Cayley extensions of maniplexes and polytopes

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Joint work with Gabe Cunningham and Elías Mochán

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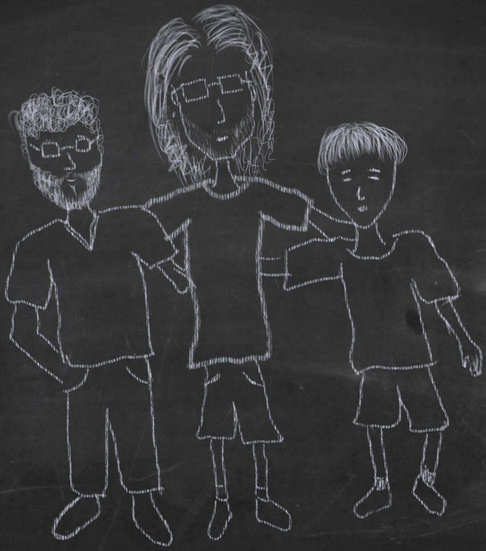
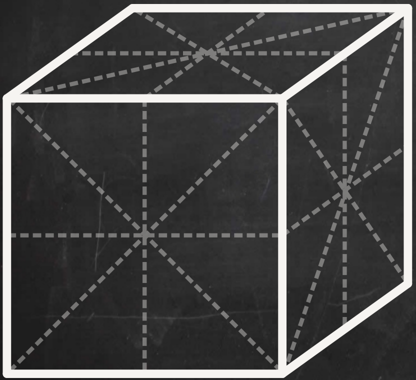


Flag-graphs of polytopes



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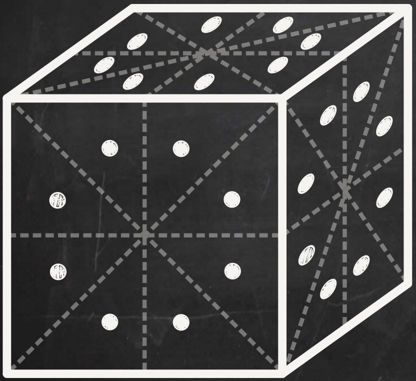

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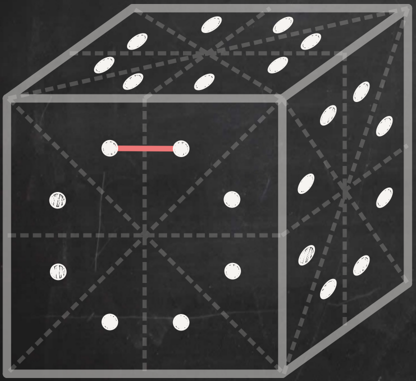

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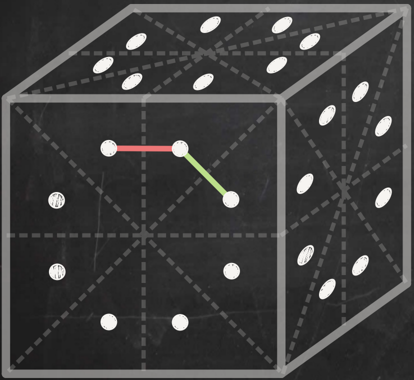

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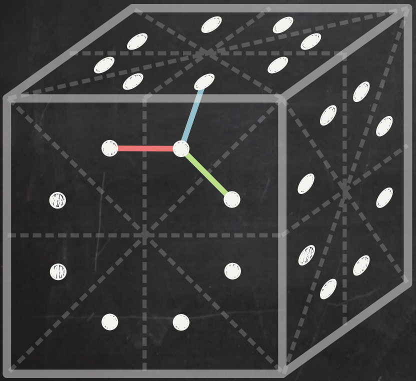

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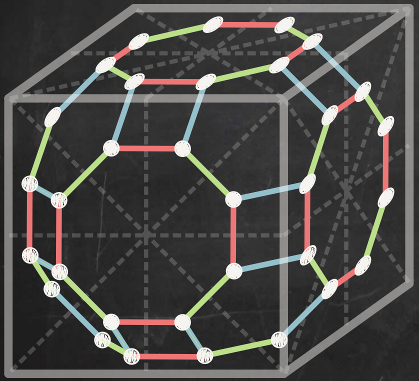

