Highly Symmetric Toroidal Polyhedra

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National Autonomous University of Mexico

9th International Slovenian Conference on Graph Theory
Bled, Slovenia - June 2019
In memory of Branko Grünbaum

The first paper I read...

... I think

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Aequationes Mathematicae **16** (1977) 1–20 Birkhäuser Verlag, Basel
University of Waterloo

Expository papers

Regular polyhedra–old and new

BRANKO GRÜNBAUM

Abstract. Although it is customary to define *polygons* as certain families of edges, when considering *polyhedra* it is usual to view polygons as 2-dimensional pieces of the plane. If this rather illogical point of view is replaced by consistently understanding polygons as 1-dimensional complexes, the theory of polyhedra becomes richer and more satisfactory. Even with the strictest definition of regularity this approach leads to 17 individual regular polyhedra in the Euclidean 3-space and 12 infinite families of such polyhedra, besides the traditional ones (which consist of 5 Platonic polyhedra, 4 Kepler–Poinsot polyhedra, 3 planar tessellations and 3 Petrie–Coxeter polyhedra). Among the many still open problems that naturally arise from the new point of view, the most obvious one is the question whether the regular polyhedra found in the paper are the only ones possible in the Euclidean 3-space.

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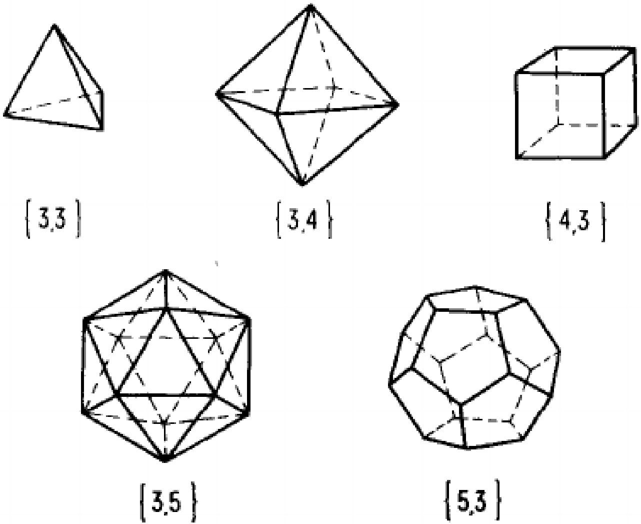

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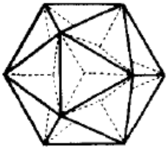
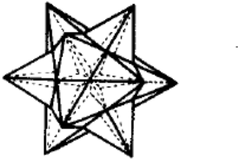


$\{5, 5/2\}$



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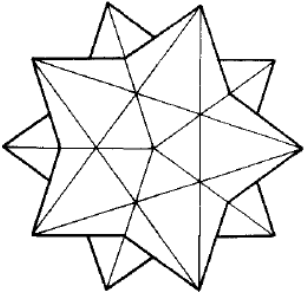

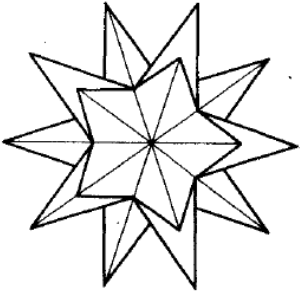

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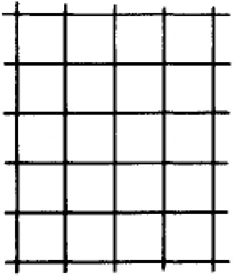
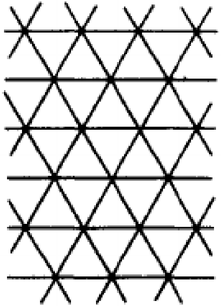
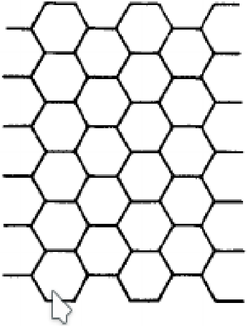


