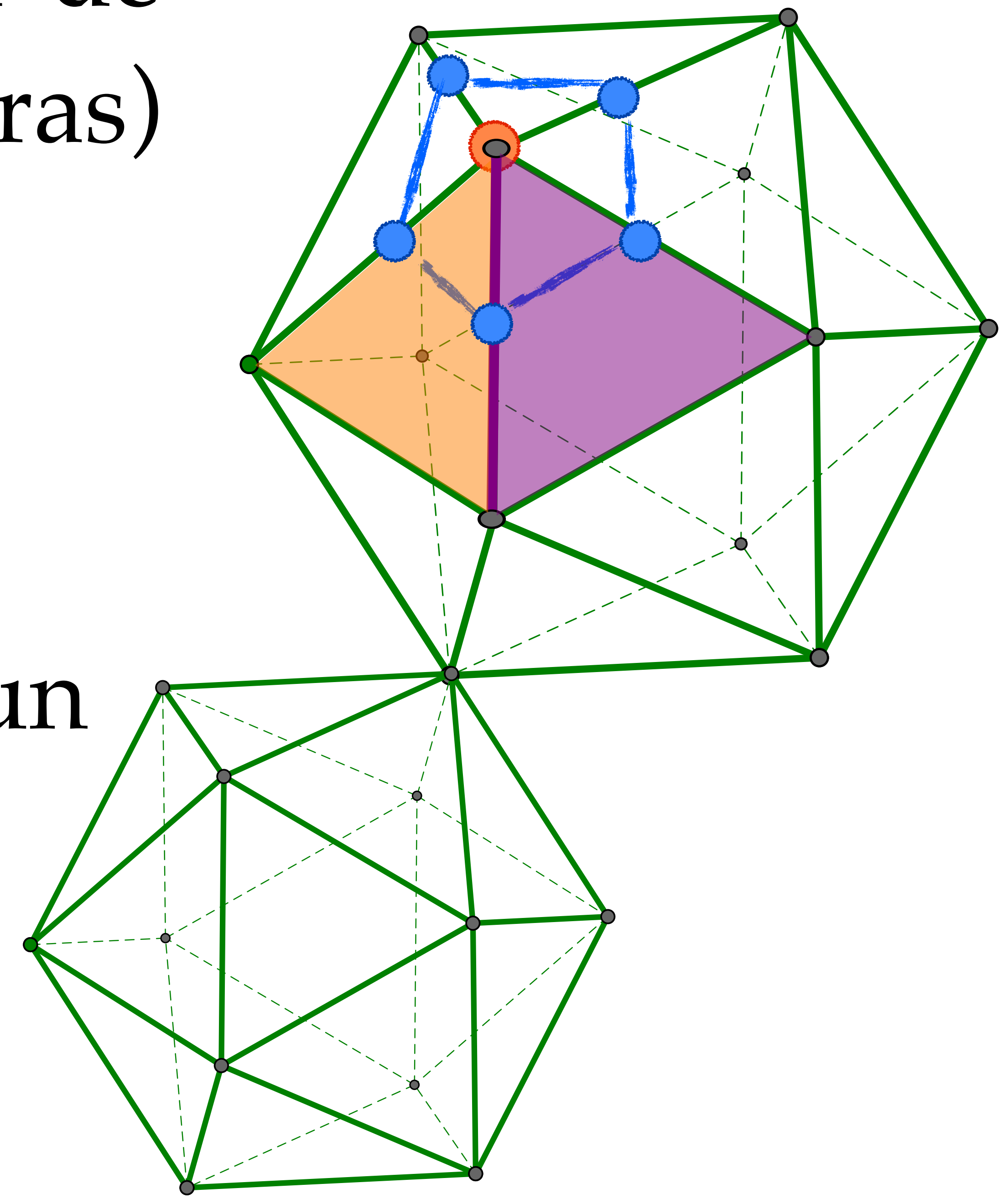






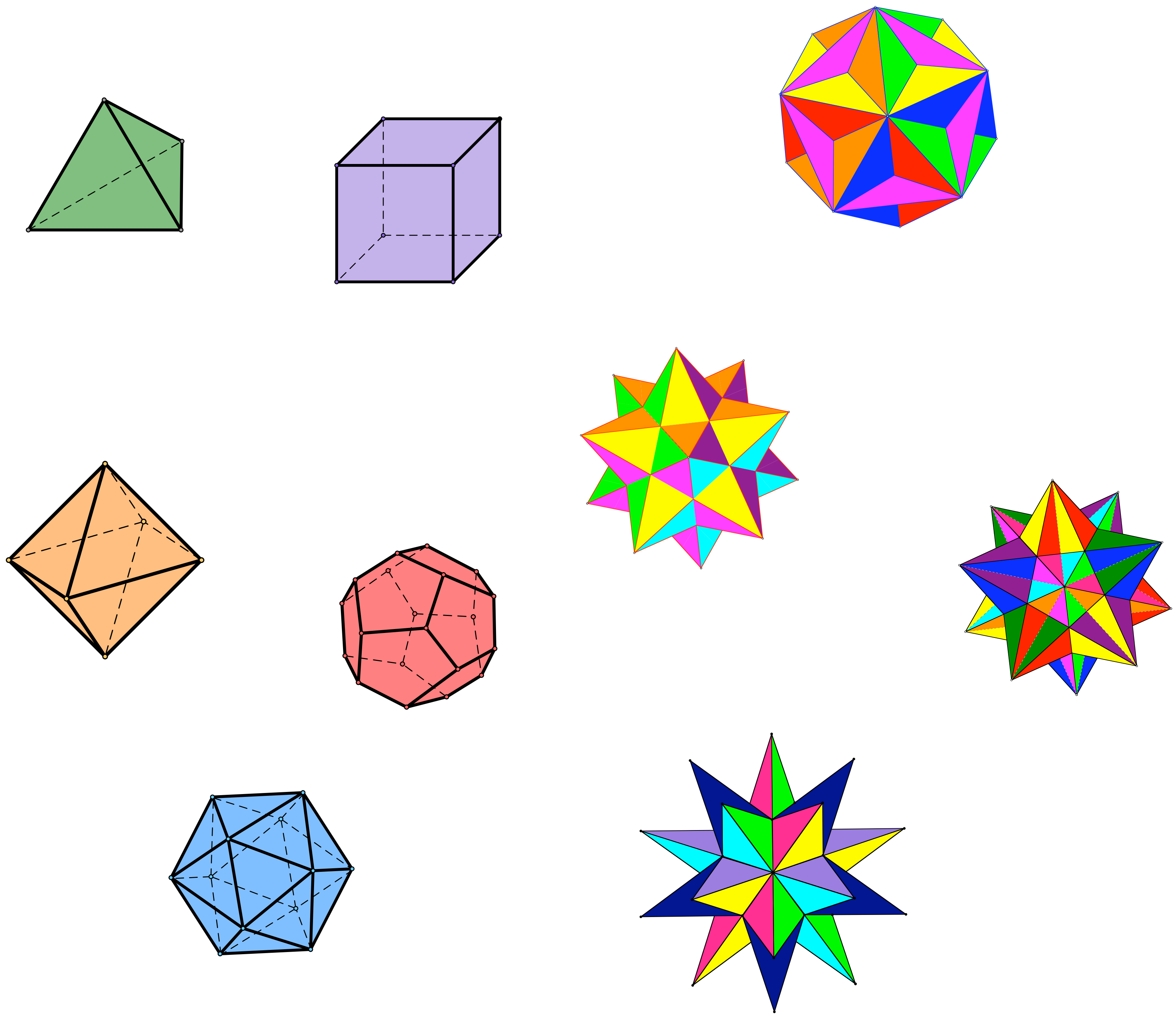
Un poliedro es una colección de polígonos (que llamamos caras) tal que:

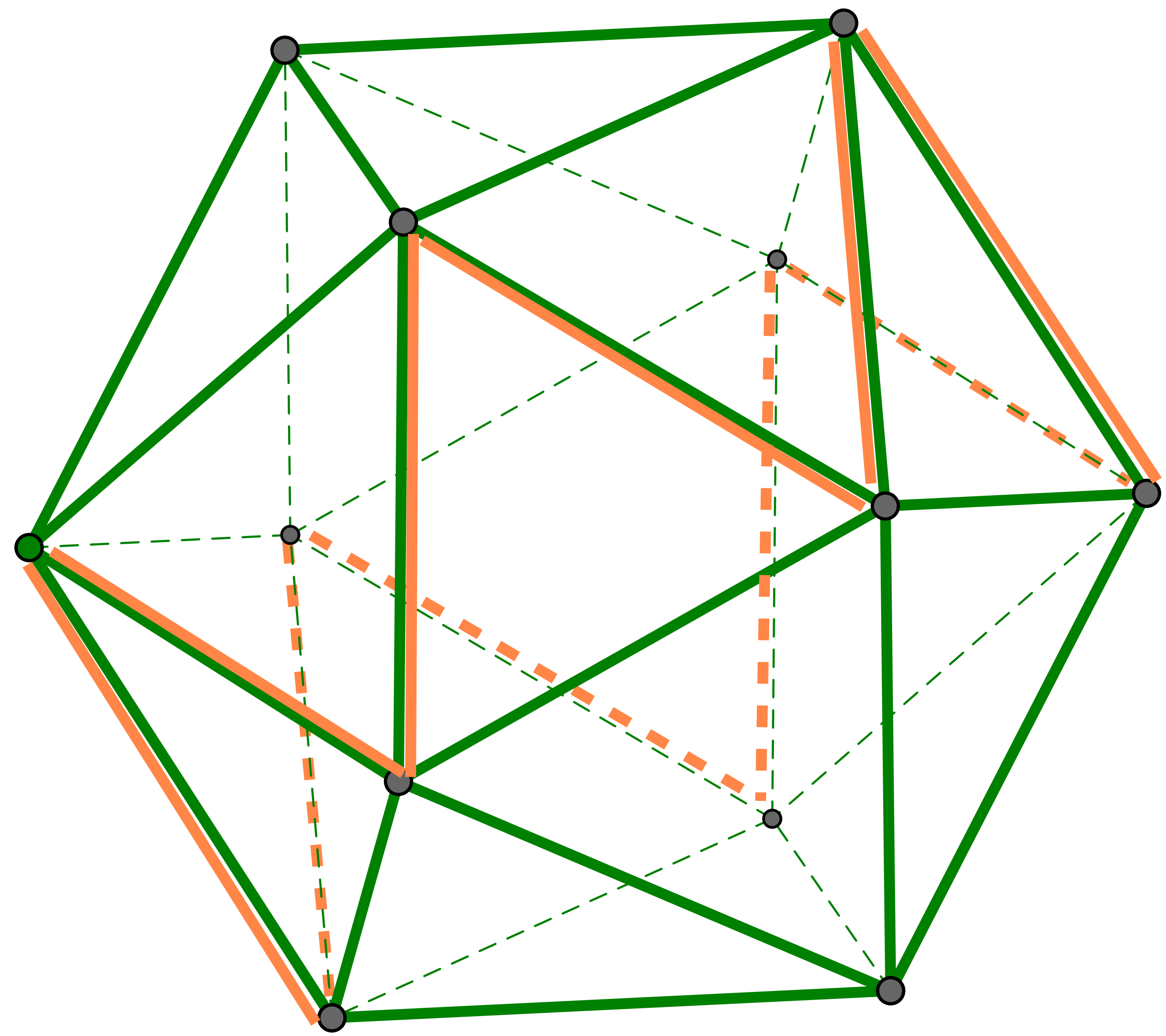
- Cada arista está en exactamente dos caras
- Cada figura de vértice es un ciclo (conexo)
- Conexo

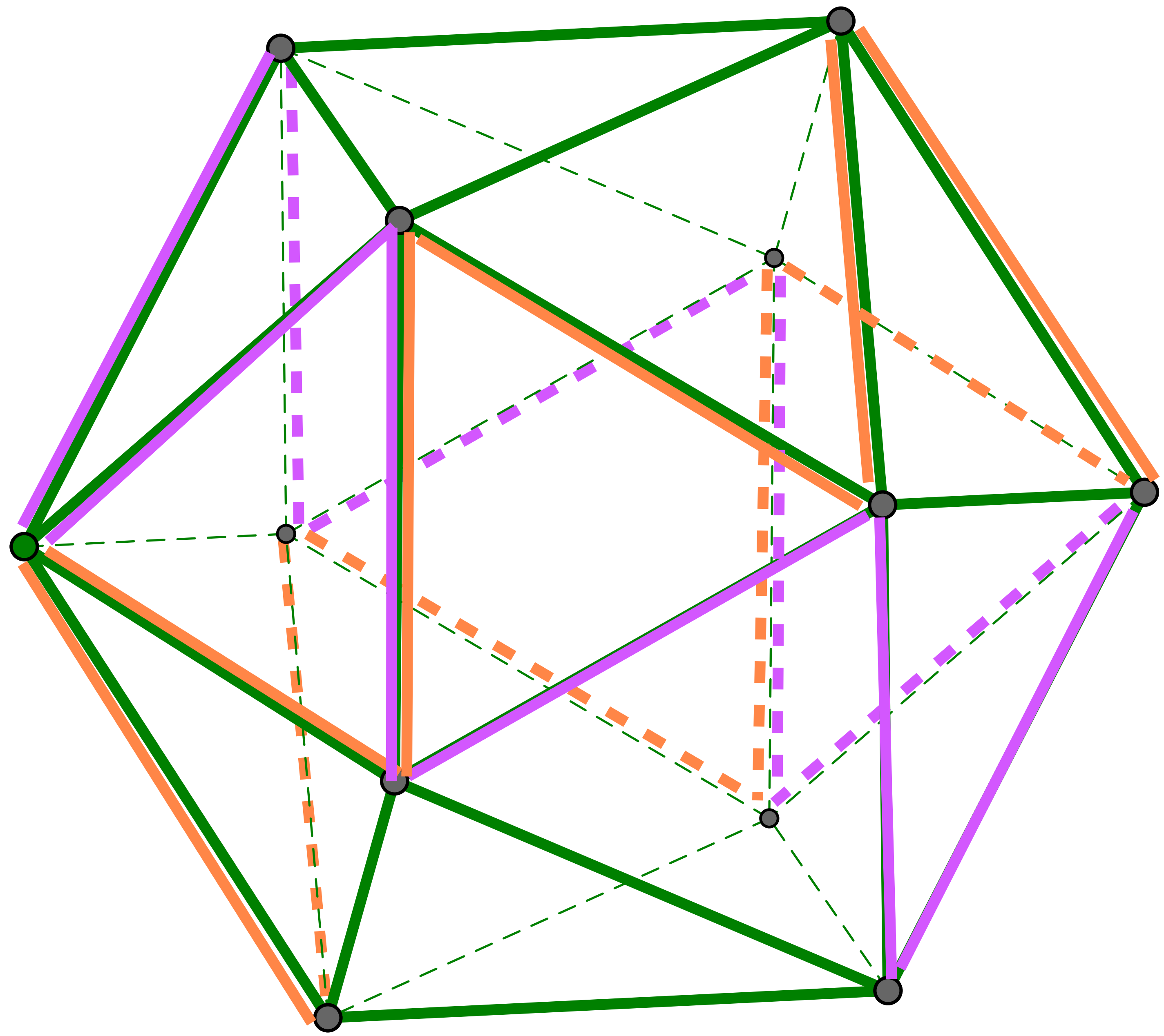


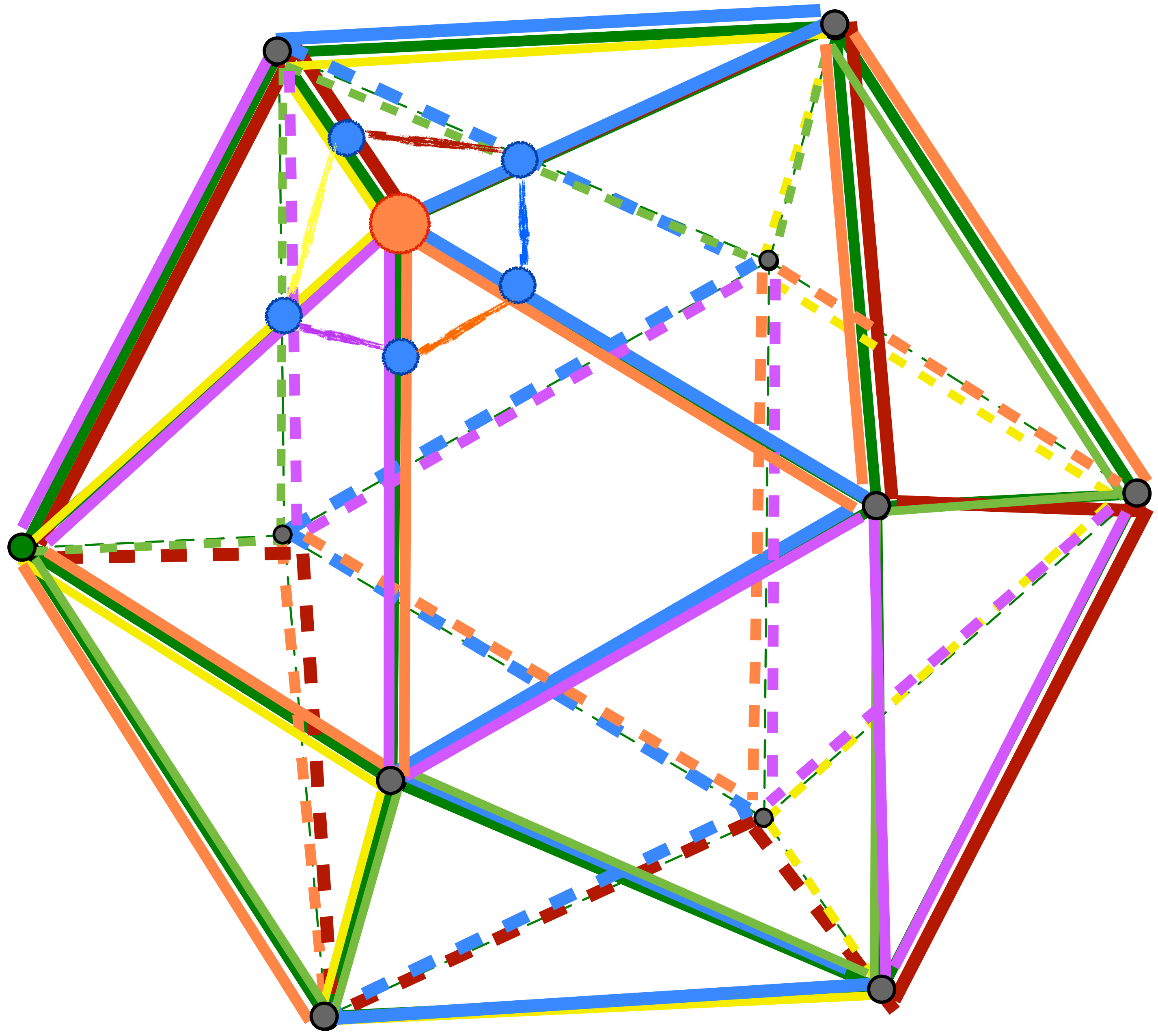
Dado un vértice v, su figura de vértice es la gráfica cuyos vértices son los puntos medios de las aristas en v y dos de ellos están conectados si las aristas correspondientes (y v) están en una misma cara.

Algunos ejemplos de poliedros







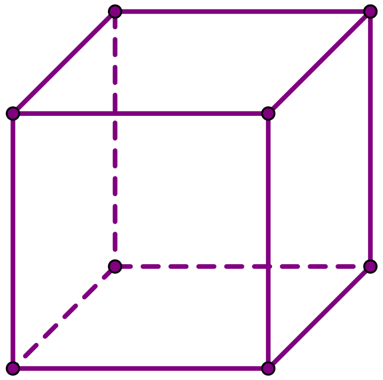


Ya sabemos qué es un poliedro, pero…

¿un poliedro regular?