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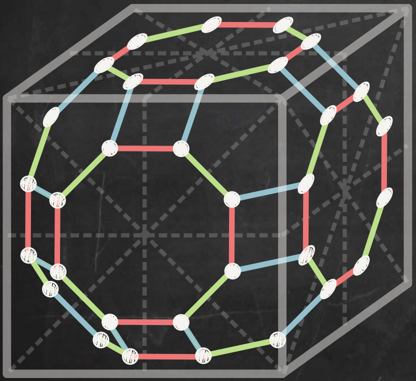

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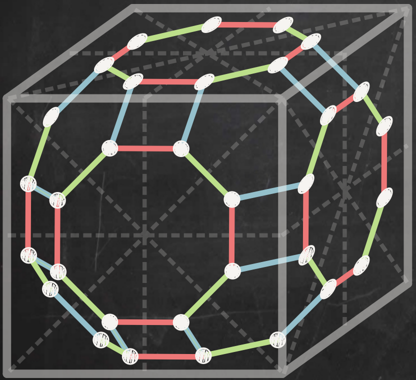

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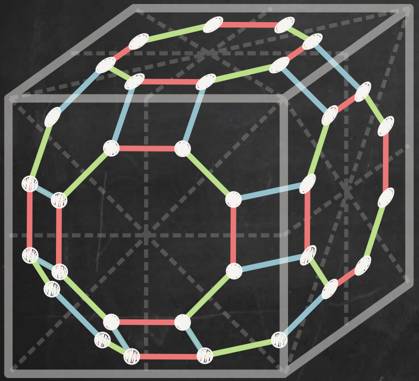

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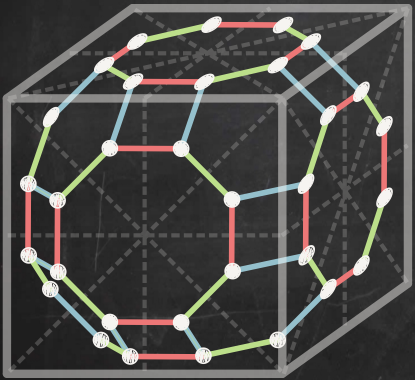

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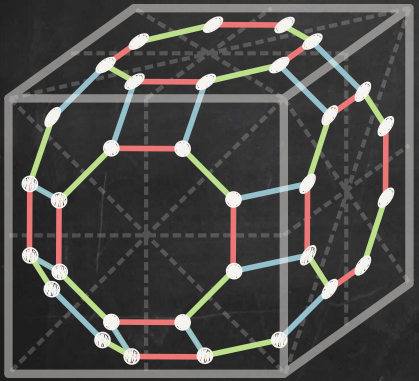

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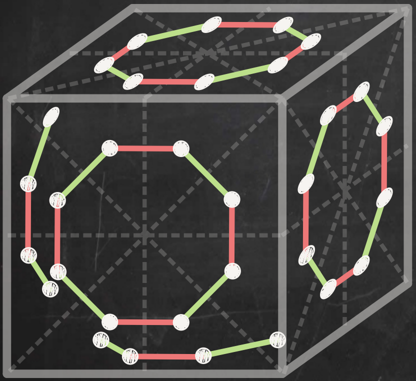

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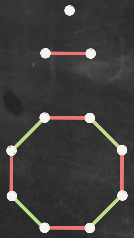

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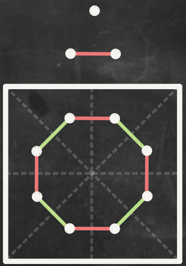

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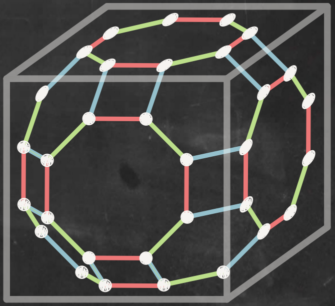

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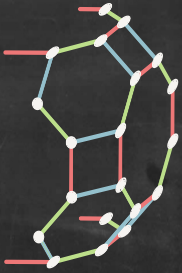

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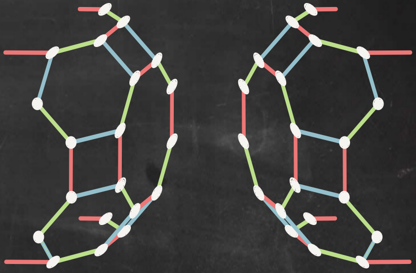