$\{4,4\}$



$\{3,6\}$



$\{6,3\}$

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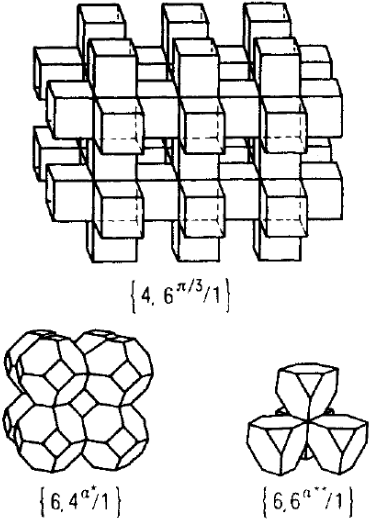

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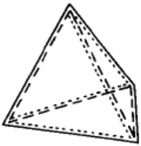
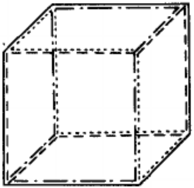
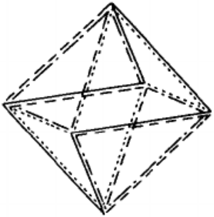


$\{4^{\pi/3}/1, 3\}$



$\{6^{\pi/2}/1, 3\}$



$\{6^{\pi/3}/1, 4\}$

What about regular polyhedra in other spaces?

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- Regular maps on the torus.

Can we classify regular polyhedra in the 3-dimensional torus $\mathbb{T}^3$?

I will not explicitly skip any definition

- The 3-dimensional torus $\mathbb{T}^3_\Lambda$ is the quotient of $\mathbb{E}^3$ by a group Λ generated by 3 independent translations.

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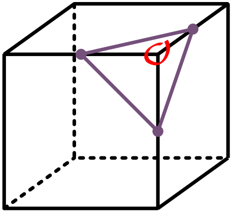

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- The symmetry group $G(\mathcal{P})$ is the set of isometries of $\mathbb{T}^3_\Lambda$ that preserve $\mathcal{P}$.

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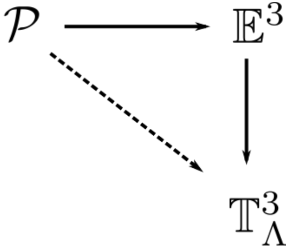
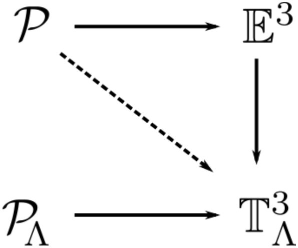
- The symmetry group $G(\mathcal{P})$ is the set of isometries of $\mathbb{T}^3_\Lambda$ that preserve $\mathcal{P}$.
- $\mathcal{P}$ is regular if $G(\mathcal{P})$ acts transitively on flags.

$$\mathcal{P} \longrightarrow \mathbb{E}^3$$

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathbb{E}^3 \\
 & & \downarrow \\
 & & \mathbb{T}^3_\Lambda
\end{array}
$$


$\mathcal{P}$
$\mathbb{E}^3$
$\mathbb{T}^3_\Lambda$


$\mathcal{P}$
$\mathbb{E}^3$
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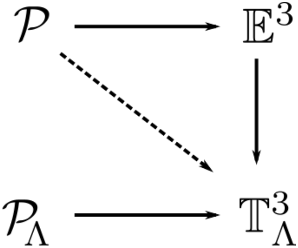
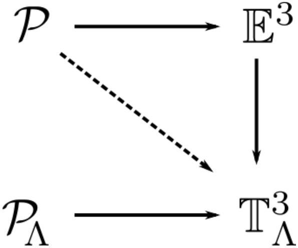


1. Is the “polyhedron” induced by $\mathcal{P}$ actually a polyhedron?



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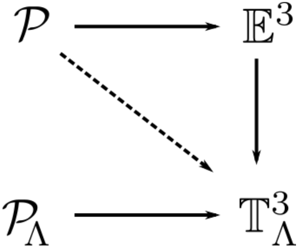
2. Is $\mathcal{P}_\Lambda$ as symmetric as $\mathcal{P}$?