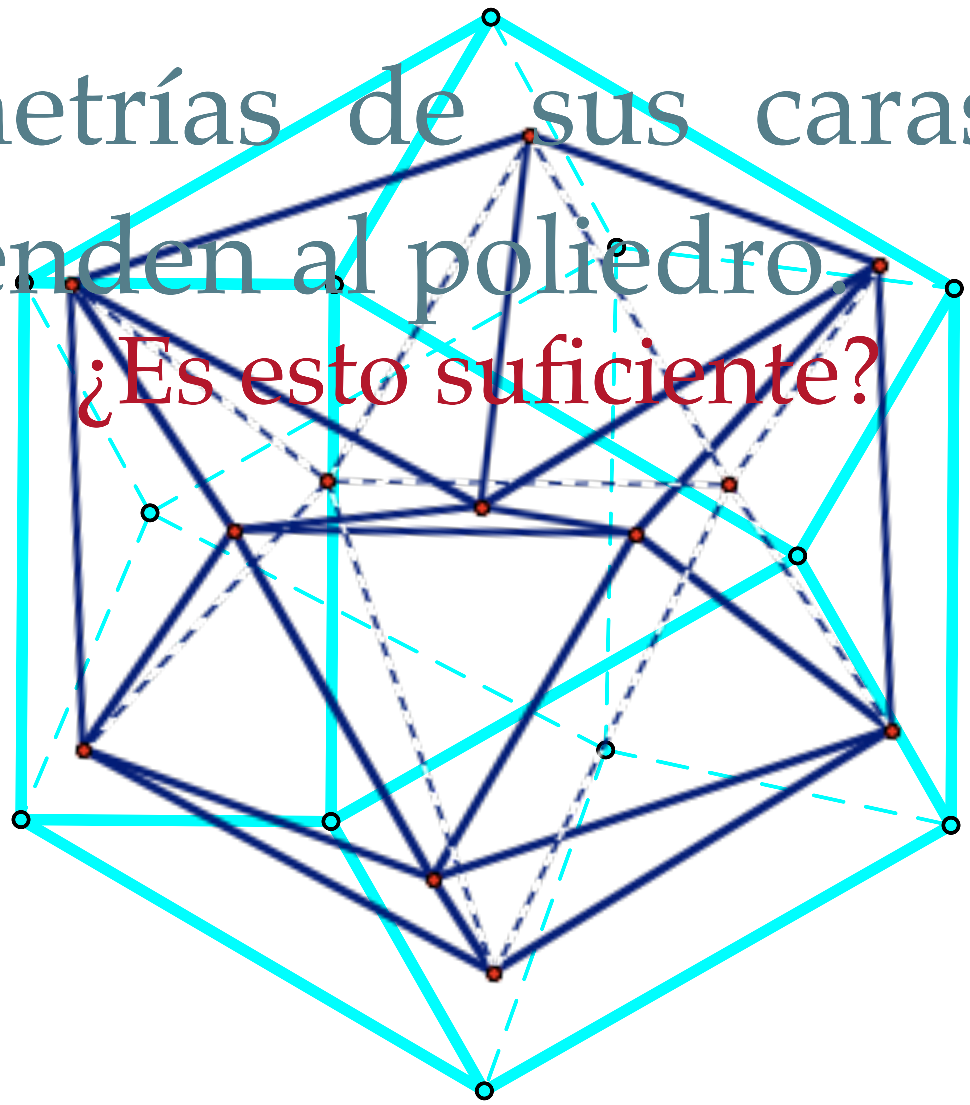
“se ve igual por todas partes”

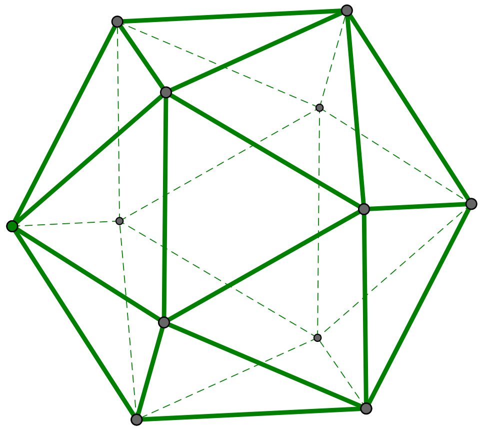
Los sólidos Platónicos cumplen:



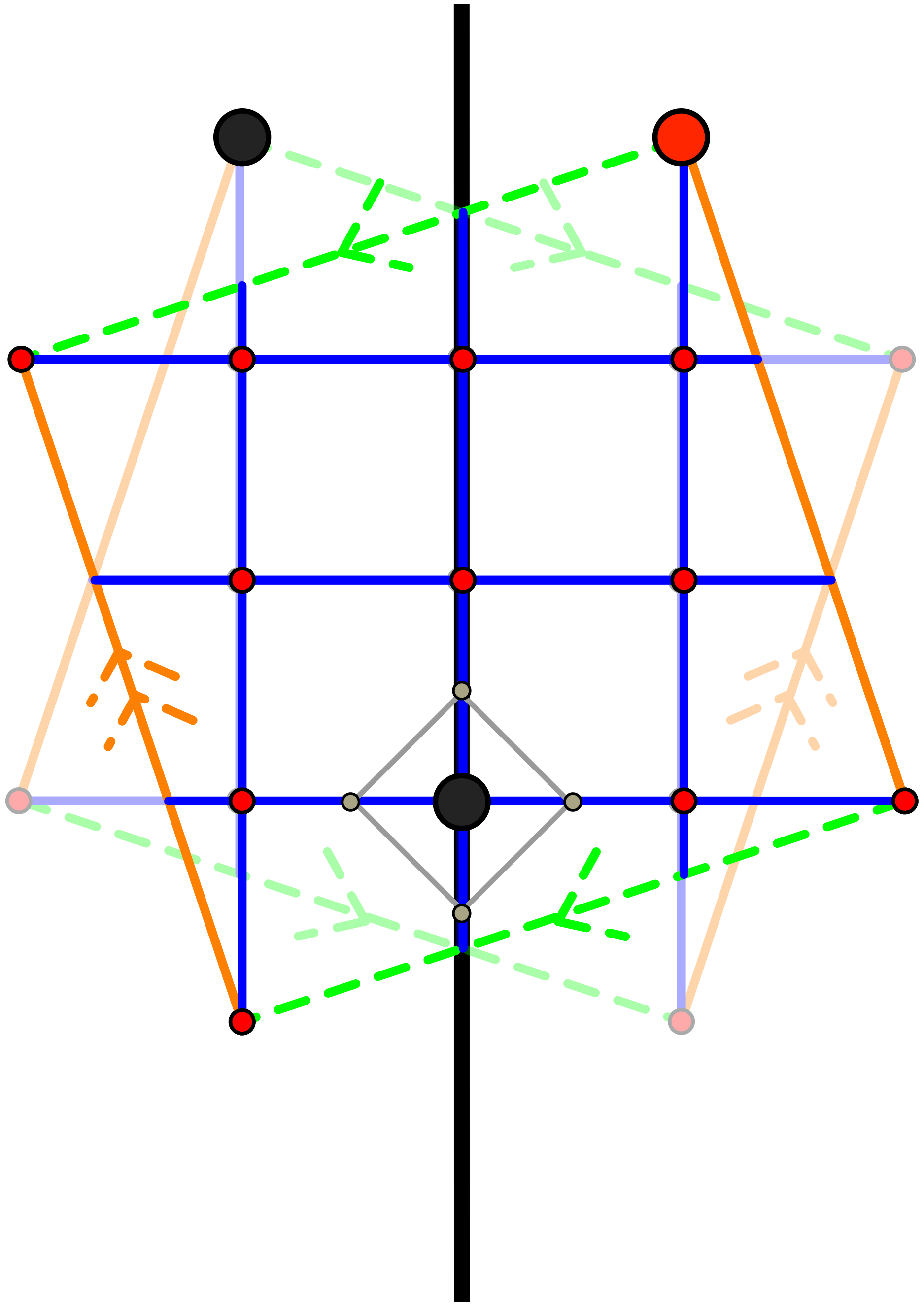
- Todas sus caras son polígonos regulares
- Todas sus caras son iguales
- Todas sus figuras de vértice son regulares (e iguales)
- Todas las simetrías de sus caras y figuras de vértice se extienden al poliedro.

¿Es esto suficiente?





¿un poliedro regular?



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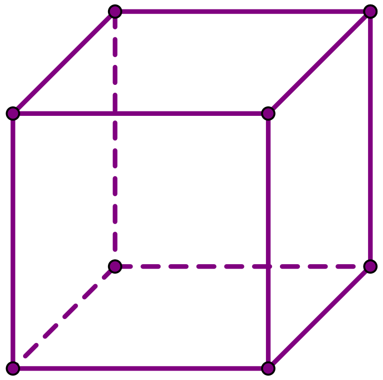

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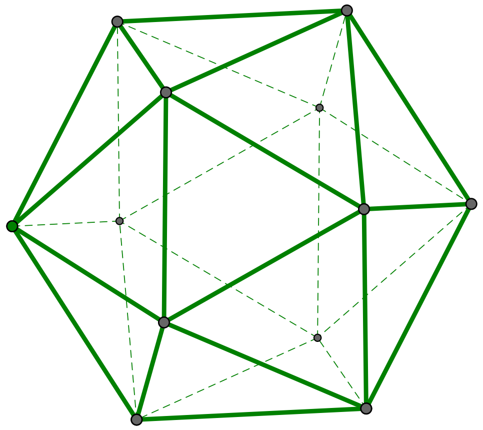
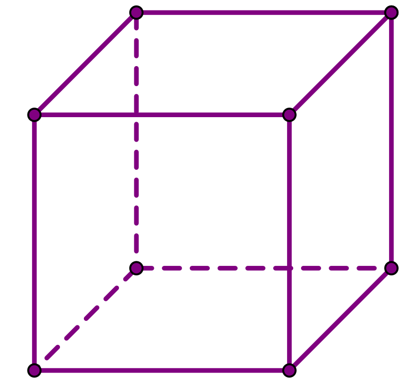
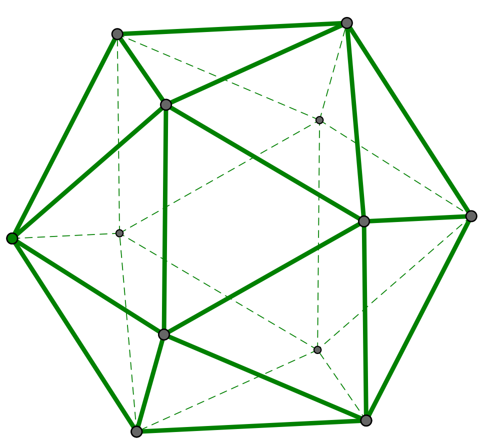




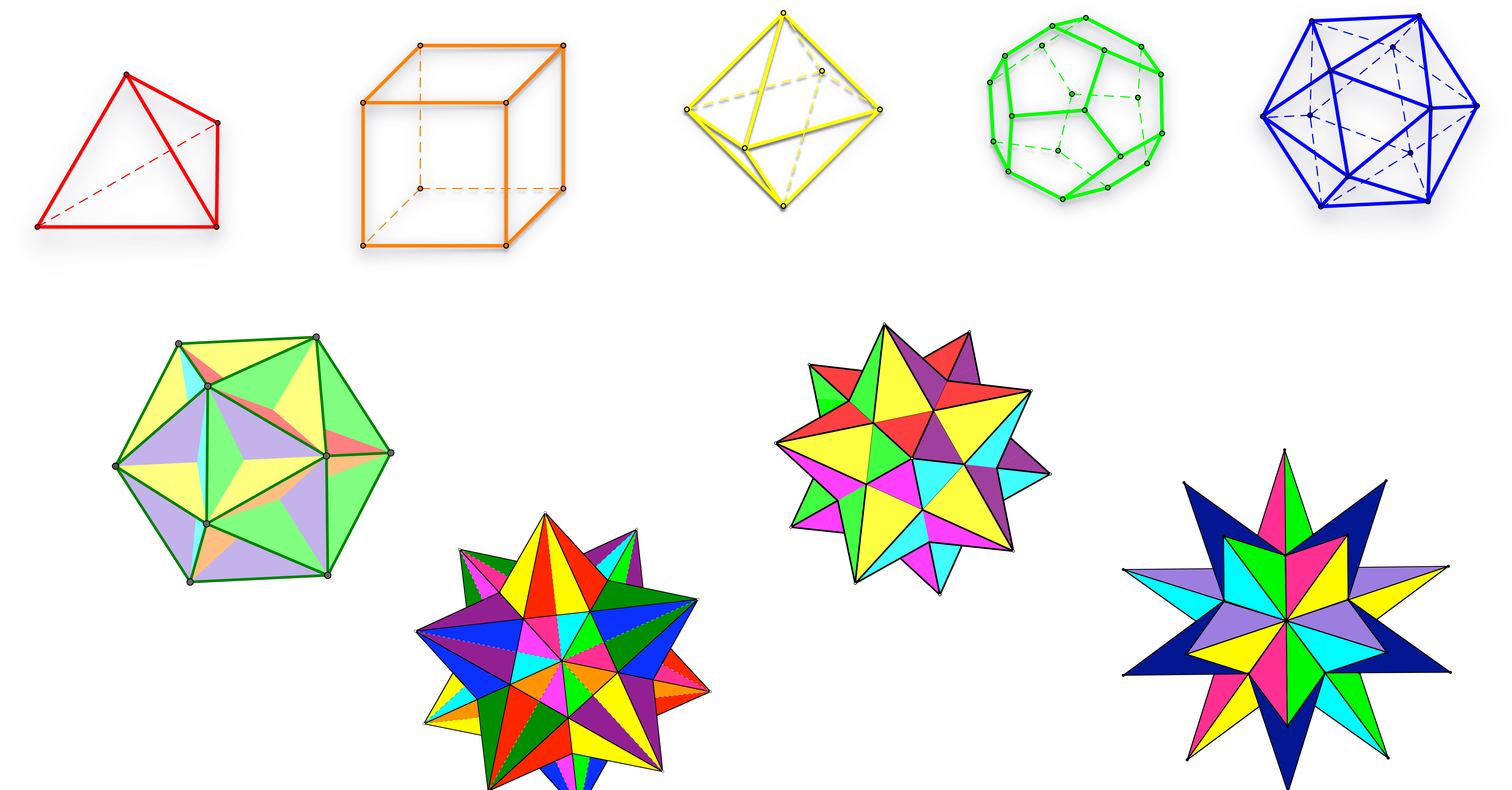
Un poliedro es regular si:

- Todas sus caras son polígonos regulares
- Todas sus caras son iguales
- Todas sus figuras de vértice son regulares (e iguales)
- Todas las simetrías de sus caras y figuras de vértice se extienden al poliedro.



Teorema (Grünbaum-Dress)

Hay exactamente 18 poliedros regulares finitos en el espacio



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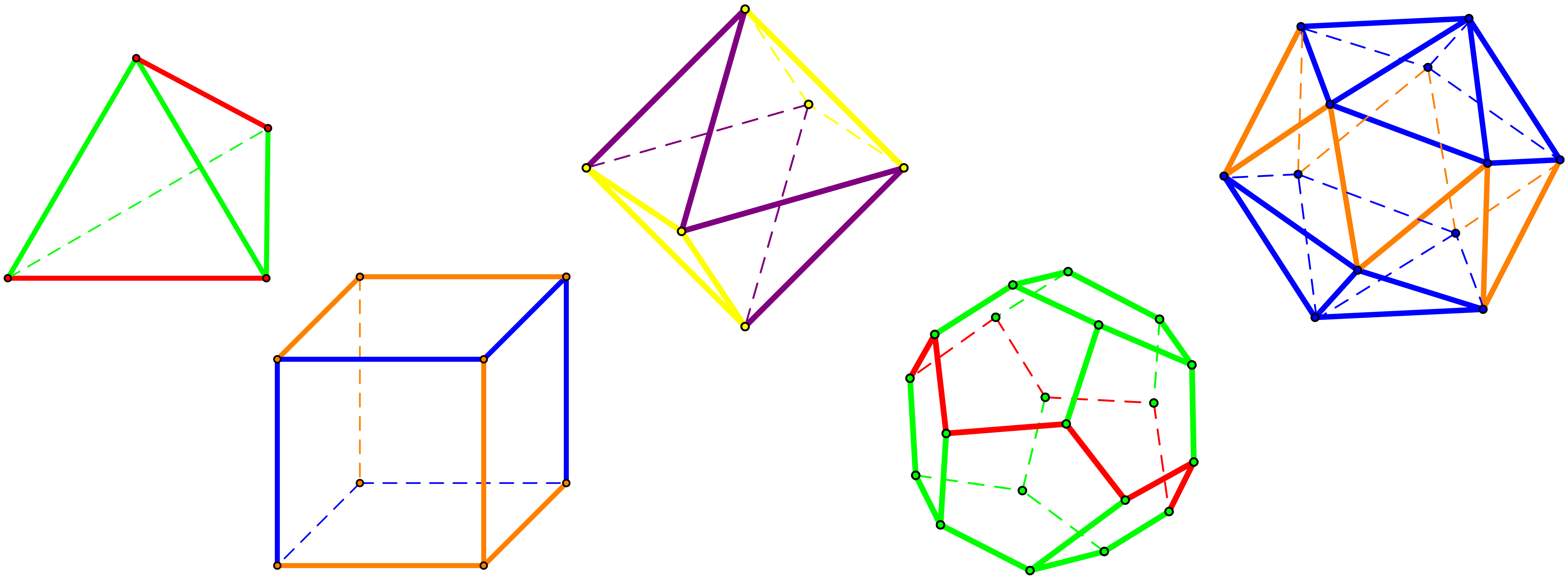

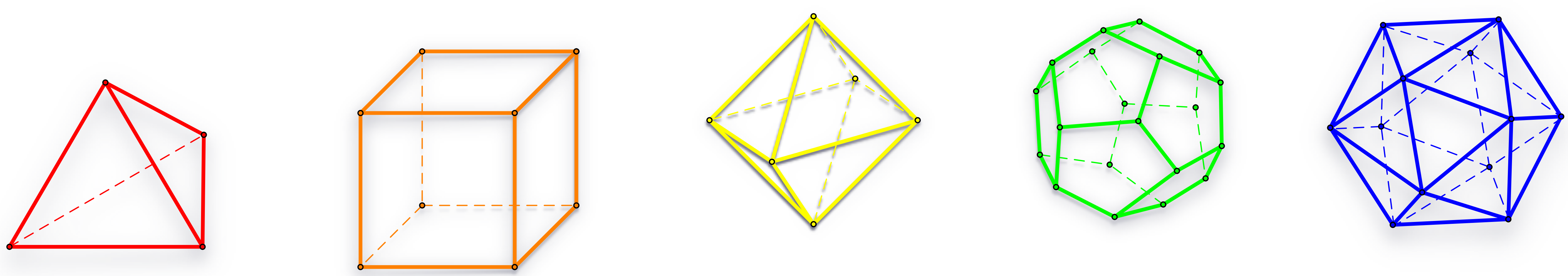

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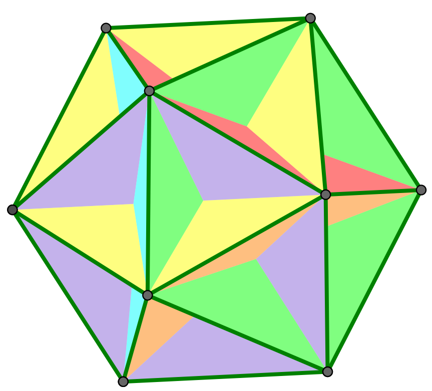