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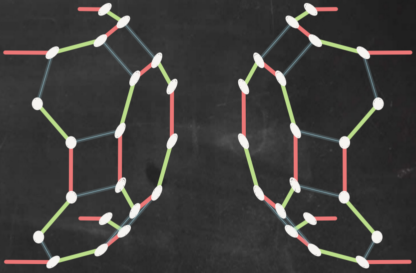

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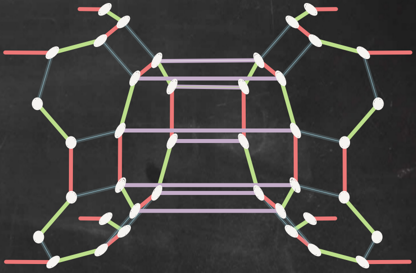

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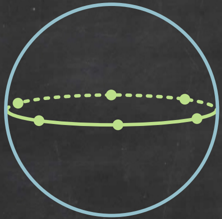
* An $(n+1)$-maniplex $\mathcal{M}$ is an extension of an n-maniplex $\mathcal{K}$ if all the facets of $\mathcal{M}$ are isomorphic to $\mathcal{K}$.

Extensions



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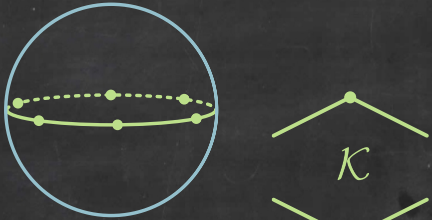

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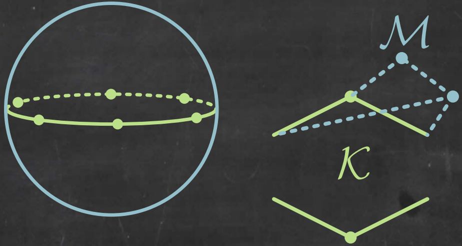

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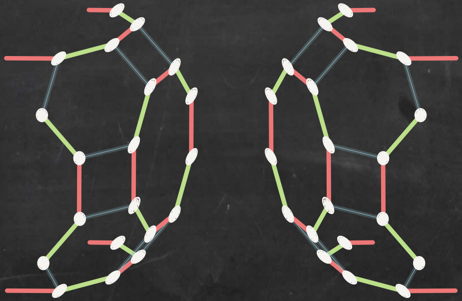

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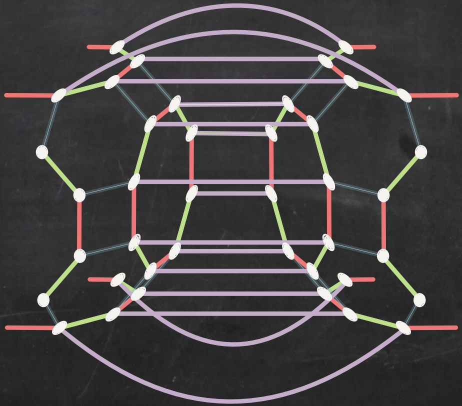

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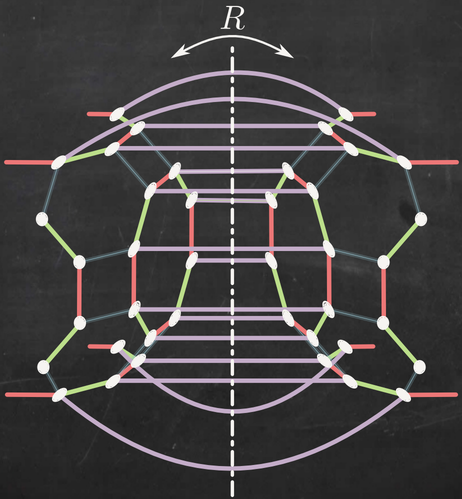

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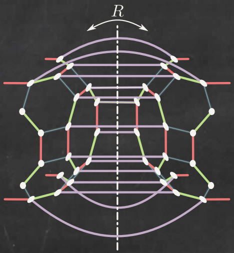

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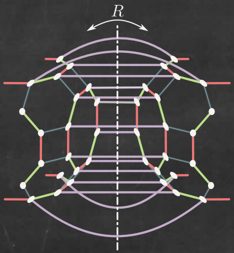

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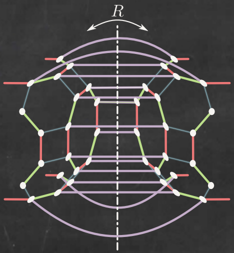