1. Is the "polyhedron" induced by $\mathcal{P}$ actually a polyhedron?
2. Is $\mathcal{P}_\Lambda$ as symmetric as $\mathcal{P}$?
3. Is every regular polyhedron $\mathcal{P}_\Lambda$ induced this way?

The usual trick

1. $G(\mathcal{P}) = \langle R_i : i \in I \rangle$,

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4. $\mathcal{P}_\Lambda$ is as symmetric as $\mathcal{P}$ if and only if Λ is preserved by $H(\mathcal{P})$.

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4. $\mathcal{P}_\Lambda$ is as symmetric as $\mathcal{P}$ if and only if Λ is preserved by $H(\mathcal{P})$.

[3, 3]	**[3, 4]**		**[3, 5]**		$\tilde{\mathbf{D}}_3$	$\tilde{\mathbf{D}}_4$	$\tilde{\mathbf{D}}_6$	
$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	$\{3,5\}$	$\{5,3\}$	$\{3,6\}\#\{\infty\}$	$\{4,4\}$	$\{3,6\}$	$\{6,3\}$
$\{6,3\}_4$	$\{4,3\}_3$	$\{6,4\}_3$	$\{10,5\}$	$\{10,3\}$	$\{\infty,6\}_3\#\{\infty\}$	$\{\infty,4\}_4$	$\{\infty,6\}_3$	$\{\infty,3\}_6$
$\{6,6\|3\}$	$\{6,4\|4\}$	$\{4,6\|4\}$	$\{6,\frac{2}{2}\}$	$\{5,\frac{5}{2}\}$	$\{6,3\}\#\{\,\}$	$\{4,4\}\#\{\,\}$	$\{3,6\}\#\{\,\}$	$\{\infty,6\}_3\#\{\}$
$\{\infty,6\}_{4,4}$	$\{\infty,4\}_{6,4}$	$\{\infty,6\}_{6,3}$	$\{\frac{5}{2},5\}$	$\{6,5\}$	$\{\infty,3\}_6\#\{\,\}$	$\{\infty,4\}_4\#\{\}$	$\{6,3\}\#\{\infty\}$	$\{\infty,3\}_6\#\{\infty\}$
$\{6,6\}_4$	$\{6,4\}_6$	$\{4,6\}_6$	$\{\frac{10}{3},3\}$	$\{\frac{5}{2},3\}$		$\{4,4\}\#\{\infty\}$		
$\{\infty,3\}^{(a)}$	$\{\infty,4\}_{\cdot,*3}$	$\{\infty,3\}^{(b)}$	$\{3,\frac{5}{2}\}$	$\{\frac{10}{3},\frac{5}{2}\}$		$\{\infty,4\}_4\#\{\}$		

Long story short

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$\mathcal{P}$	$a\Lambda_{(1,0,0)}$			$a\Lambda_{(1,1,0)}$			$a\Lambda_{(1,1,1)}$		
	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0, \alpha_1]$
$\{3,3\}$	2	2	–	2	2	–	1	2	$(1,2]$
$\{3,4\}$	1	2	$(1,2)$	1	1	–	$\frac{1}{2}$	1	$(\frac{1}{2},1)$
$\{4,3\}$	2	2	–	1	2	$(1,2)$	1	2	$(1,2)$

Table 6: Parameters for finite polyhedra

Long story short

$\mathcal{P}$	$a\Lambda_{(1,0,0)}$			$a\Lambda_{(1,1,0)}$			$a\Lambda_{(1,1,1)}$		
	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0$
$\{3,3\}$	2	2	–	2	2	–	1	2	(1
$\{3,4\}$	1	2	$(1,2)$	1	1	–	$\frac{1}{2}$	1	($\frac{1}{}$
$\{4,3\}$	2	2	–	1	2	$(1,2)$	1	2	(1