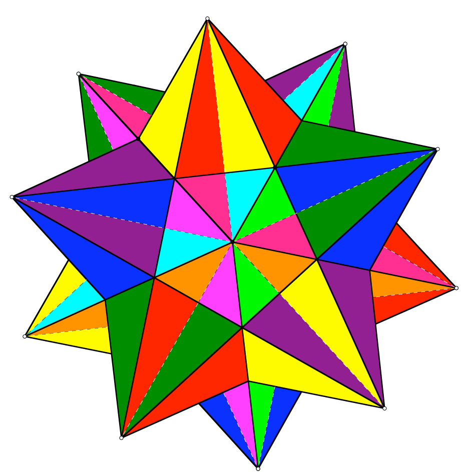

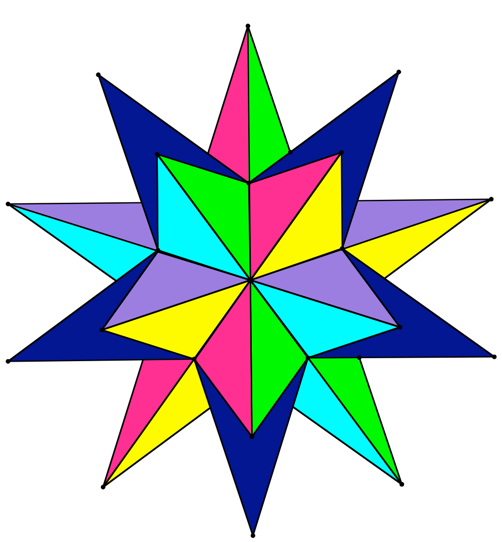

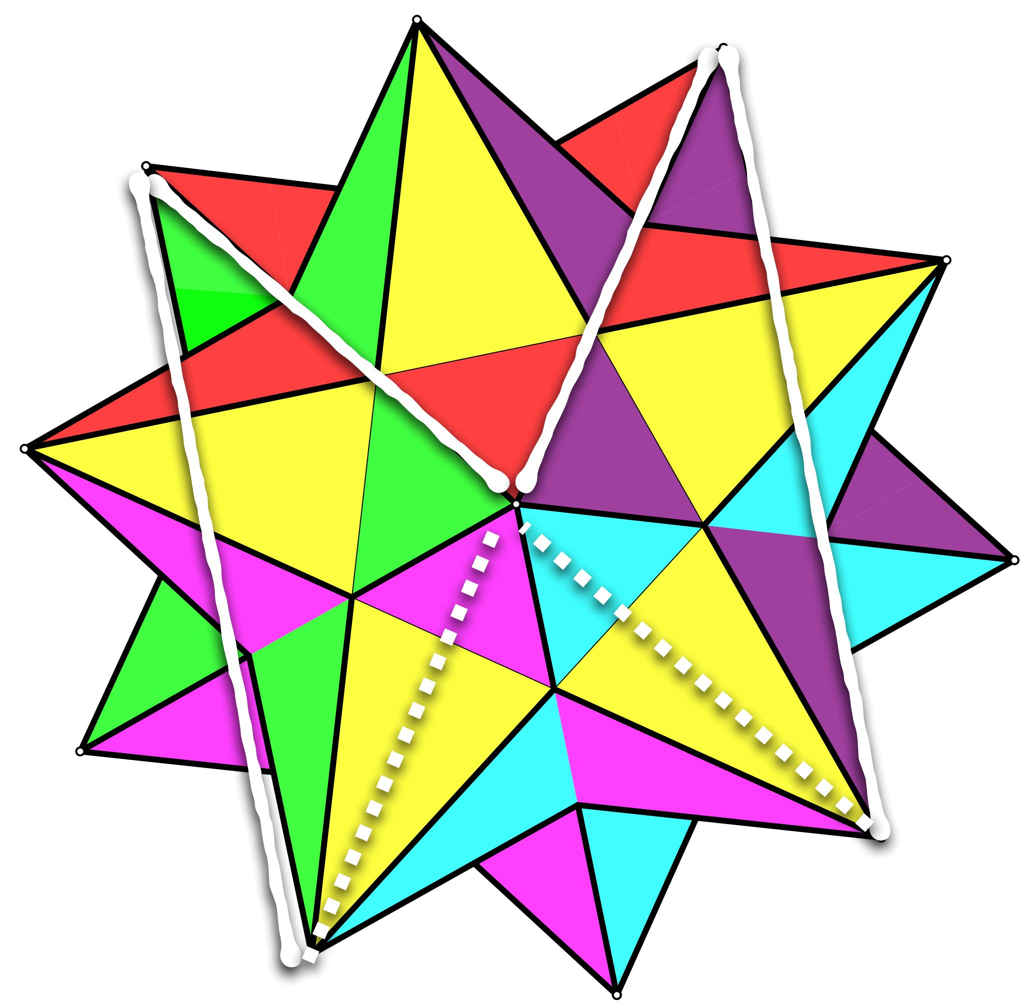

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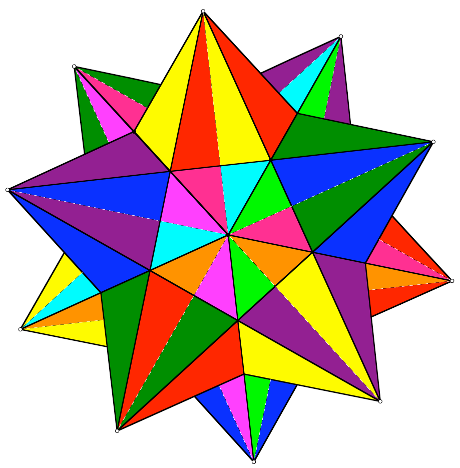
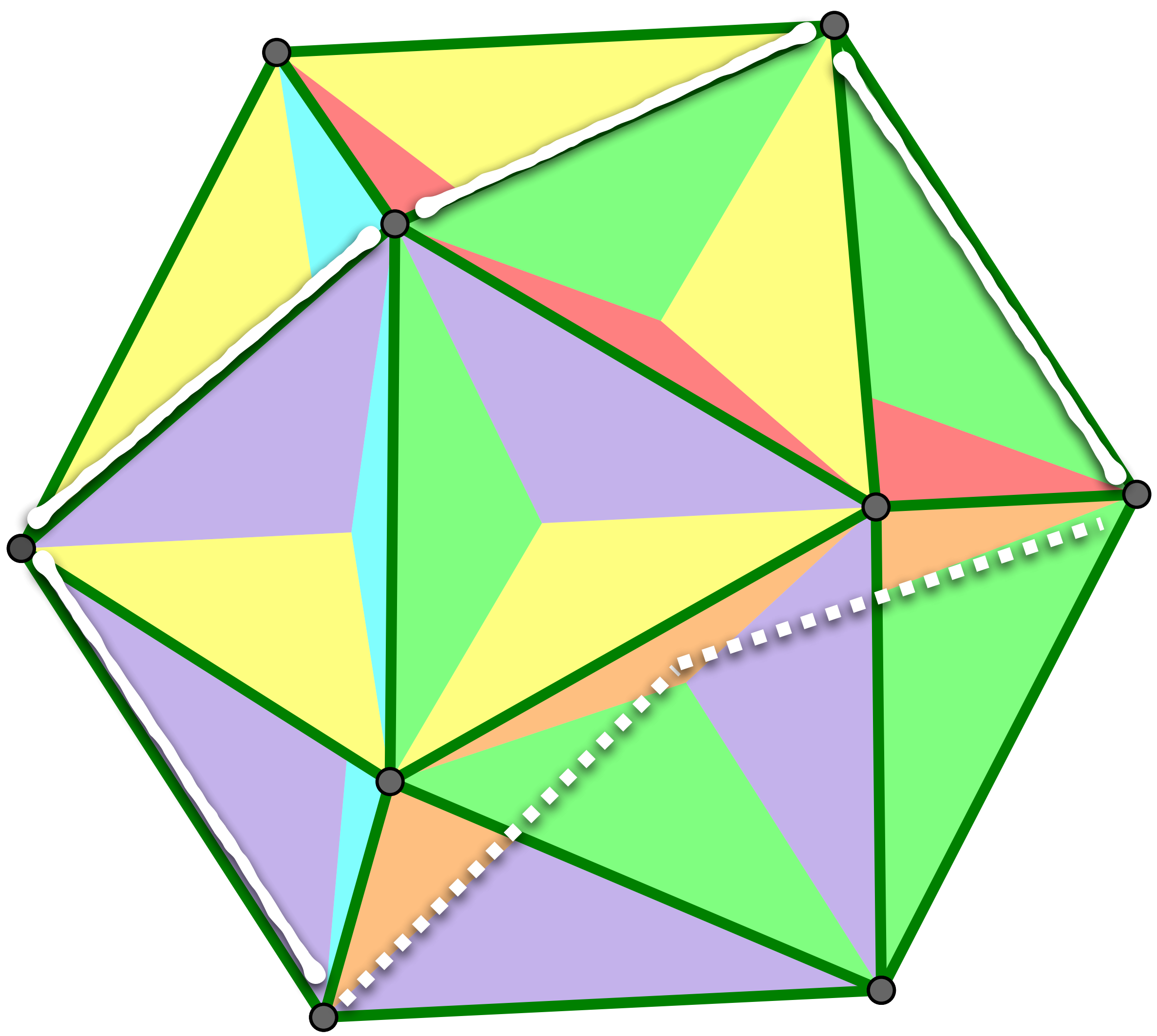
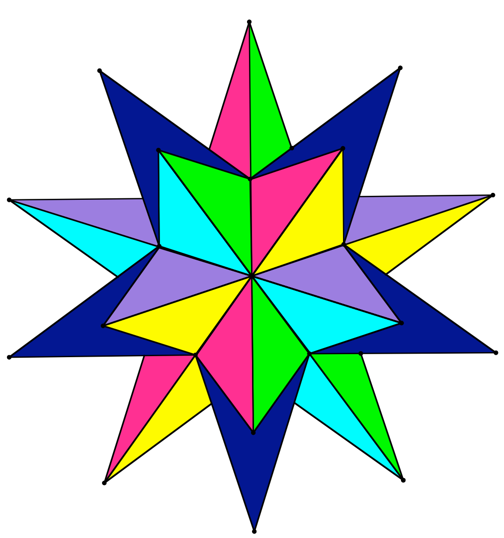


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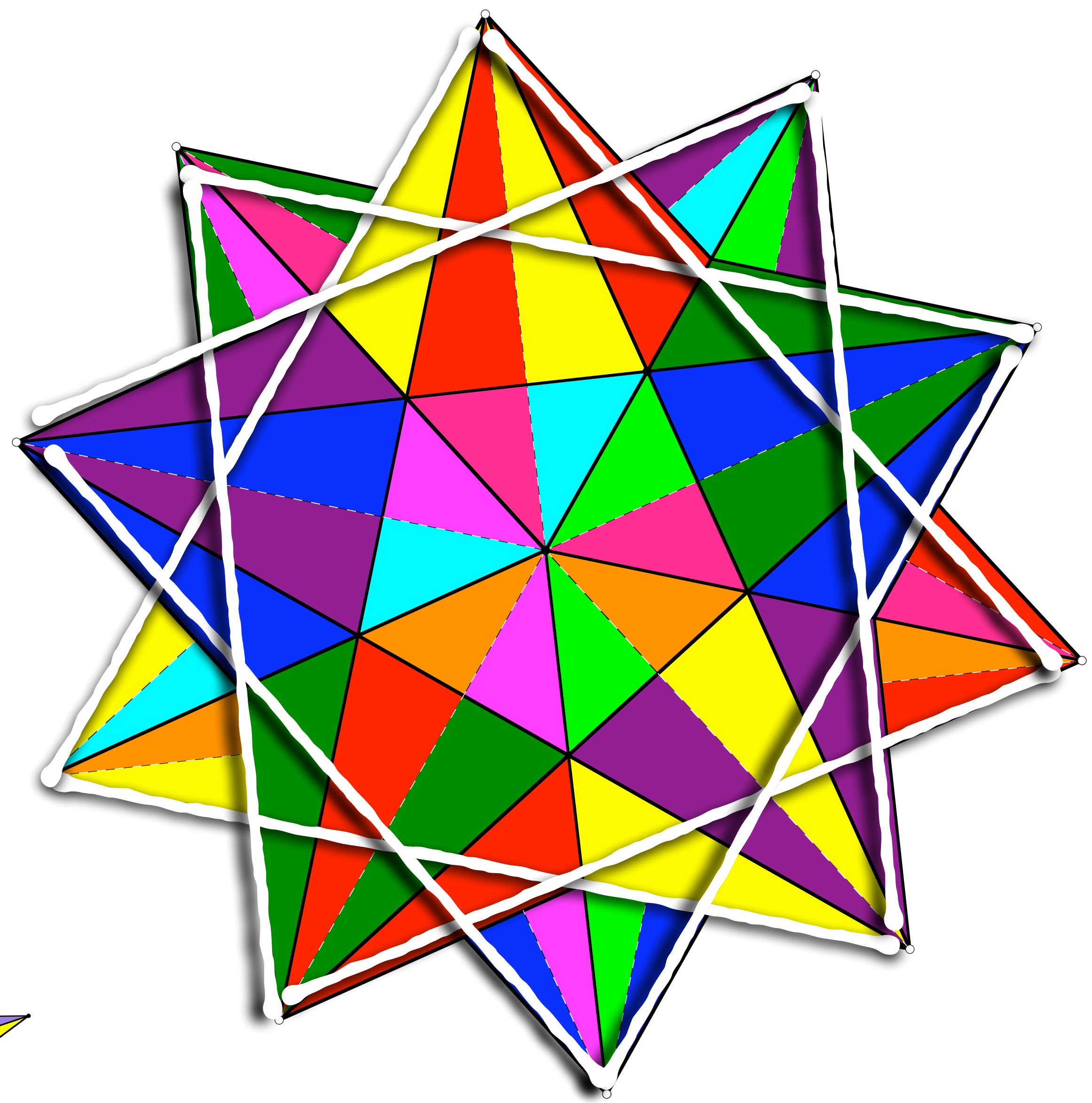
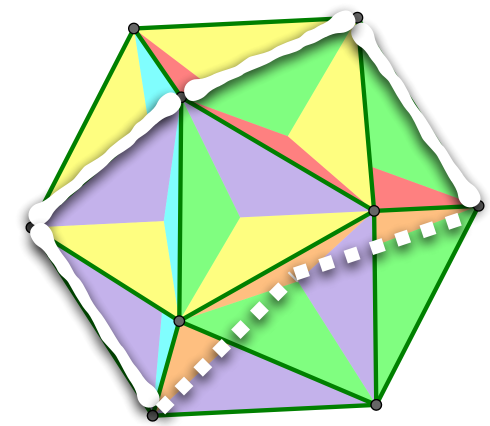
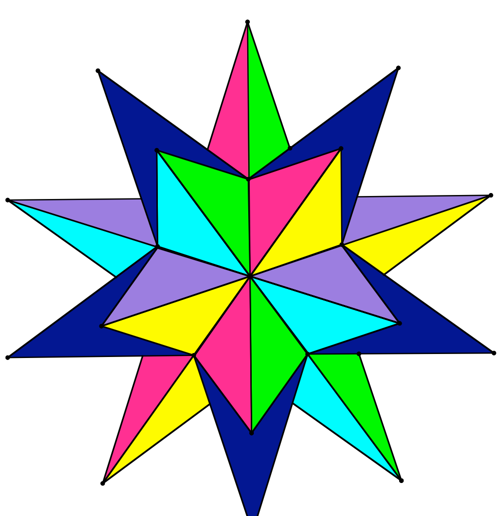


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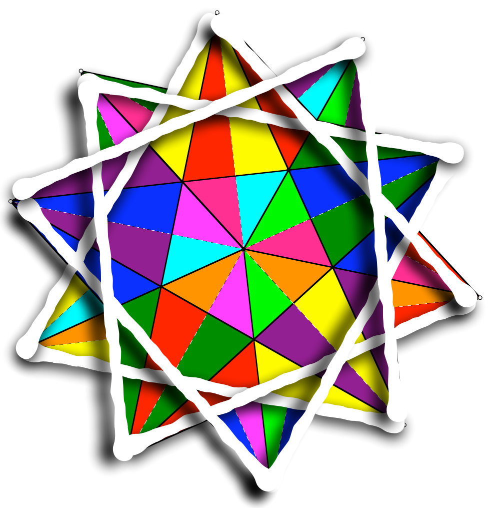

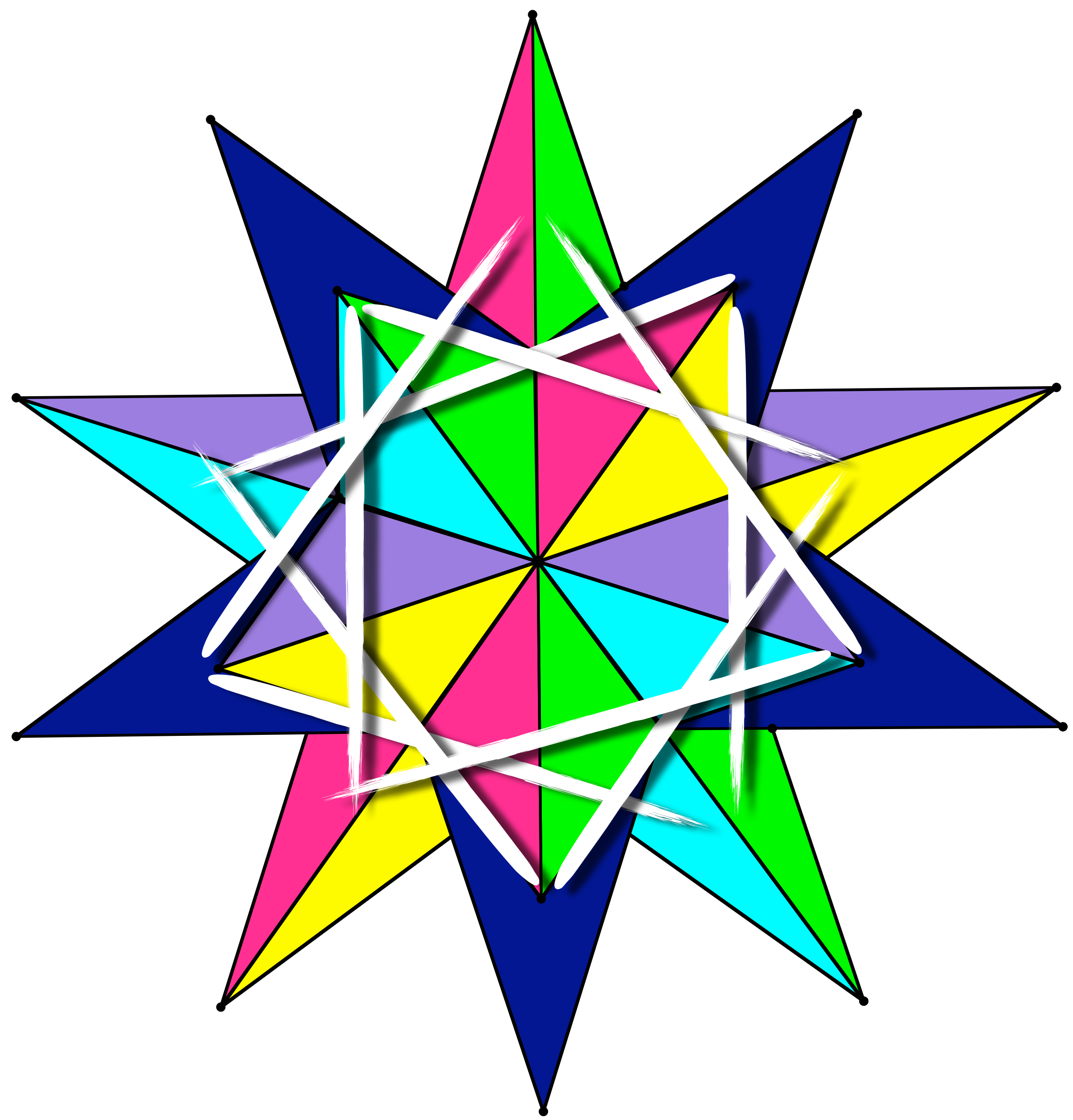

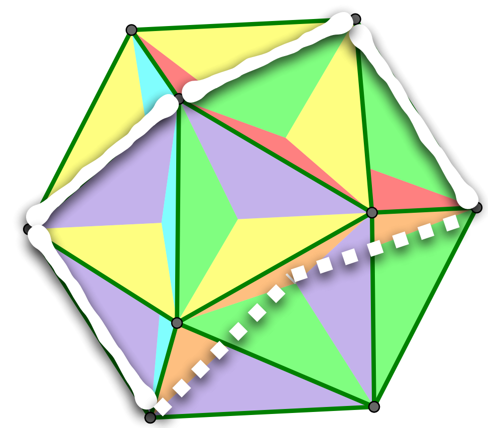

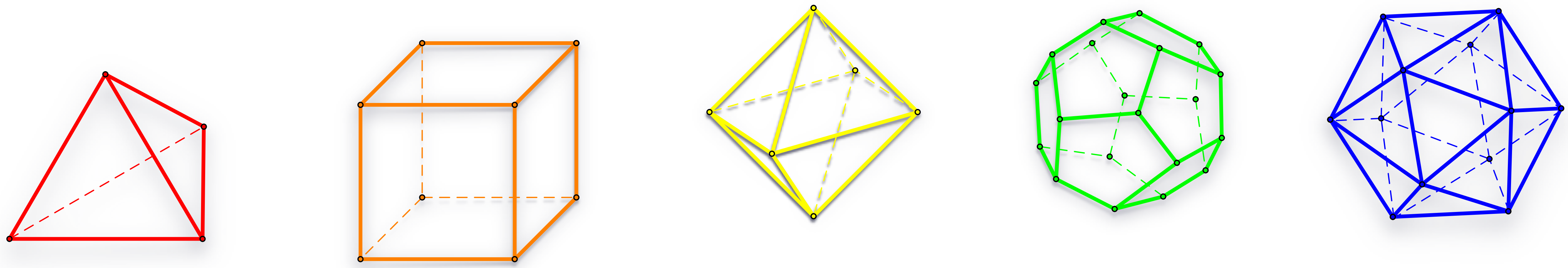

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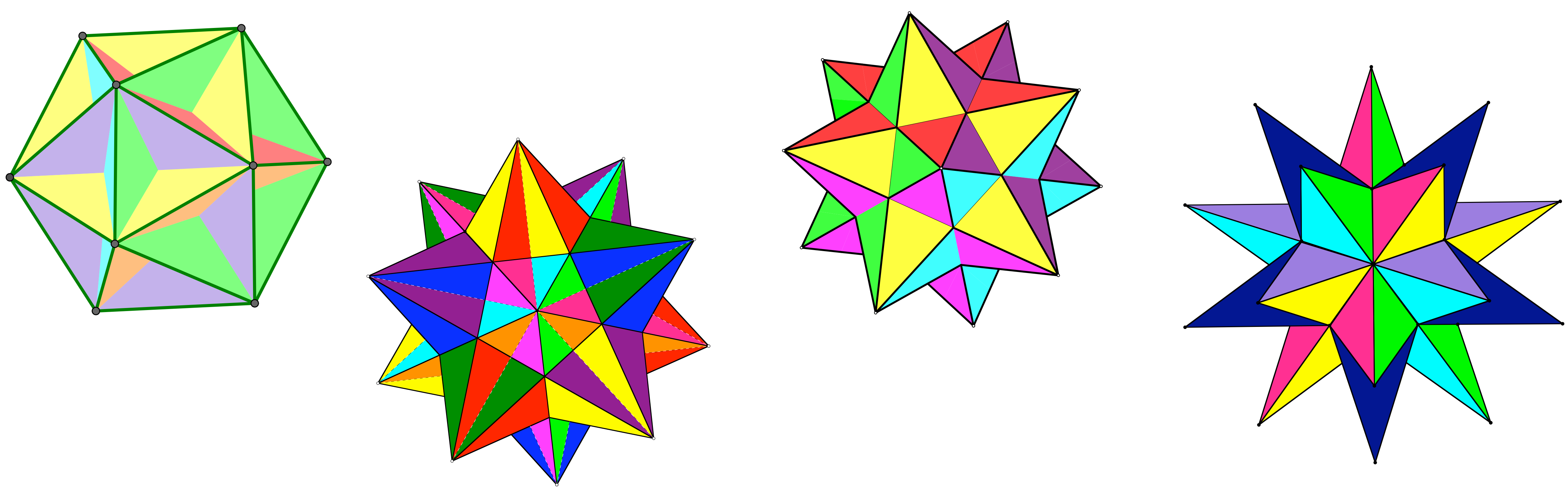
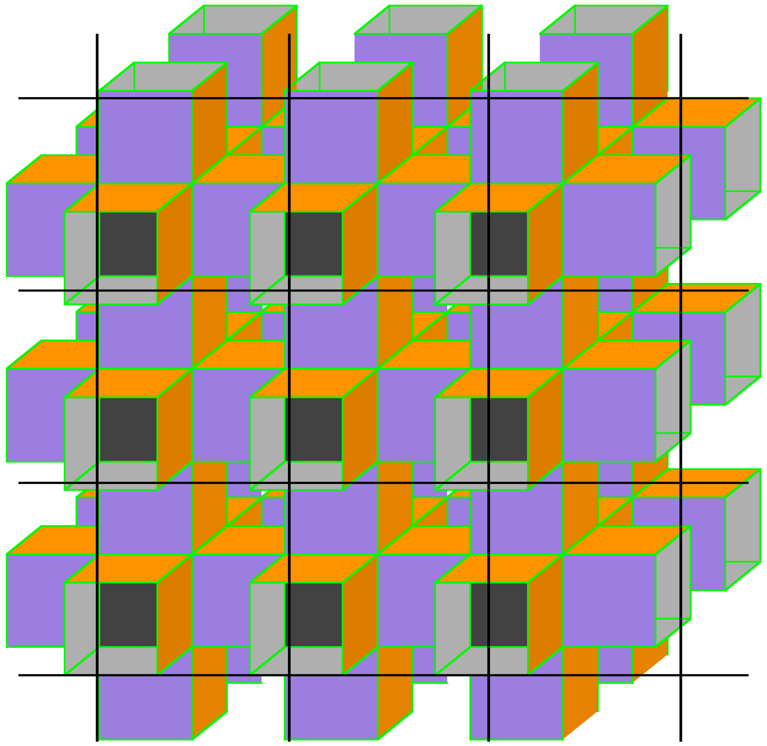


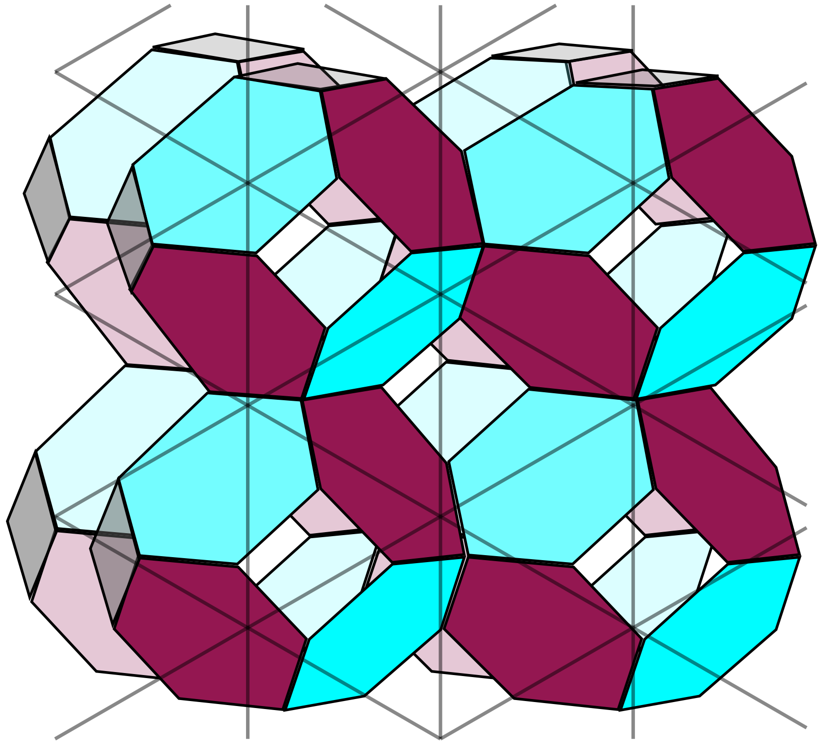
Teorema (Grünbaum-Dress)

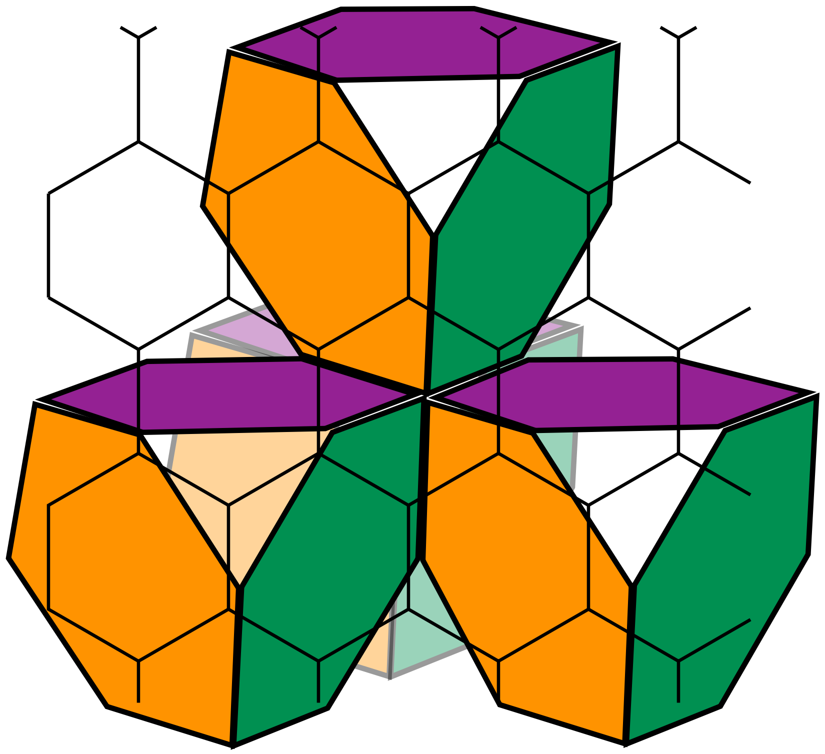
Hay exactamente 18 poliedros regulares finitos en el espacio

¿Poliedros regulares infinitos?