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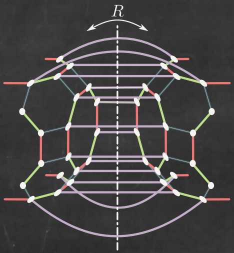

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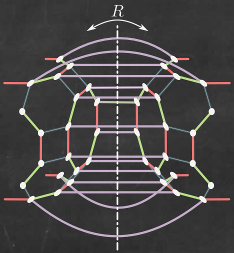

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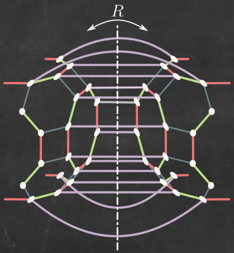

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A Cayley extender (extension) is canonical if $r_n = Id$

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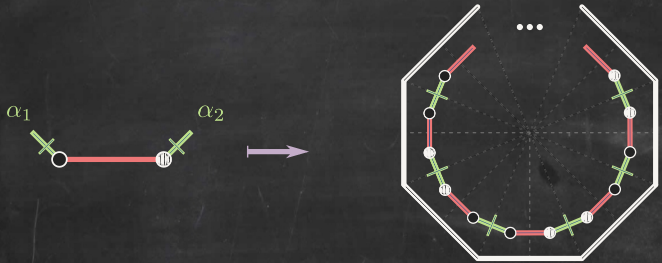

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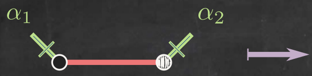

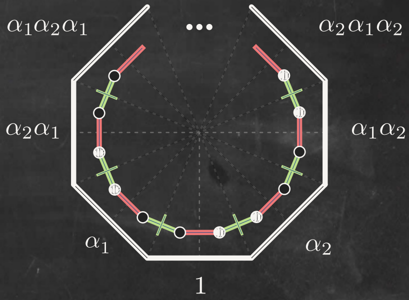

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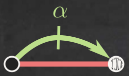

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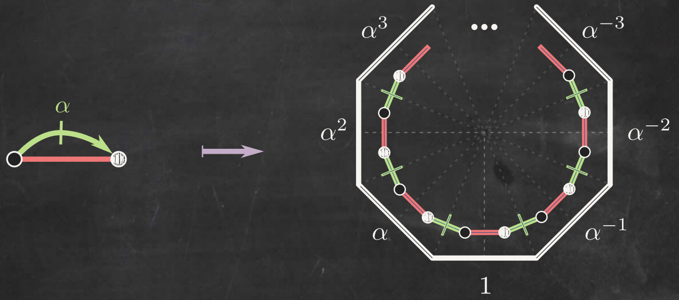

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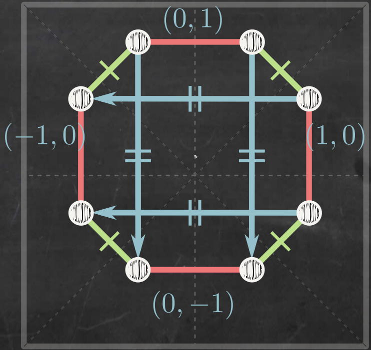

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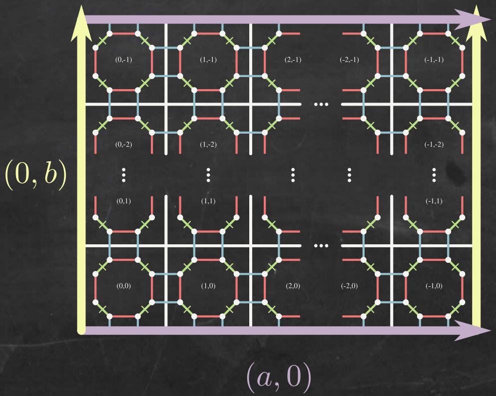

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Corollary

There are no chiral maniplexes that are canonical Cayley extensions.

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Proposition
An ARP $\mathcal{P}$ satisfies the flat-amalgamation propety (with respect of its facets) if and only if $\mathcal{P}$ is a canonical Cayley extension.