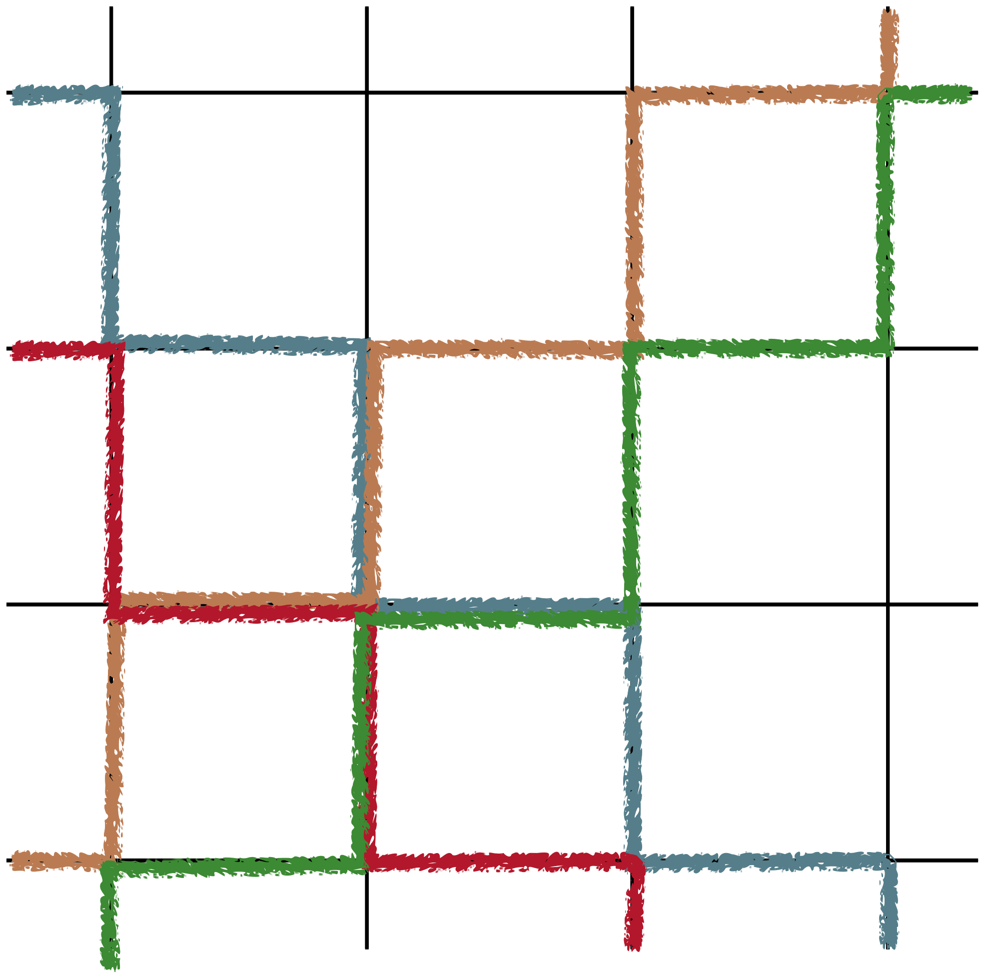
Poliedros regulares infinitos

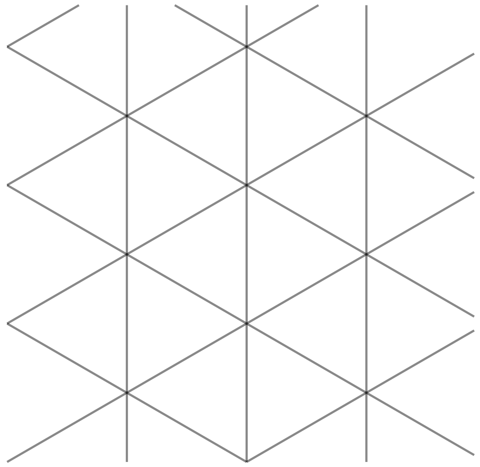


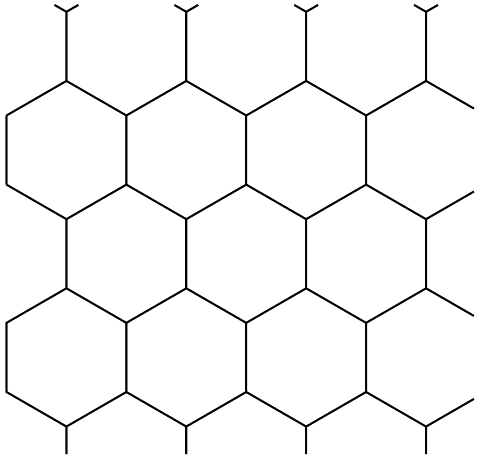




Poliedros regulares infinitos







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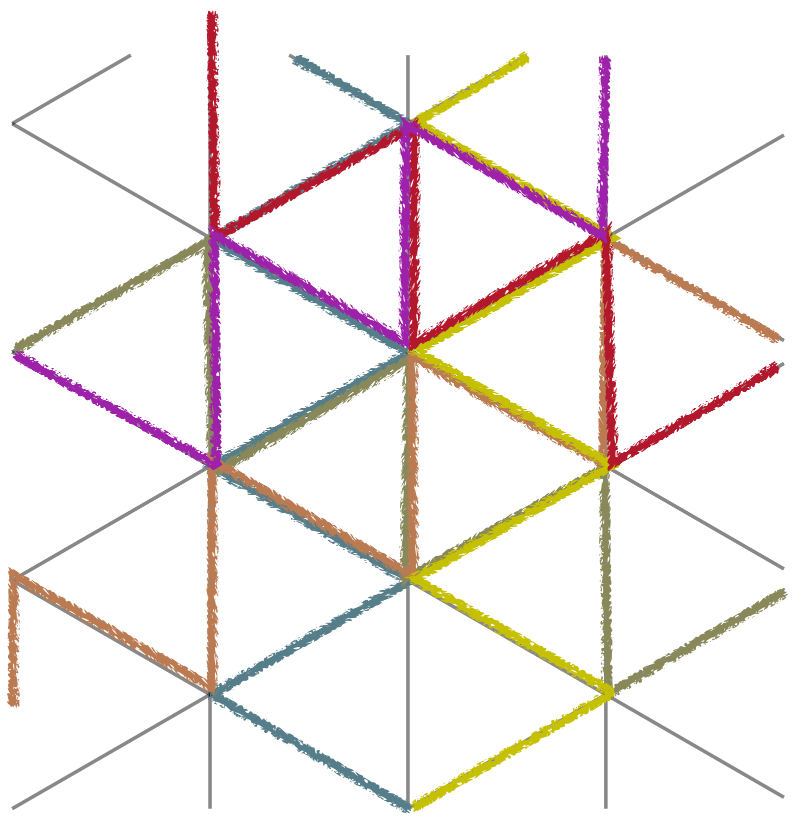

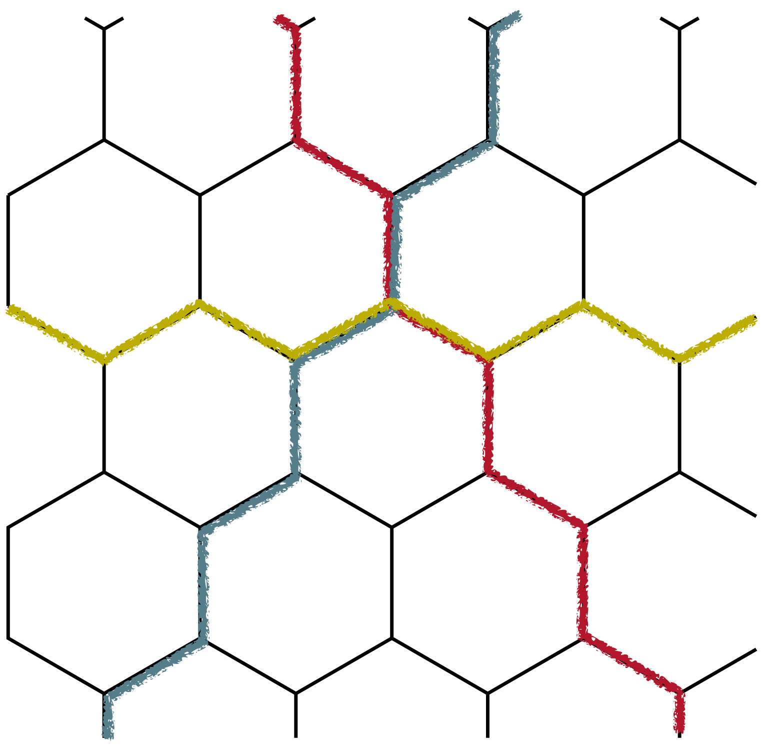

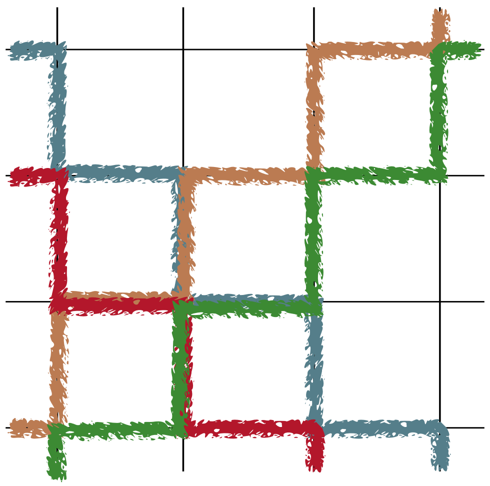

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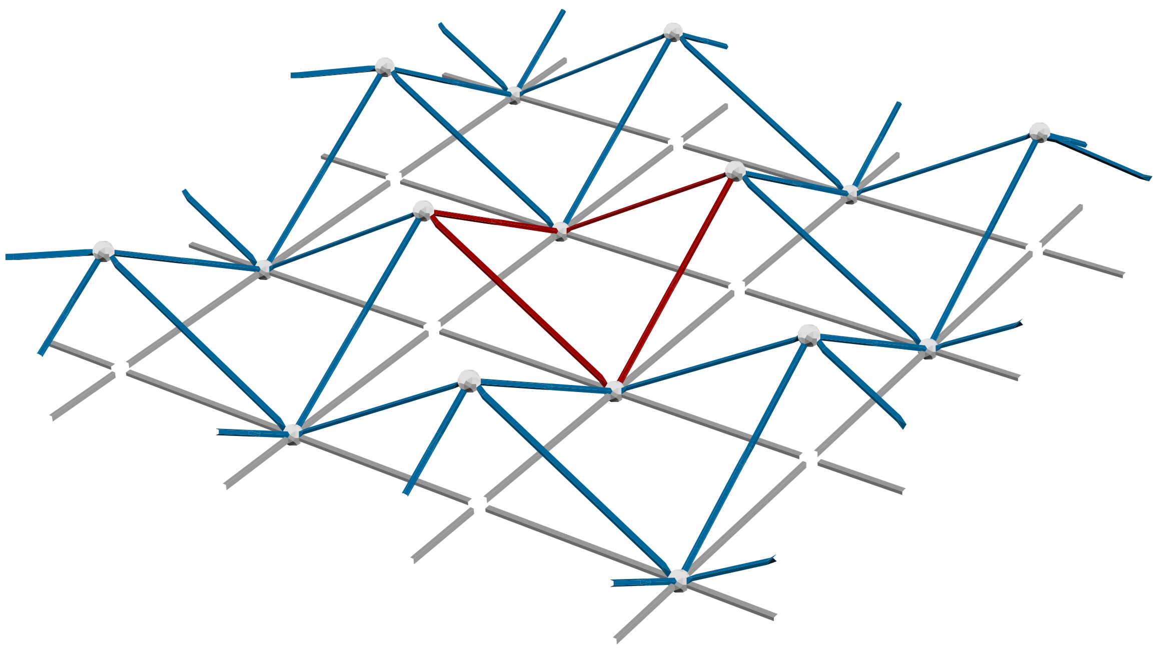

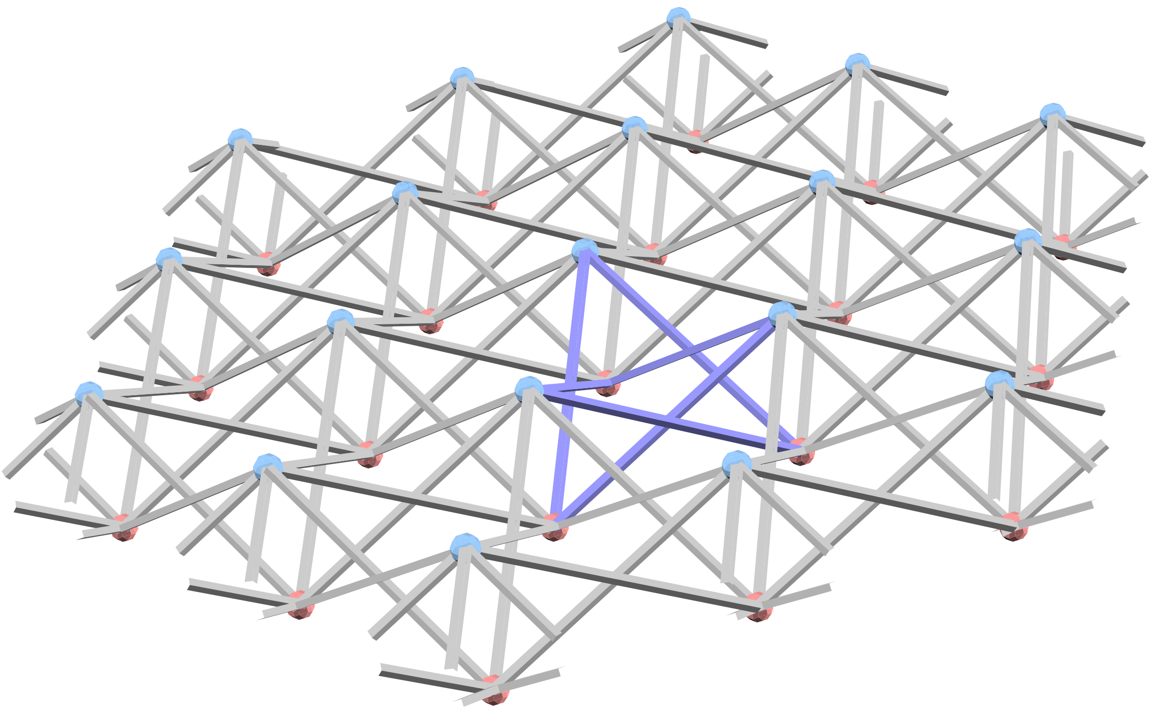

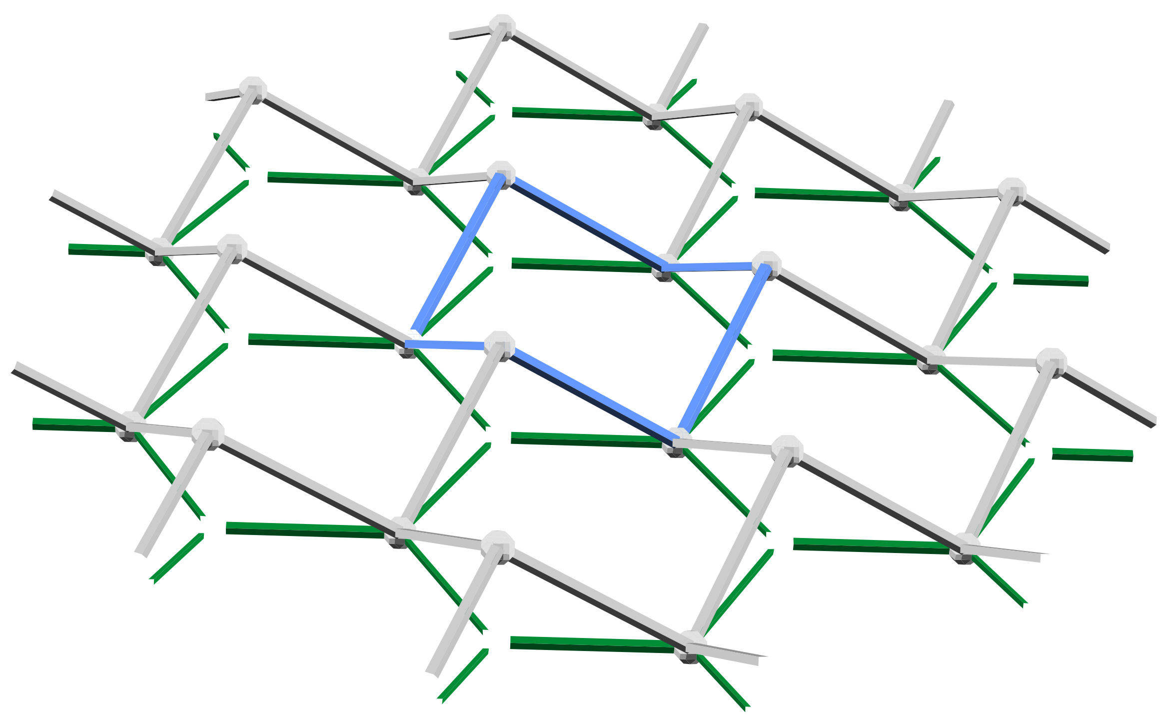

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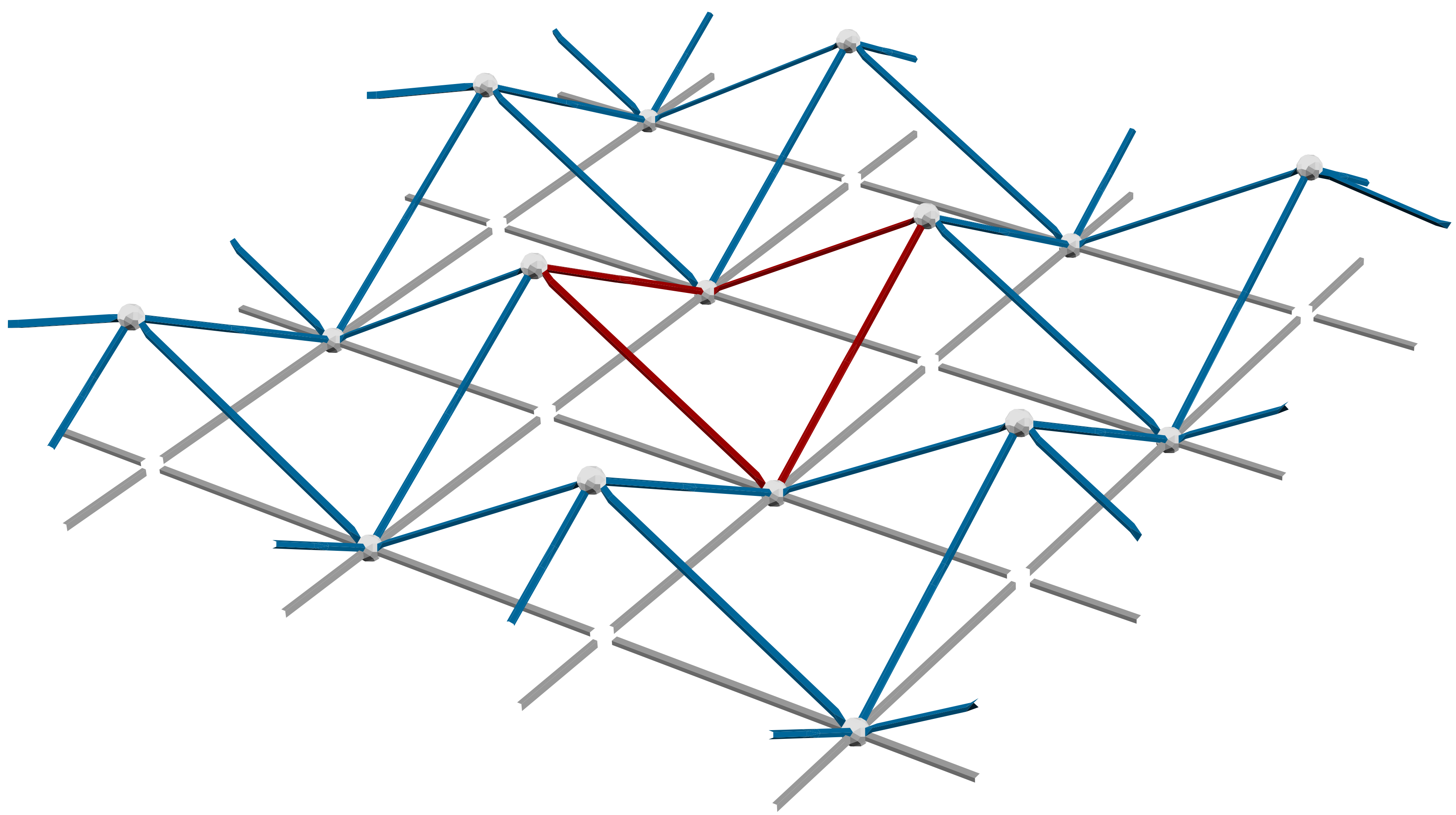

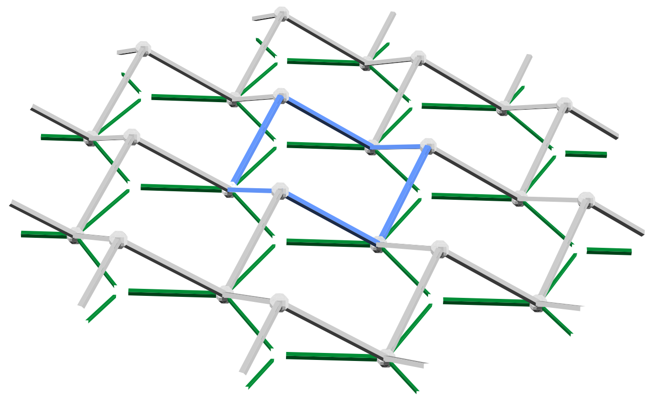

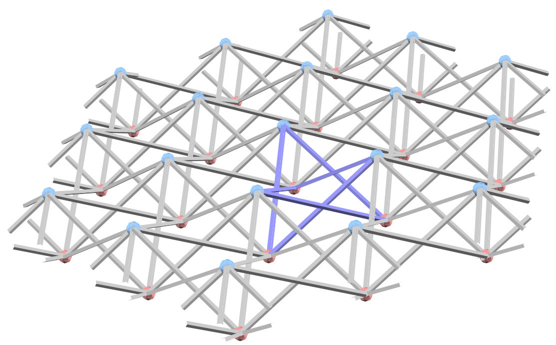

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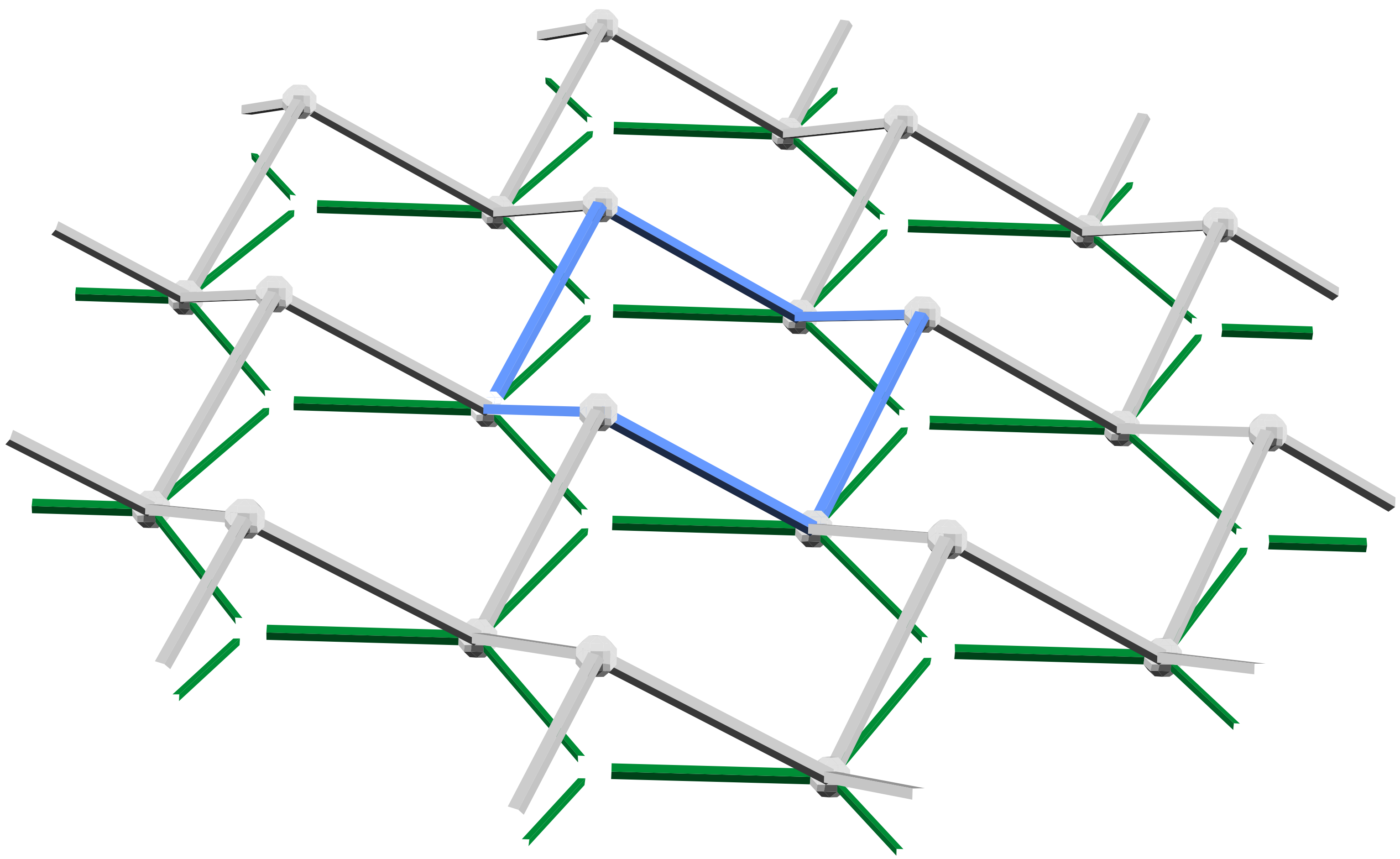

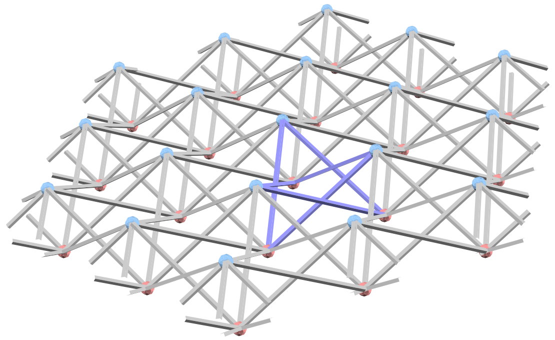

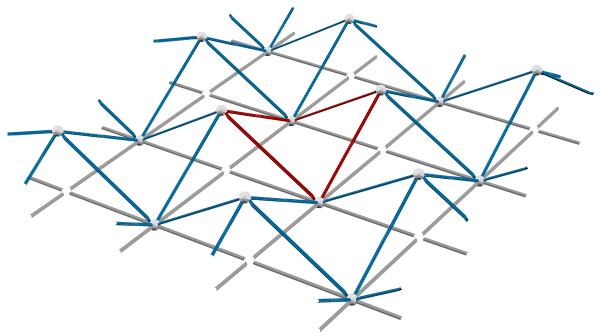

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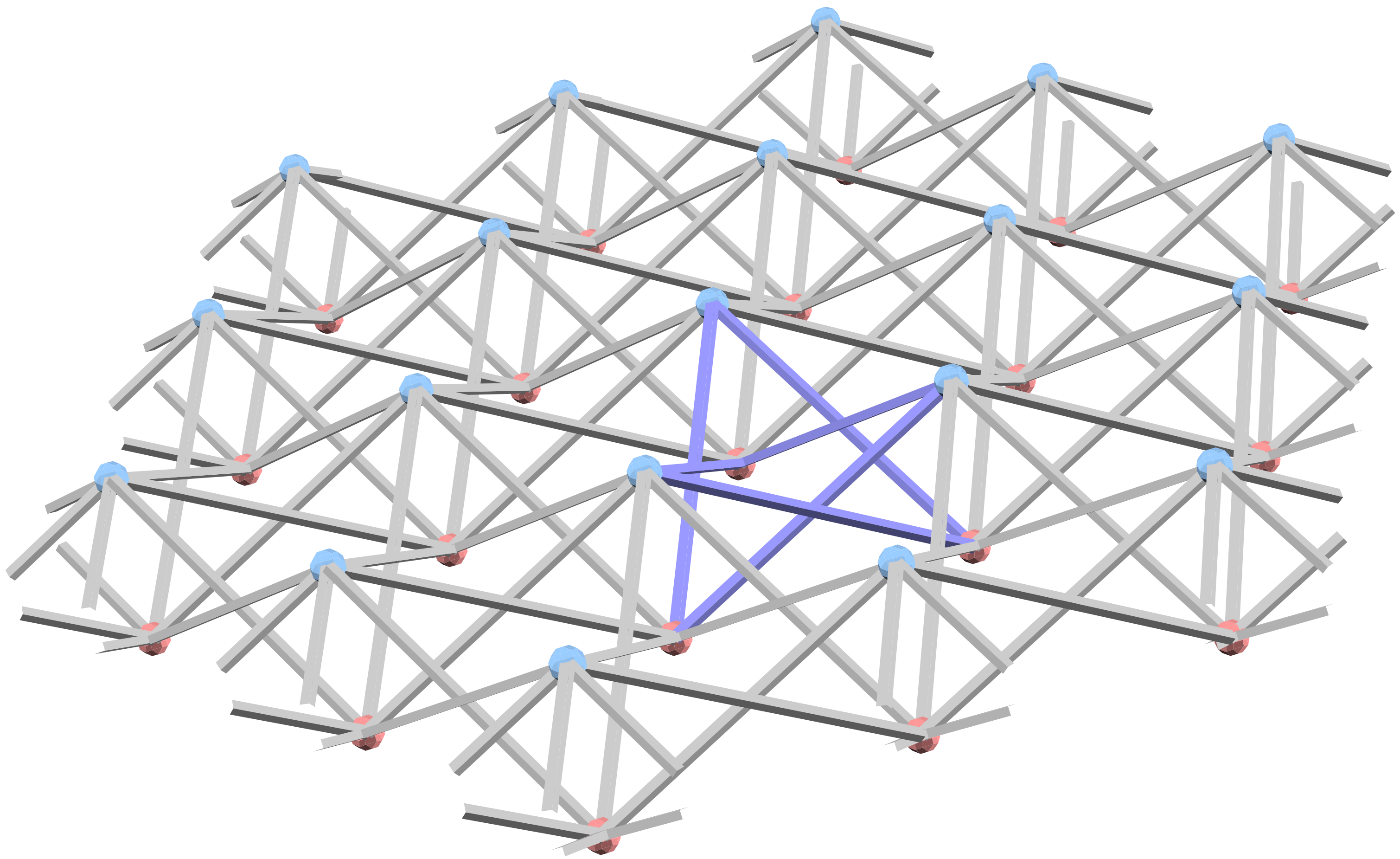
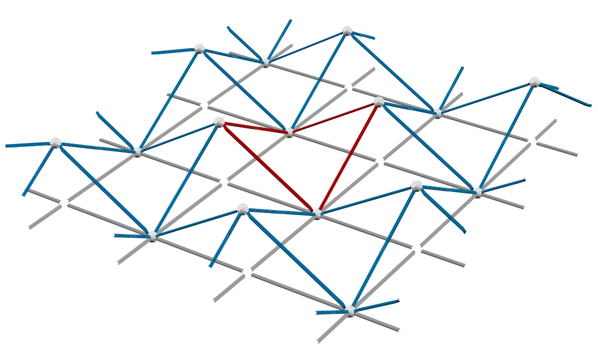
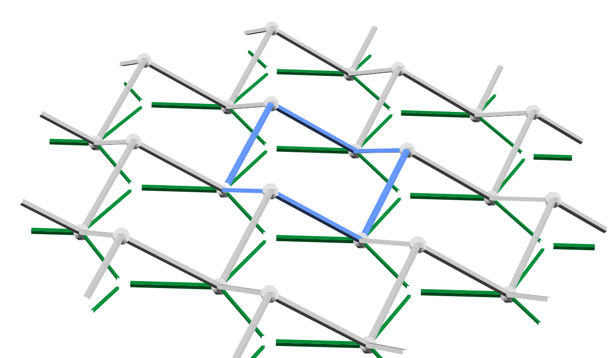
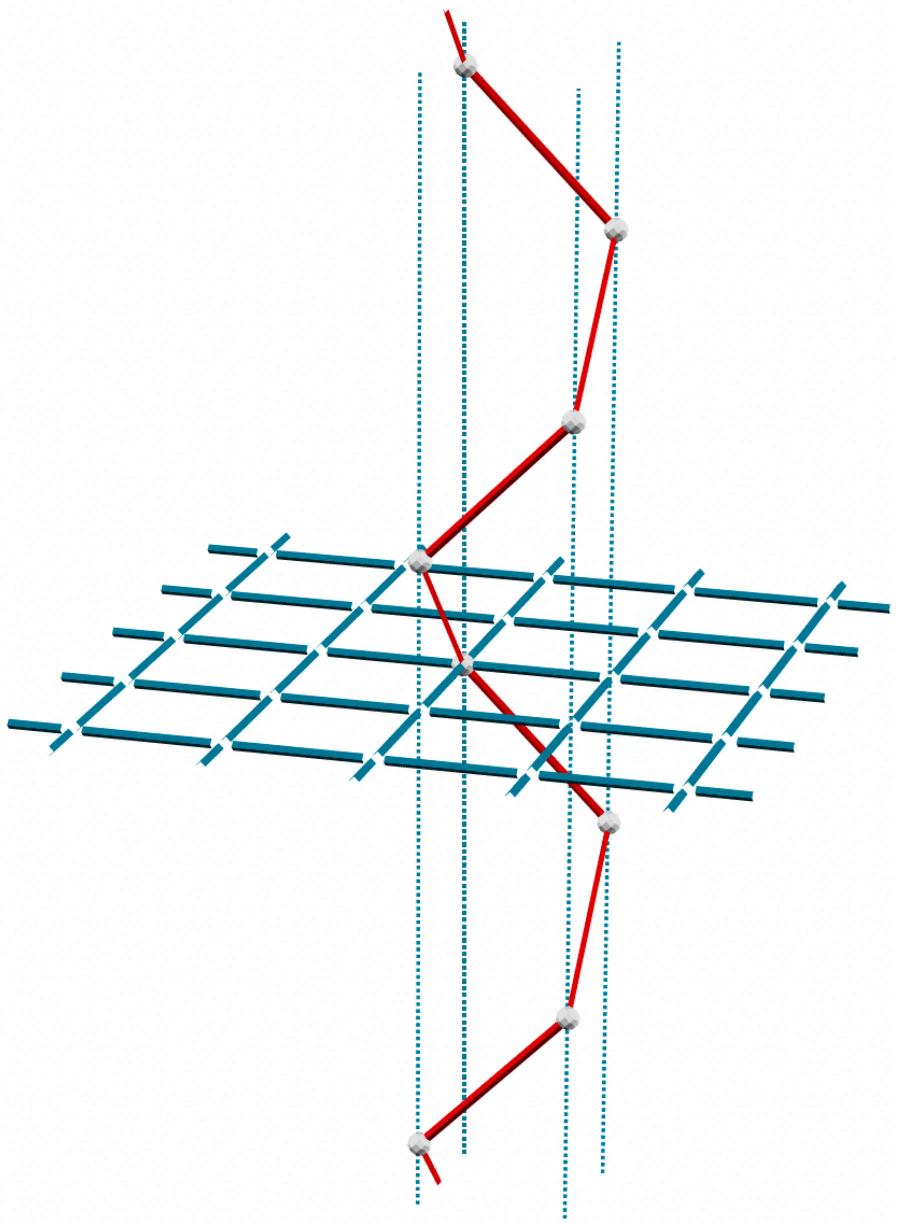
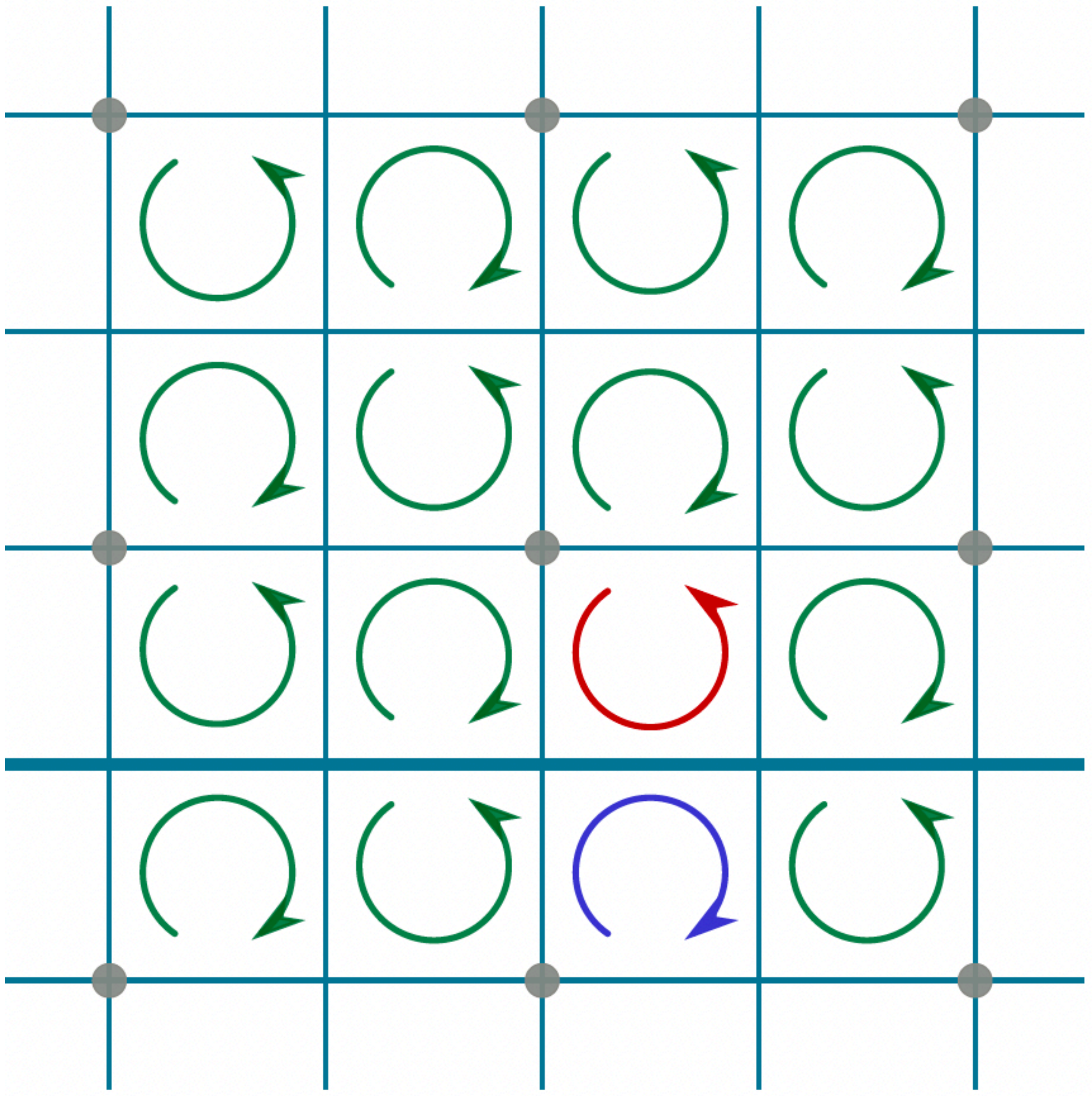