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Politopality of Cayley extensions

Theorem (Elías paid me to write this theorem down)
Let $(\mathcal{K}, r_n, \xi, G)$ a Cayley extender, then the maniplex $\mathcal{K}_{r_n}^{\xi}$ is polytopal if and only if

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Corollary
Let $(\mathcal{K}, id, \xi, G)$ a canonical Cayley extender, then the maniplex $\mathcal{K}_{r_n}^{\xi}$ is polytopal if and only if $\mathcal{K}$ is polytopal.

Quotients and examples

Proposition

Let $\pi : G \to H$ and $(\mathcal{K}, r_n, \xi, G)$ a Cayley extender, then $\pi\xi : \mathrm{Fac}(\mathcal{K}) \to H$, $(\mathcal{K}, r_n, \pi\xi, H)$ is a Cayley extender and

$$\mathcal{K}^{\xi}_{r_n} \searrow \mathcal{K}^{\pi\xi}_{r_n}$$

Quotients and examples

Let us take $(\mathcal{K}, id, \xi, \_)$ a canonical Cayley extender and denote $\xi(F) = \alpha_F$.

$$G = \langle \alpha_F : \alpha_F^2 = 1 \rangle$$

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\end{aligned}$$

then

Quotients and examples

Let us take $(\mathcal{K}, id, \xi, \_\_)$ a canonical Cayley extender and denote $\xi(F) = \alpha_F$.

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then

$$\mathcal{U}(\mathcal{K}) \searrow \mathcal{U}_{2s}(\mathcal{K}) \searrow \mathcal{U}'_{2s}(\mathcal{K}) \searrow \hat{2}s^{\mathcal{K}-1} \searrow \mathcal{K}|2s$$

Quotients and examples

Extension	Size	Also known as
$\mathcal{U}(\{4\})$	∞	$\{4,\infty\}$
$\mathcal{U}_{2s}(\{4\})$	∞ for $s \geqslant 2$	
$\mathcal{U}'_{2s}(\{4\})$	∞ for $s \geqslant 3$	
$\hat{2}s^{\{4\}-1}$	$16s^3$	$\{4,4\}_{(4,0)}$ for $s=2$ $\{4,6\} * 432b$ for $s=3$
$\{4\}\|2s$	$16s$	$\{4,2s \mid 2\}$

* We used RAMP and so should you...

Symmetry type graphs

Proposition

If every automorphism of $\mathcal{K}_{r_n}$ induces an automorphism of G, then $\mathcal{K}^{\xi}_{r_n}$ is hereditary, $\Gamma(\mathcal{K}^{\xi}_{r_n}) = G \rtimes \Gamma(\mathcal{K})$ and the STG of $K^{\xi}_{r_n}$ is $\mathcal{K}_{r_n} / \Gamma(\mathcal{K})$.

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Corollary
For a canonical Cayley extension, if every automorphism of $\mathcal{K}$ induces an automorphism of G, then $\mathcal{K}^{\xi}_{id}$ is hereditary, $\Gamma(\mathcal{K}^{\xi}_{id}) = G \rtimes \Gamma(\mathcal{K})$ and the STG is obtained by adding n-semiedges to each vertex of the STG of $\mathcal{K}$.

Symmetry type graphs

Theorem

For any Cayley extender $(\mathcal{K}, r_n, \xi, G)$ there exists a group $\heartsuit(\mathcal{K}, r_n)$ (called the r_n-friendly group) such that

- $\Gamma(\mathcal{K}_{r_n}) \leqslant \heartsuit(\mathcal{K}, r_n) \leqslant \Gamma(\mathcal{K})$

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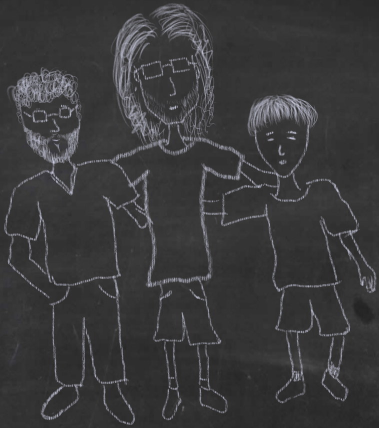
- $\Gamma(\mathcal{K}_{r_n}) \leqslant \heartsuit(\mathcal{K}, r_n) \leqslant \Gamma(\mathcal{K})$
- The symmetry type graph of $\mathcal{K}^{\xi}_{r_n}$ is $\mathcal{K}_{r_n} / H$ for some $H \leqslant \heartsuit(\mathcal{K}, r_n)$



Thank you!