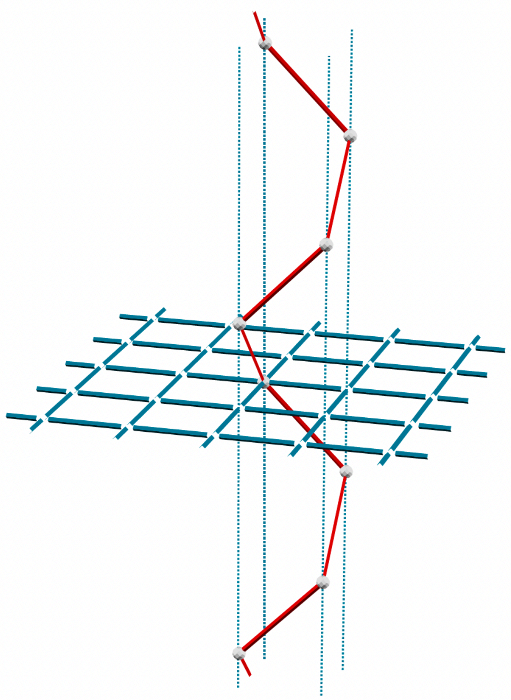


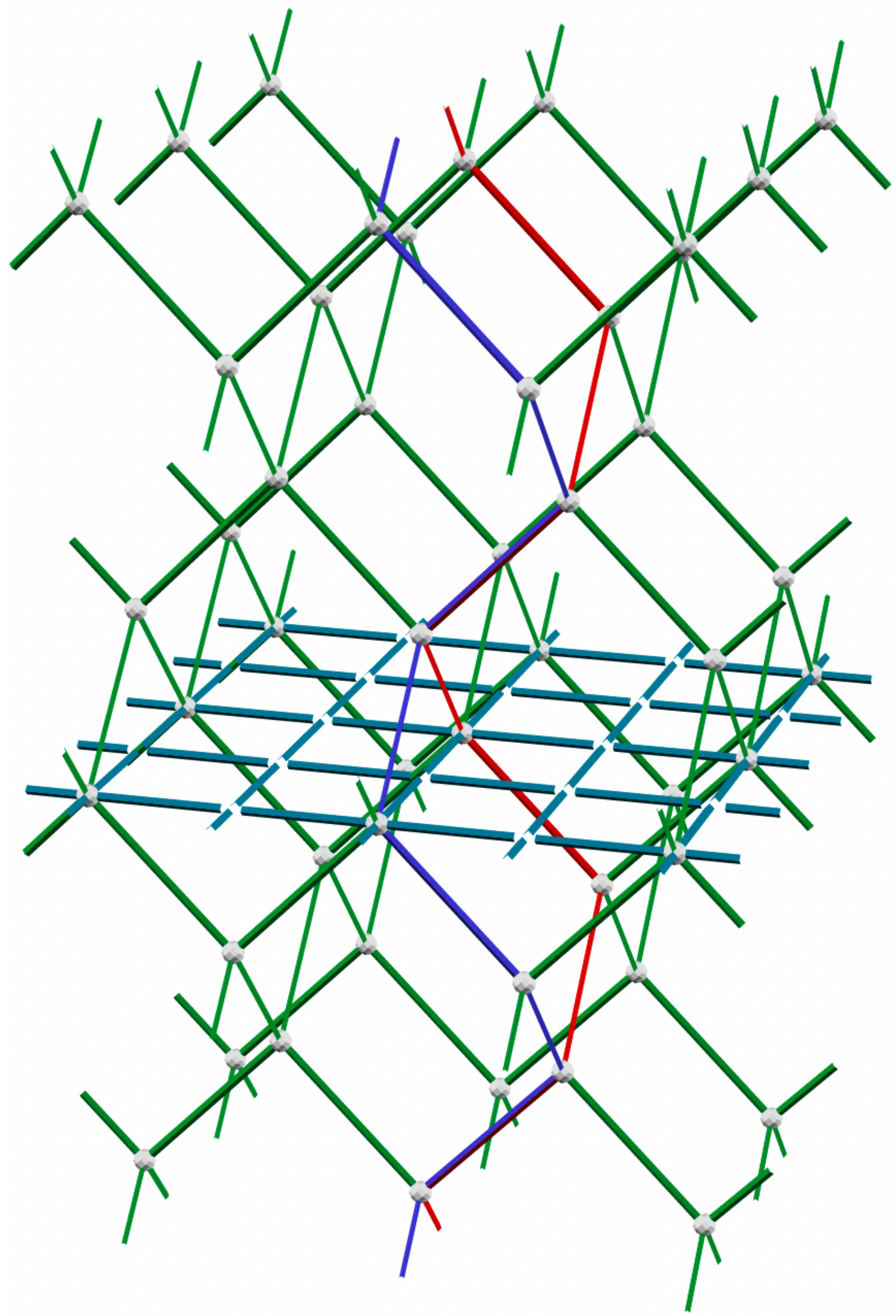
Poliedros regulares infinitos

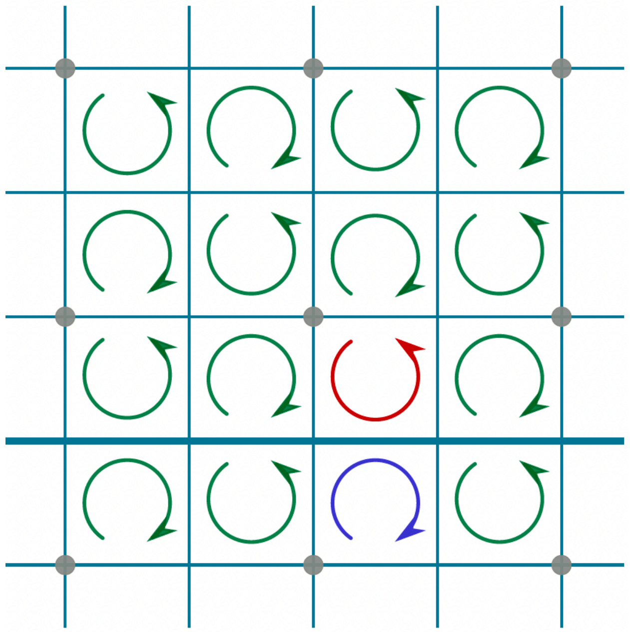




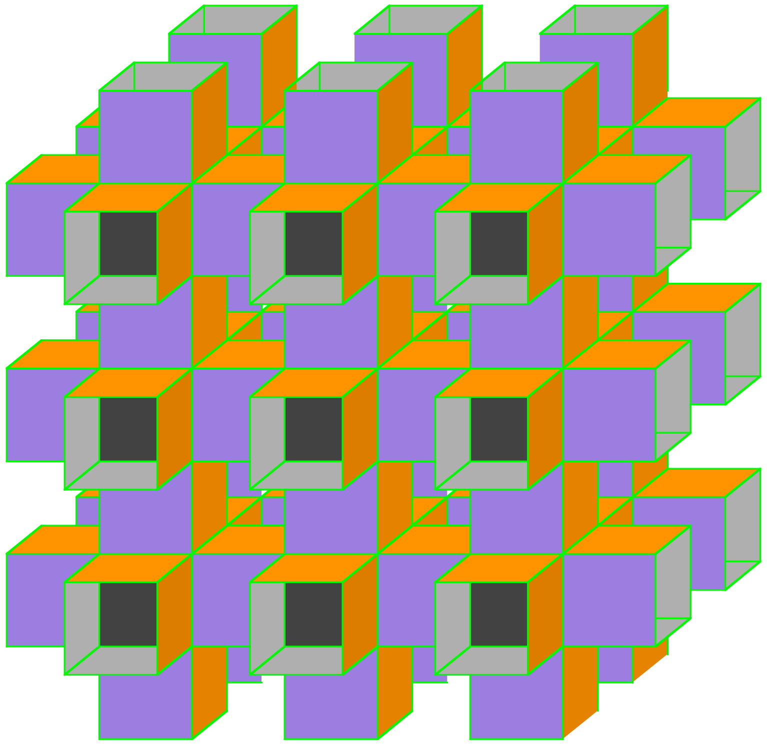
Poliedros regulares infinitos

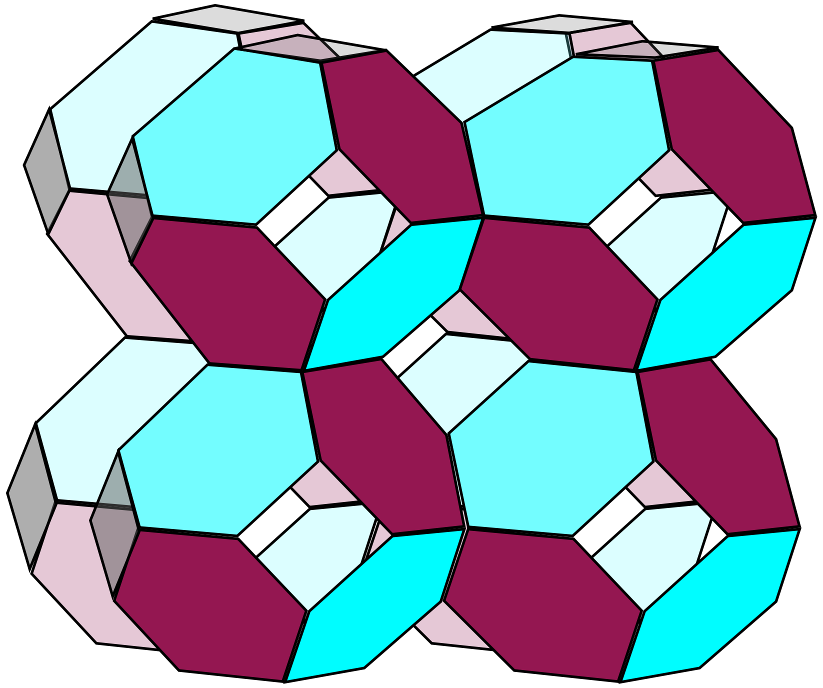


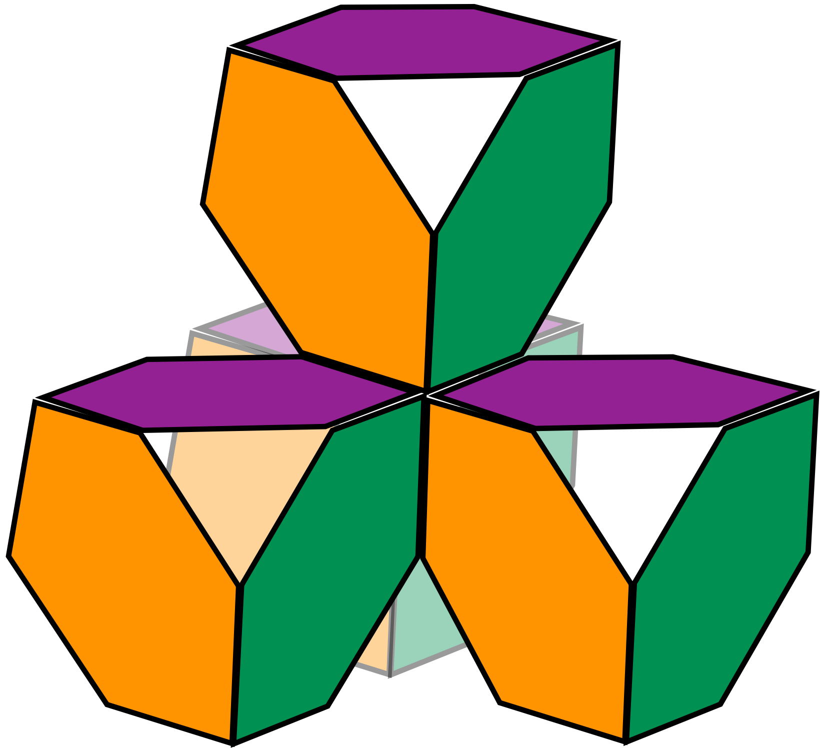




Poliedros regulares infinitos







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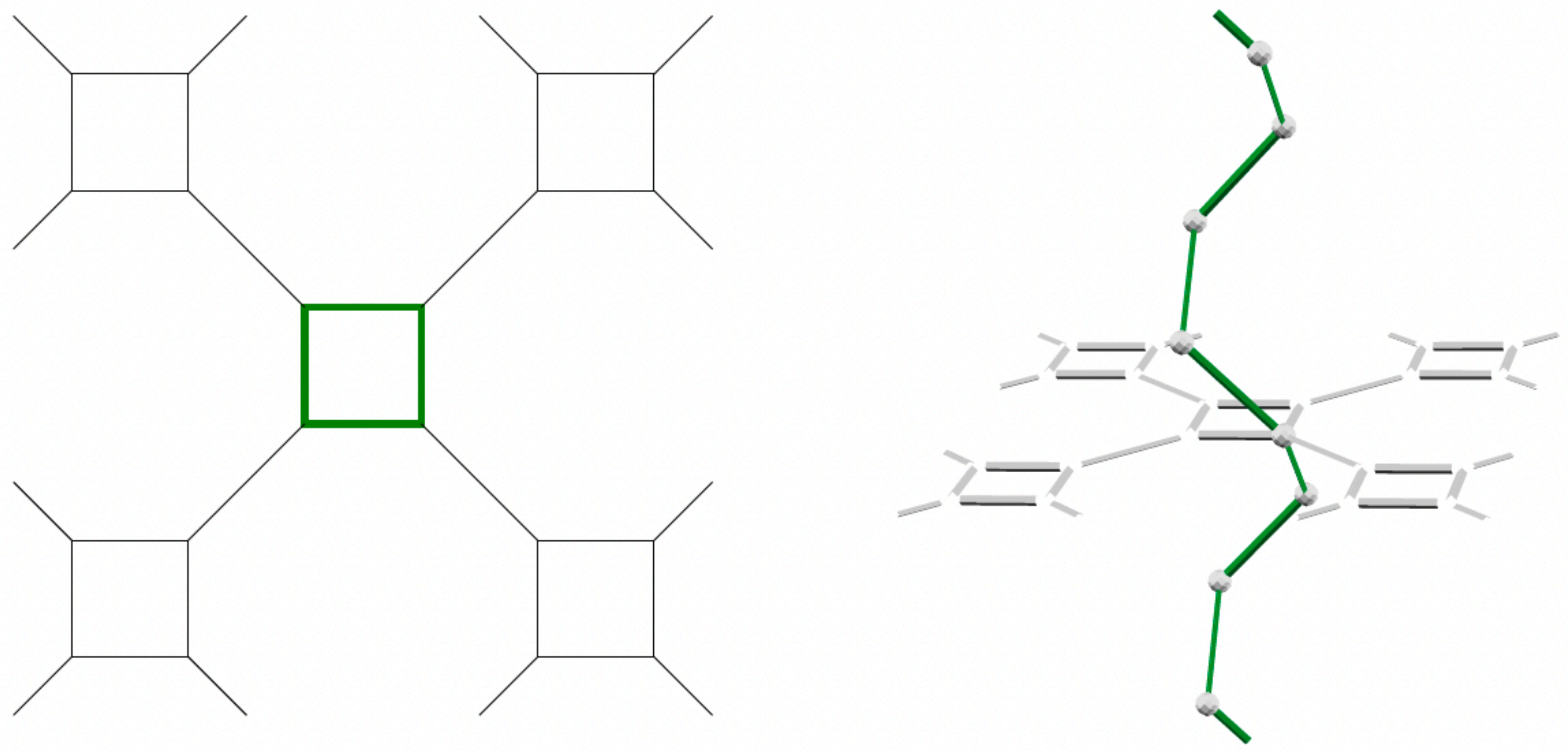

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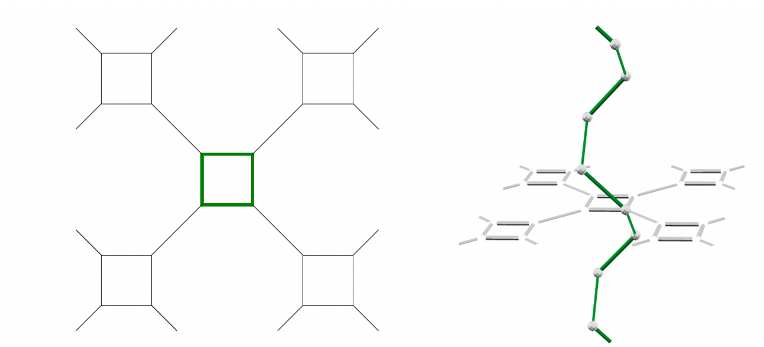

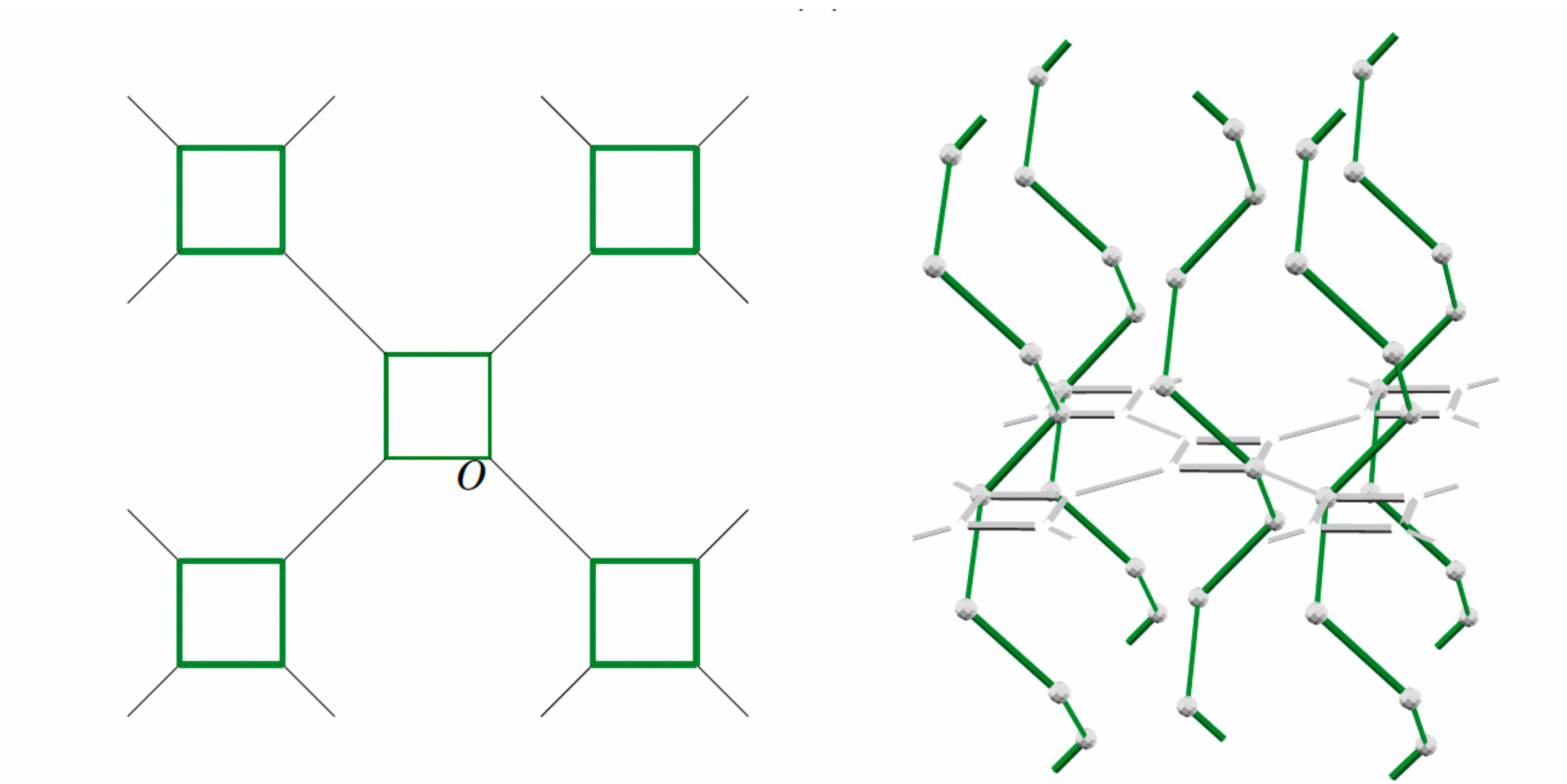

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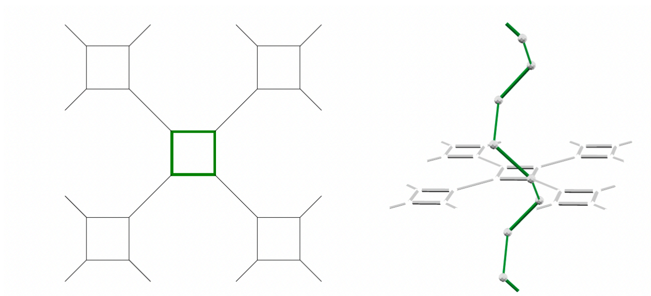

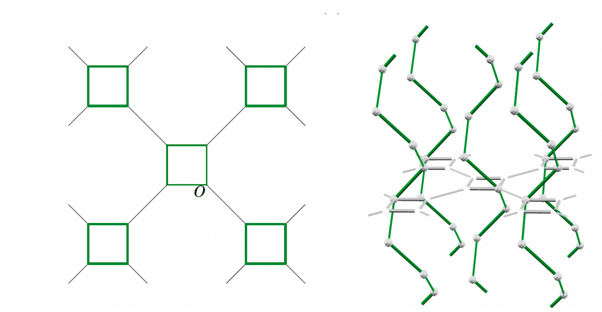

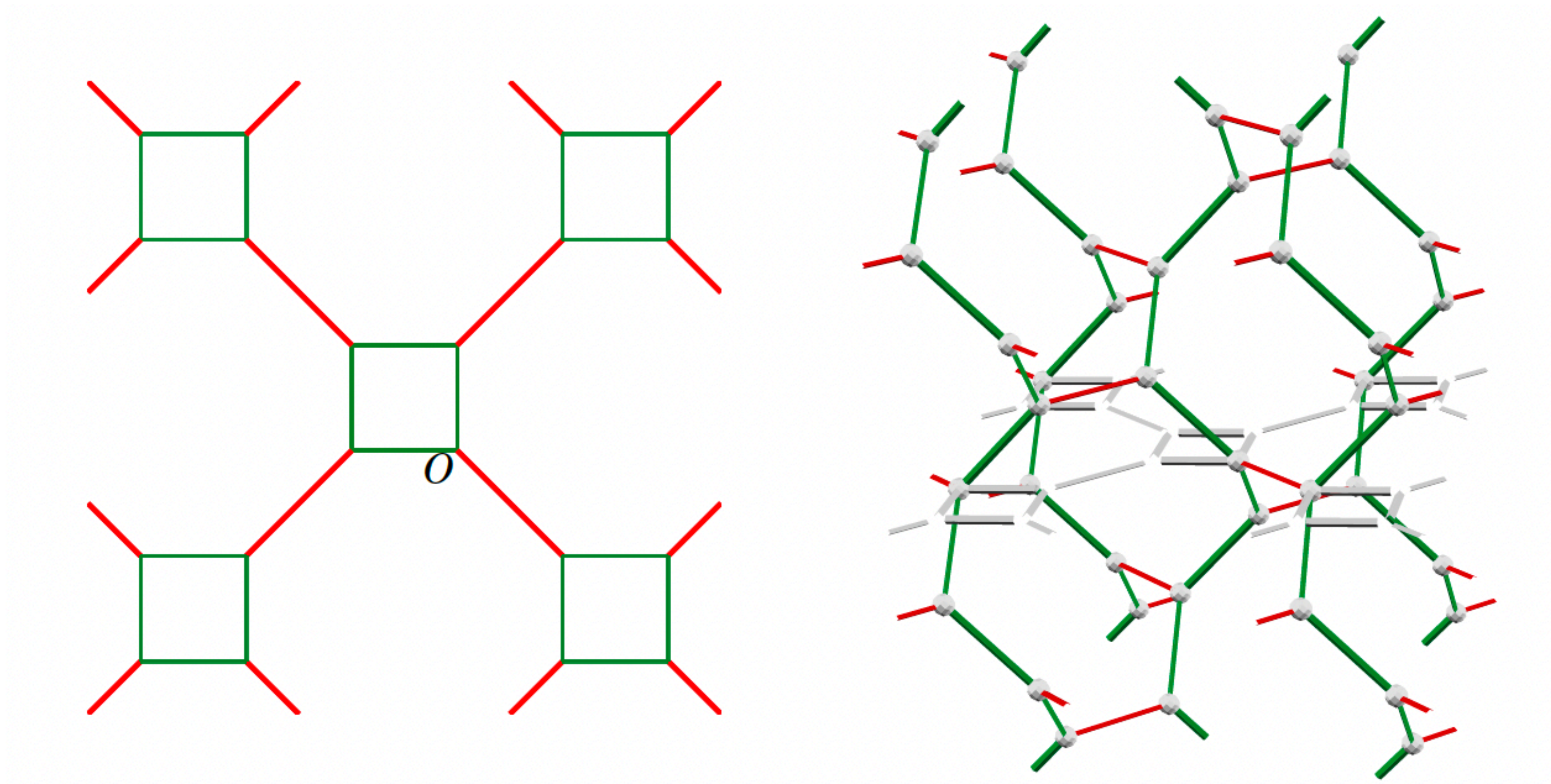

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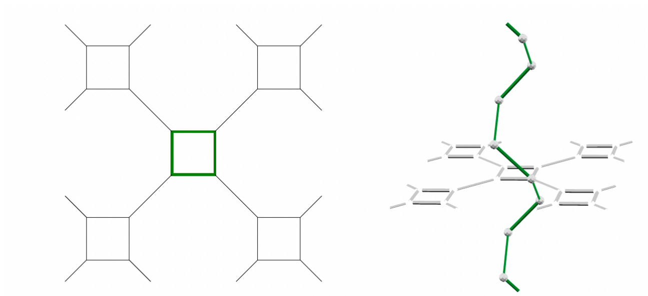

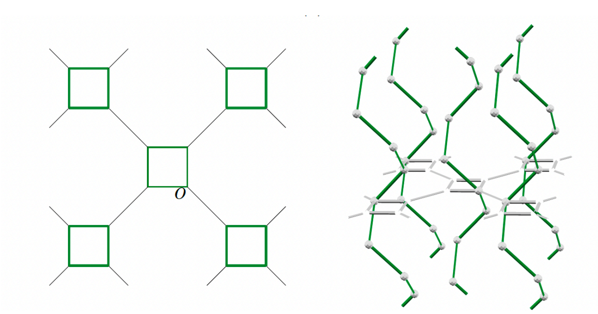

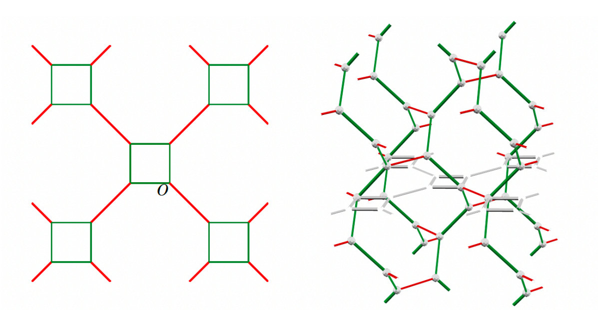

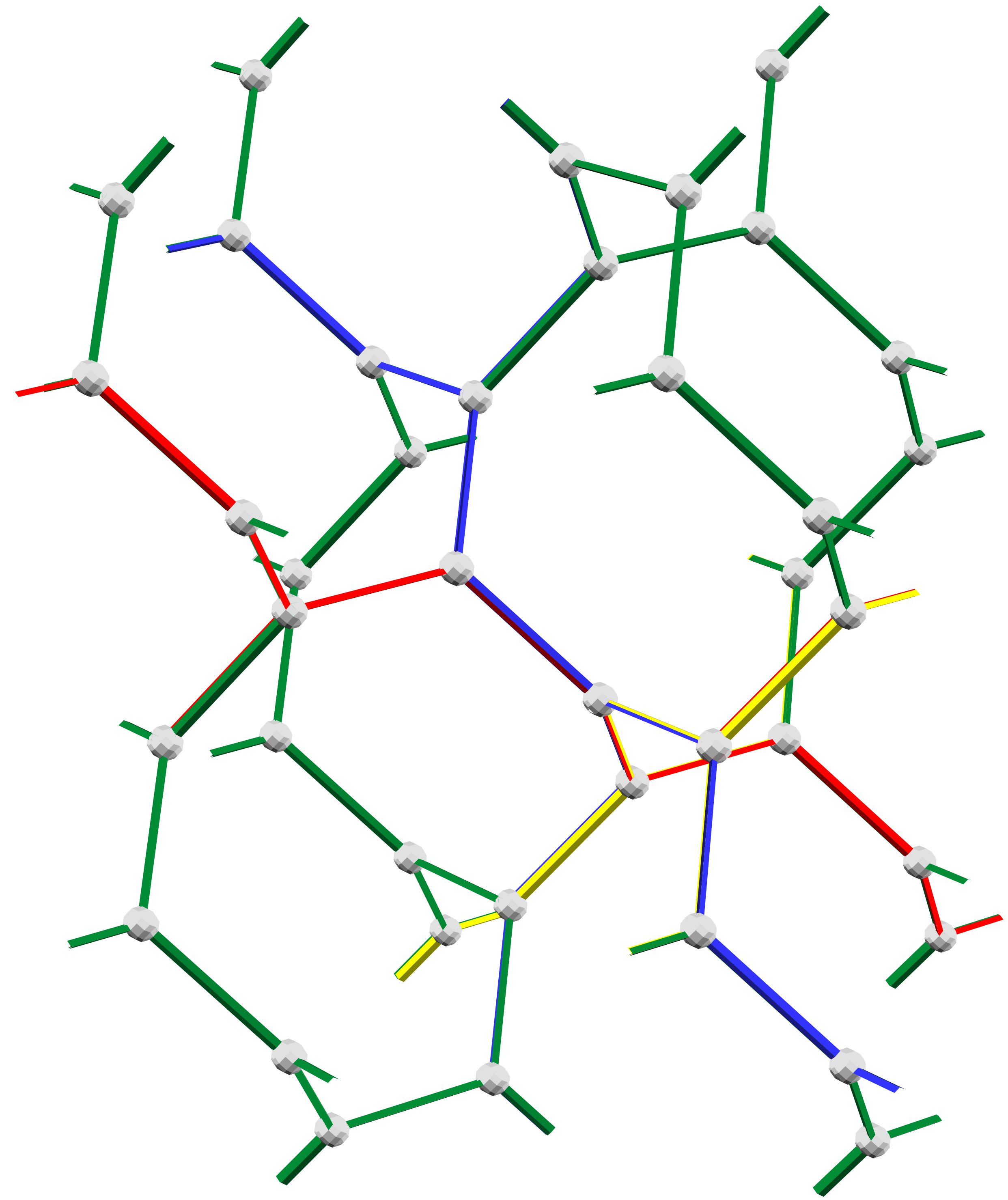

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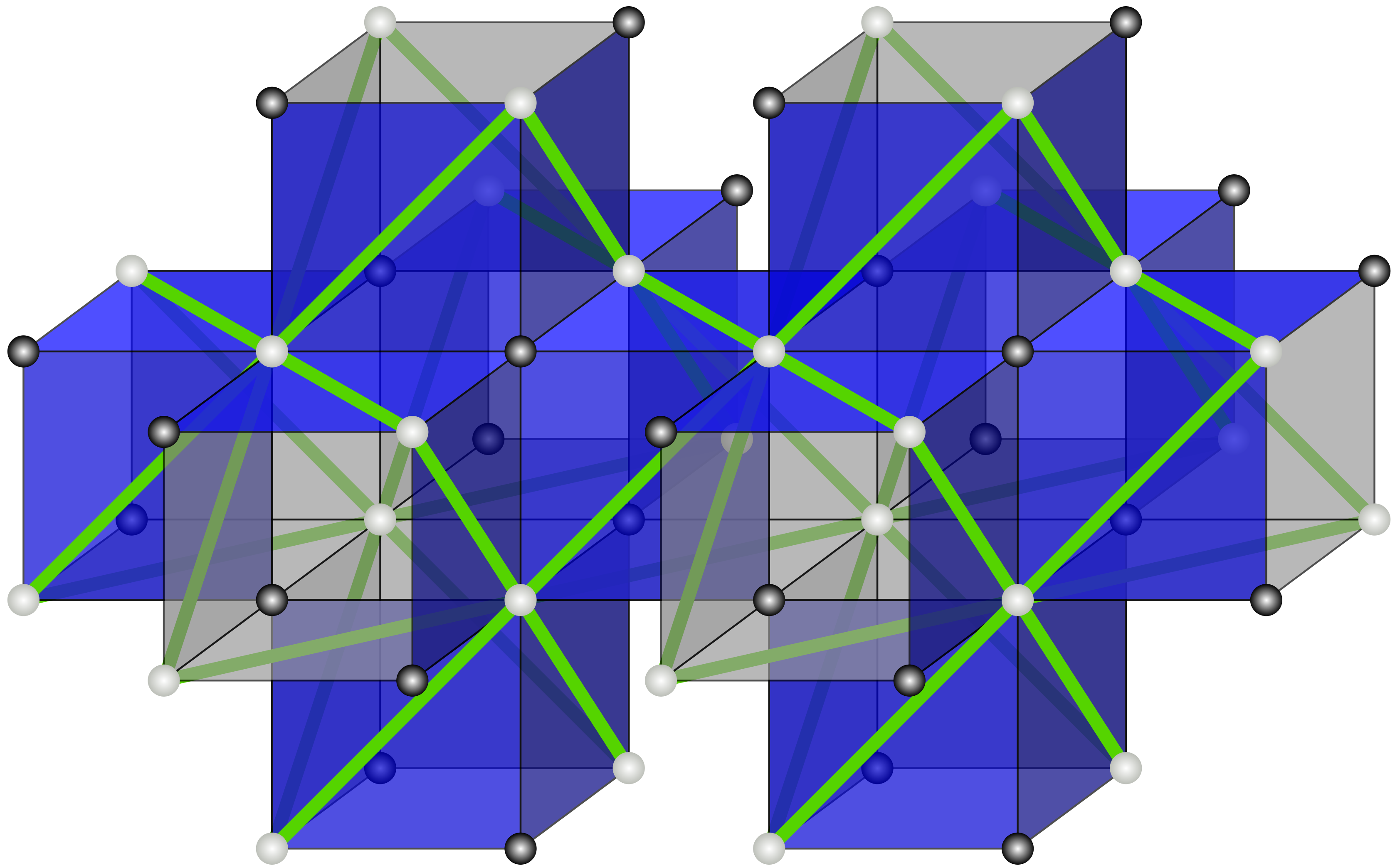

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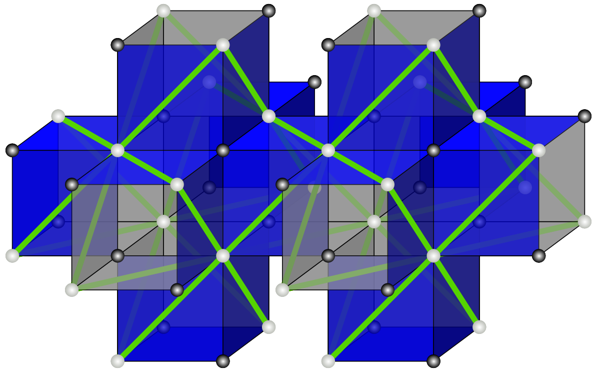

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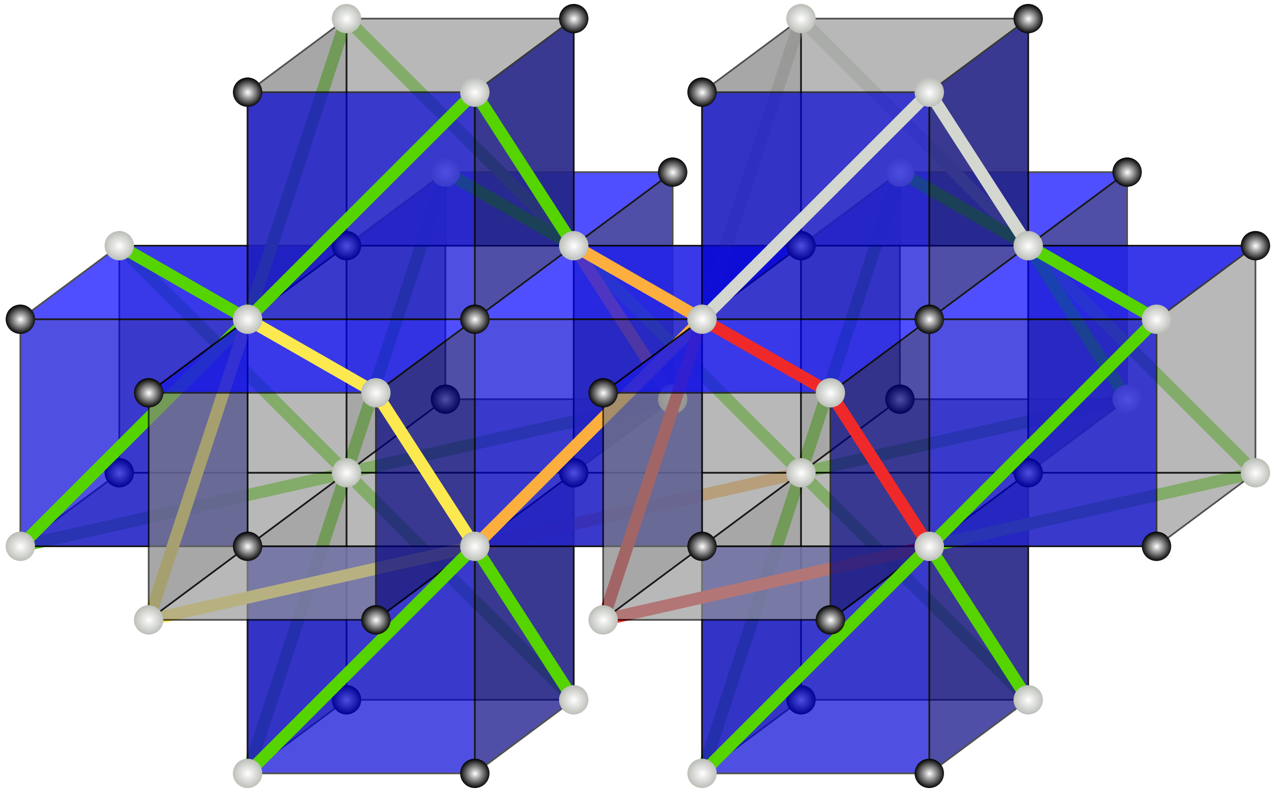

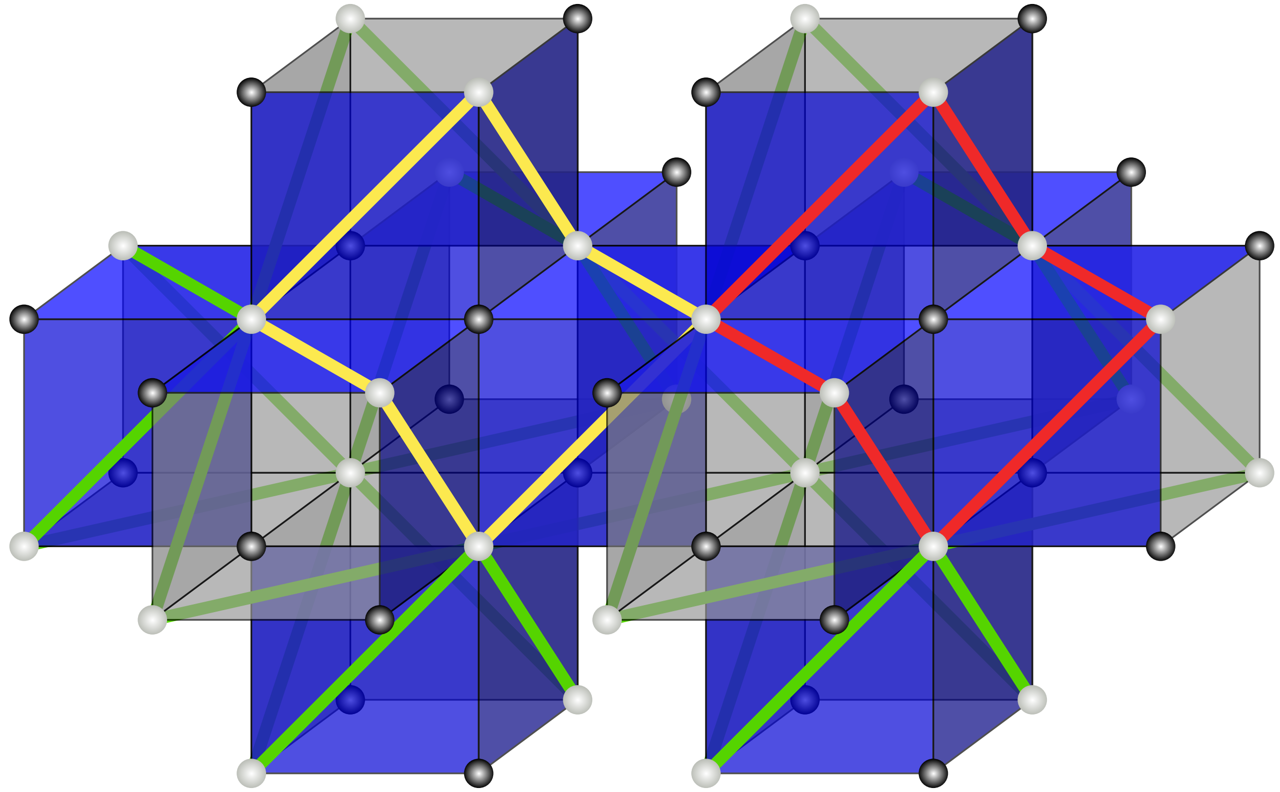

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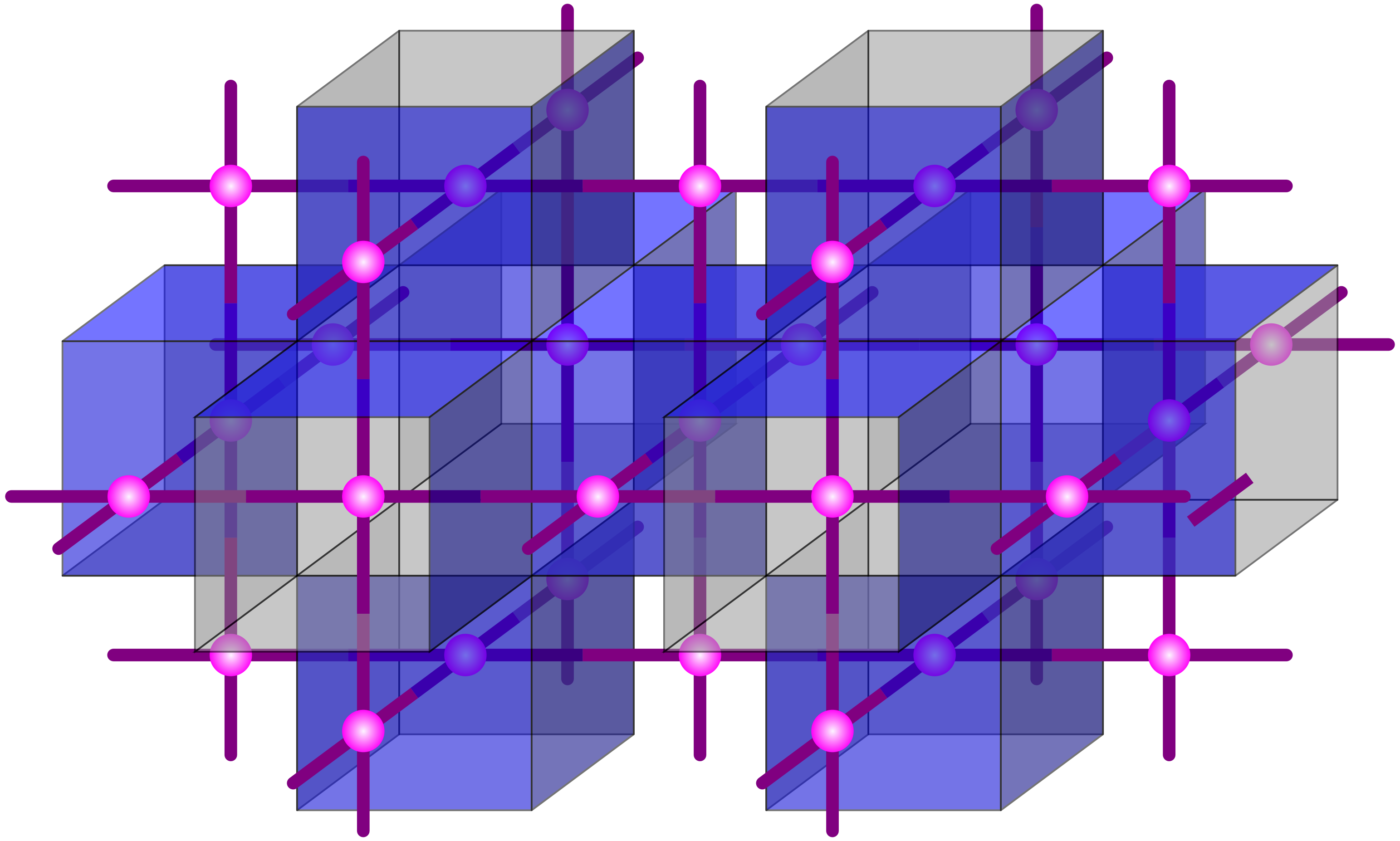

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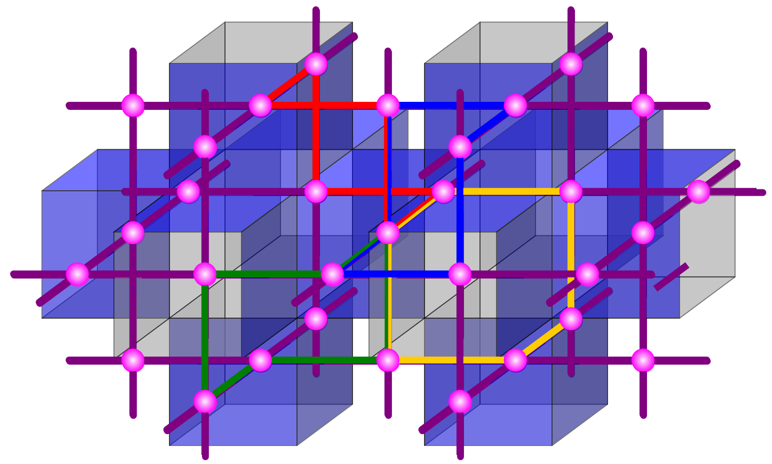

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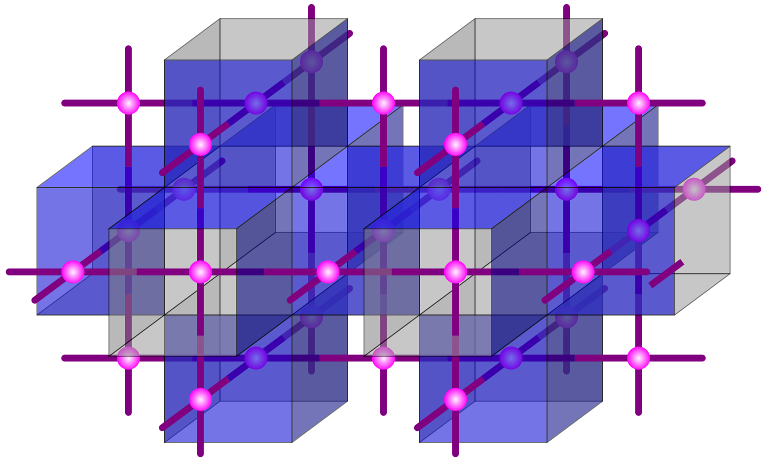

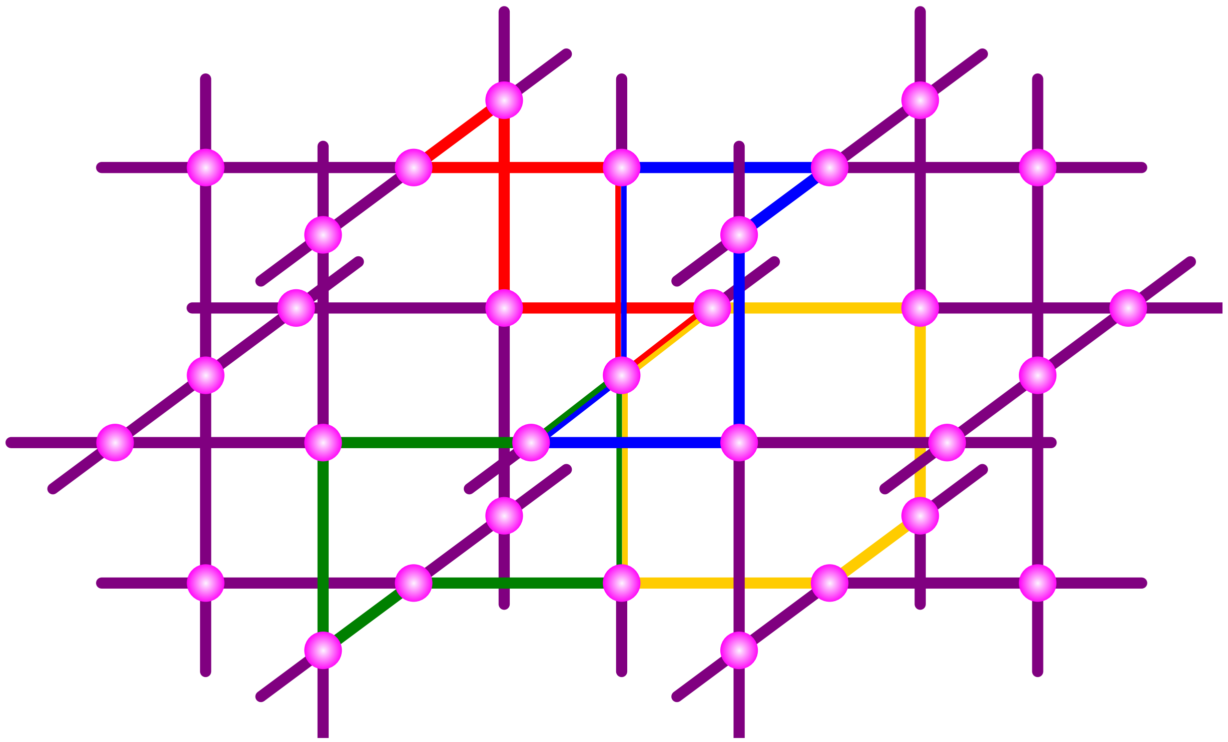

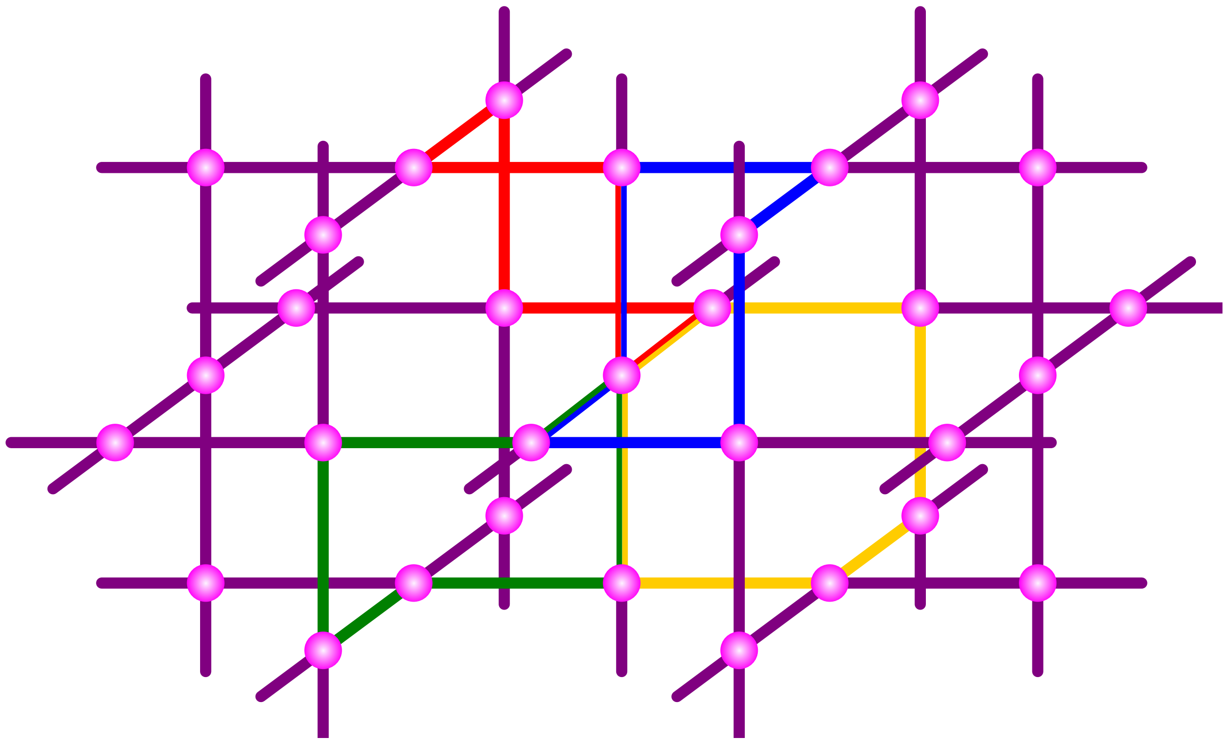

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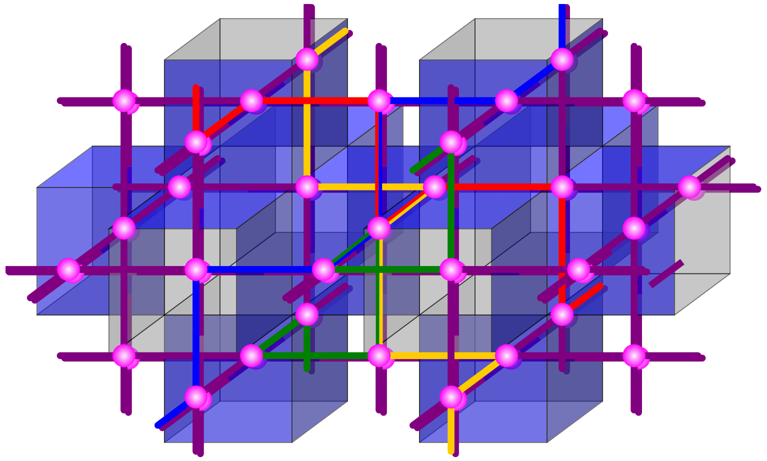

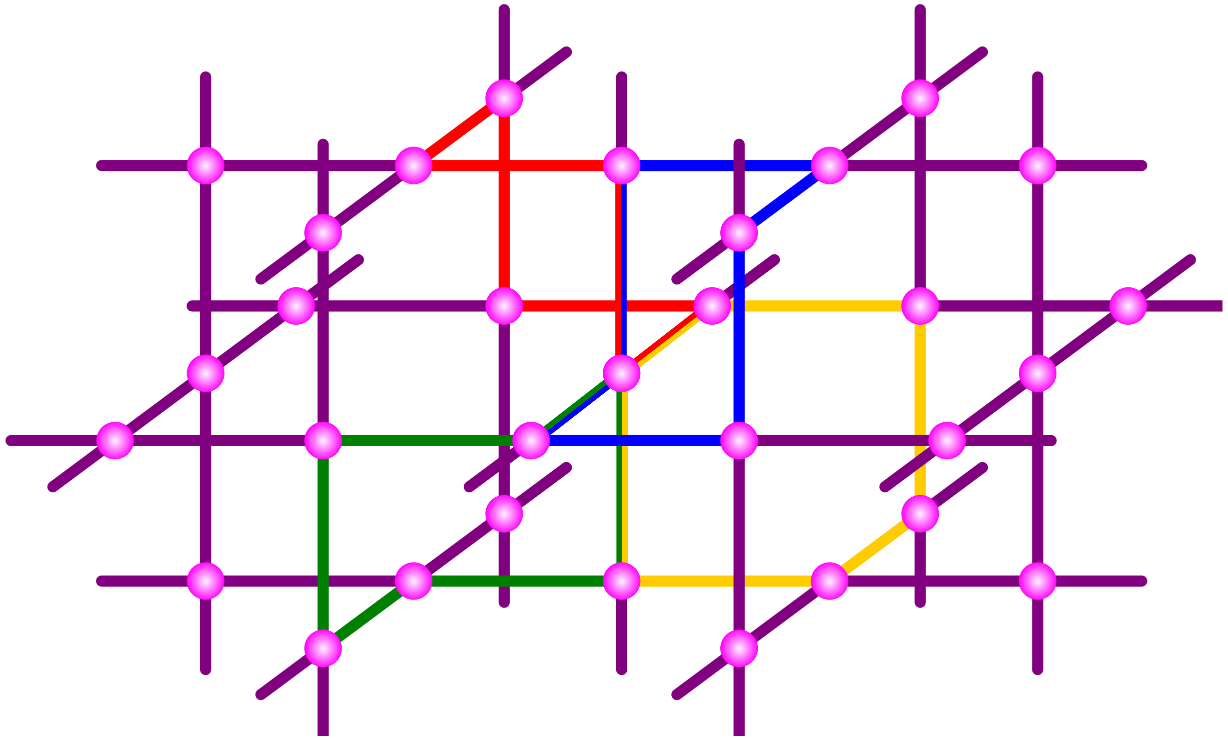

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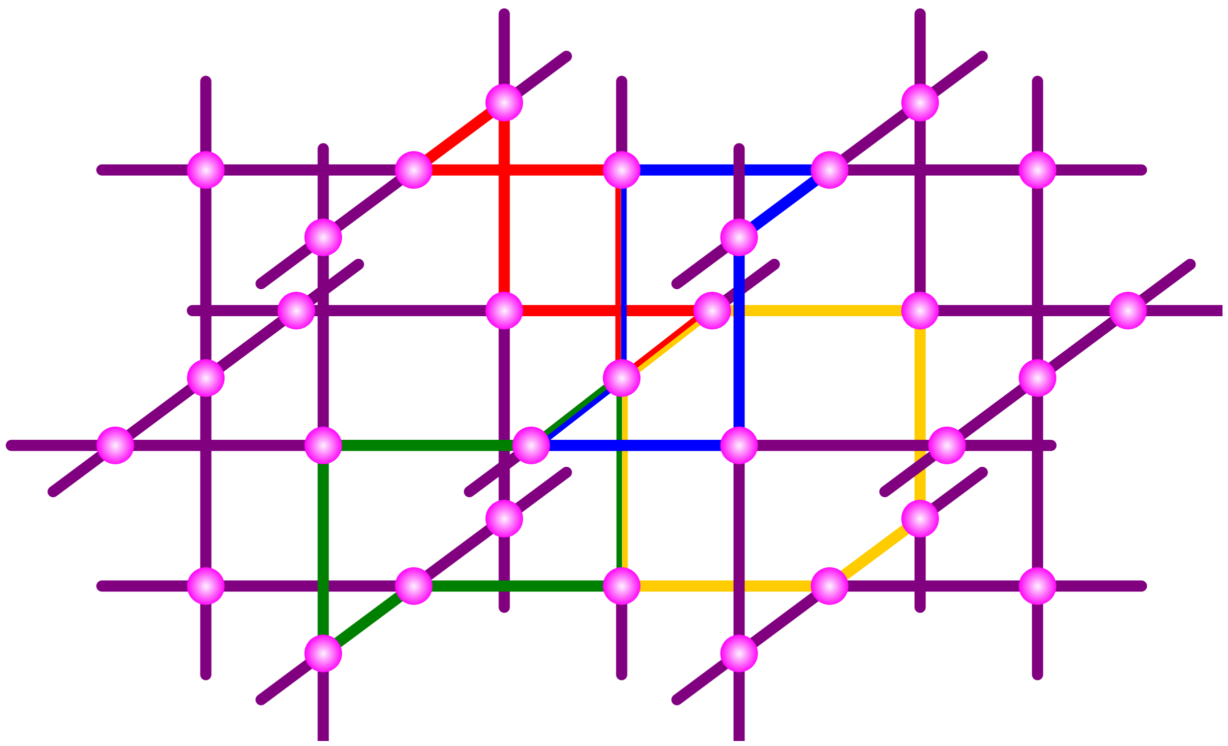
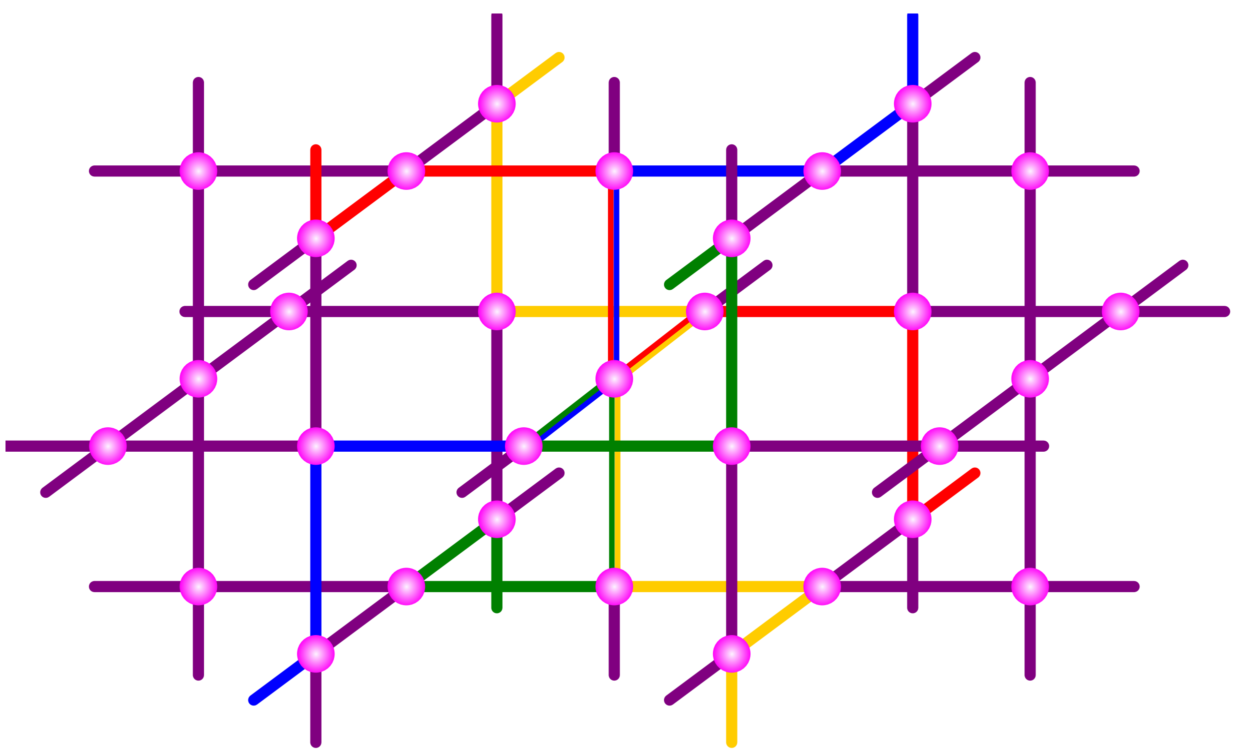
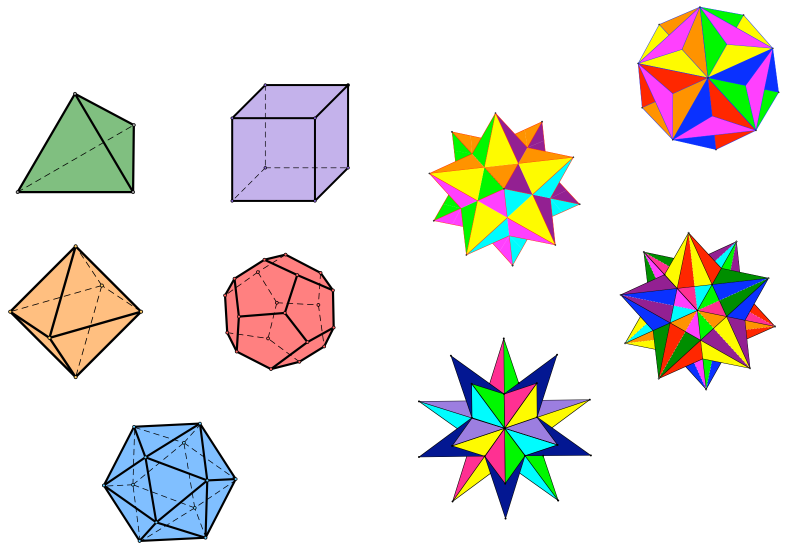
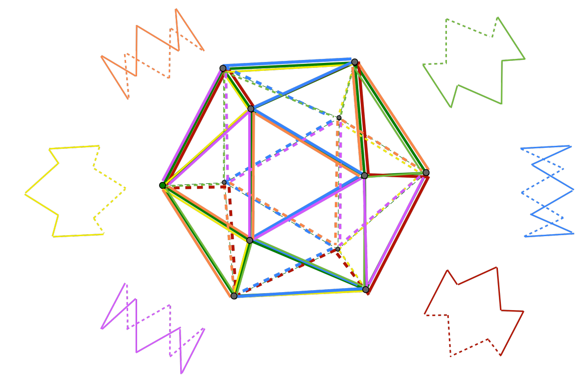
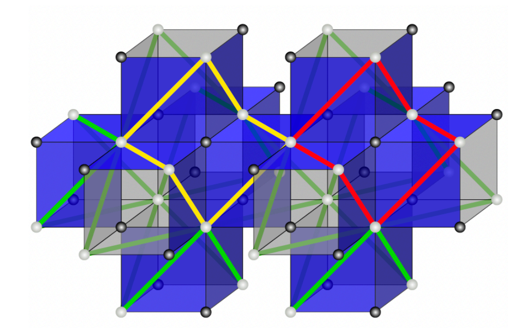
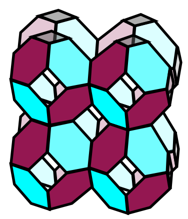


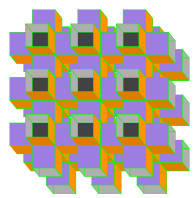


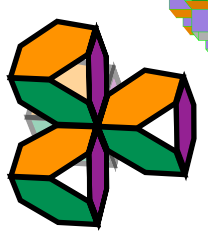






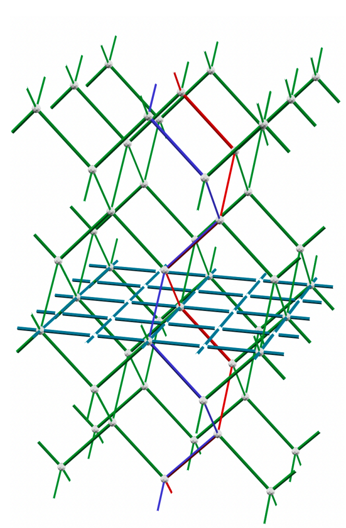


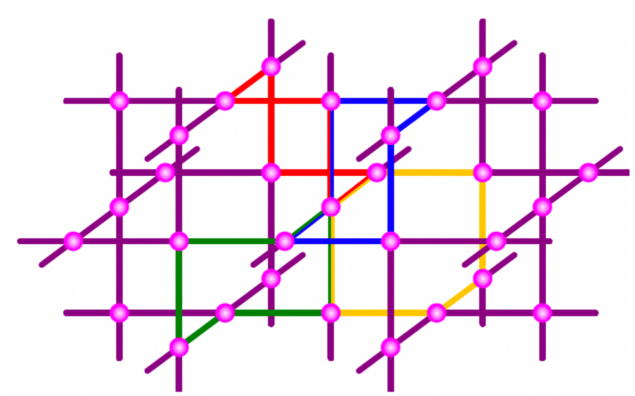


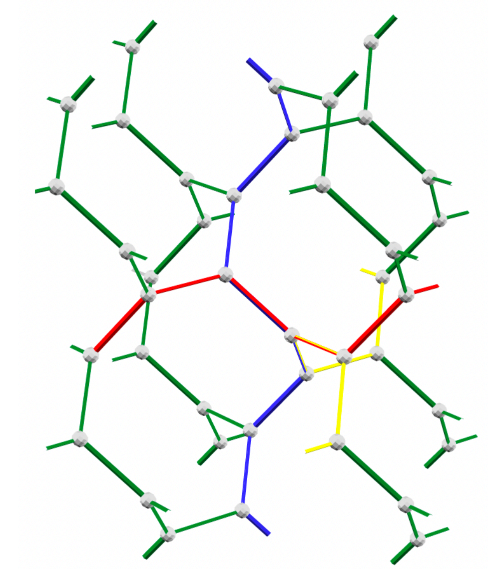


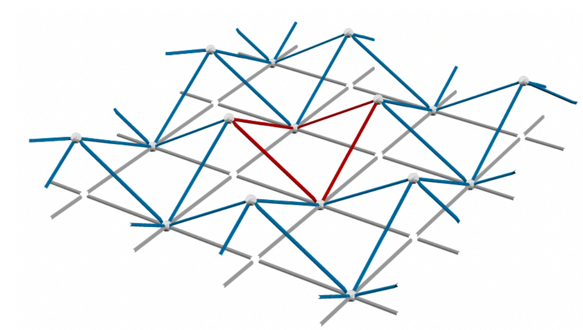
Teorema (A. Dress 1984)

Existen 48 poliedros regulares en el espacio euclidiano tridimensional









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Coloquio del CCM

EN LA BÚSQUEDA DE

POLIEDROS (ESQUELÉTICOS) QUIRALES

Isabel Hubard Escalera
Instituto de Matemáticas, UNAM

17 Noviembre
12:00hrs

Auditorio del
CCM - IRyA

¡Muchas gracias!