Table 6: Parameters for finite polyhedra

Polyhedron	Possible values of a	
$\{4,4\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,0,0)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1,2\})$	if $\Lambda = \Lambda_{(1,1,0)}{}^D$
	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,1,1)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1,2\})$	if $\Lambda = \langle \Lambda_{(1,1)}, t_3 \rangle^D$
$\{3,6\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \langle \Lambda_{(1,0)}^{\{3,6\}}, t_3 \rangle^D$
	$a \in \frac{1}{3}(\mathbb{N} \setminus \{1\})$	if $\Lambda = \langle \Lambda_{(1,1)}^{\{3,6\}}, t_3 \rangle^D$
$\{6,3\}$	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1,2\})$	if $\Lambda = \langle \Lambda_{(1,0)}^{\{3,6\}}, t_3 \rangle^D$
	$a \in \frac{1}{2}\mathbb{N}$	if $\Lambda = \langle \Lambda_{(1,1)}^{\{3,6\}}, t_3 \rangle^D$

Table 8: Parameters for planar apeirohedra

Long story short

$\mathcal{P}$	$a\Lambda_{(1,0,0)}$			$a\Lambda_{(1,1,0)}$			$a\Lambda_{(1,1,1)}$		
	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0, \alpha_1]$	α_0	α_1	$(\alpha_0$
$\{3,3\}$	2	2	–	2	2	–	1	2	(1
$\{3,4\}$	1	2	$(1,2)$	1	1	–	$\frac{1}{2}$	1	(

Polyhedron	Possible values of a	
$\{4,4\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,0,0)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1,2\})$	if $\Lambda = \Lambda_{(1,1,0)}{}^D$
	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,1,1)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1,2\})$	if $\Lambda = \langle \Lambda_{(1,1)}, t_3 \rangle^D$
$\{3,6\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,0)}, t_3 \rangle^D$
	1})	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,1)}, t_3 \rangle^D$
	1, 2})	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,0)}, t_3 \rangle^D$
		if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,1)}, t_3 \rangle^D$

nar apeirohedra

$\mathcal{P}$	$T(\mathcal{P})$	Possible values of a		
		$\Lambda = a\Lambda_{(1,0,0)}$	$\Lambda = a\Lambda_{(1,1,0)}$	$\Lambda = a\Lambda_{(1,1,1)}$
$\{4,6\|4\}$	$\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N}$
$\{6,4\|4\}$	$\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in \mathbb{N}$, $a = \frac{p}{6}$, $p \equiv \pm 4 \pmod{12}$
$\{6,6\|3\}$	$2\Lambda_{(1,1,0)}$	$a \in 2(\mathbb{N} \setminus \{1\})$	$a \in 2(\mathbb{N} \setminus \{1\})$	$a \in \mathbb{N} \setminus \{1\}$
$\{\infty,6\}_{4,4}$	$\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N}$
$\{\infty,4\}_{6,4}$	$\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in \mathbb{N}$, $a = \frac{p}{6}$, $p \equiv \pm 4 \pmod{12}$
$\{\infty,6\}_{6,3}$	$2\Lambda_{(1,1,0)}$	$a \in 2\mathbb{N}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N}$
$\{6,6\}_4$	$2\Lambda_{(1,0,0)}$	$a \in 2(\mathbb{N} \setminus \{1\})$	$a \in 2\mathbb{N}$	$a \in \mathbb{N}$
$\{4,6\}_6$	$2\Lambda_{(1,0,0)}$	$a \in 2\mathbb{N}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N}$
$\{\infty,3\}^{(b)}$	$2\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in \mathbb{N}$, $a = \frac{p}{6}$, $p \equiv \pm 4 \pmod{12}$
$\{\infty,3\}^{(a)}$	$2\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in 2\mathbb{N}$, $a = \frac{p}{3}$, $p \equiv \pm 4 \pmod{12}$	$a \in \mathbb{N}$, $a = \frac{p}{6}$, $p \equiv \pm 4 \pmod{12}$
$\{6,4\}_6$	$2\Lambda_{(1,0,0)}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N} \setminus \{1\}$	$a \in \mathbb{N} \setminus \{1\}$
$\{\infty,4\}_{\cdot,*3}$	$\Lambda_{(1,1,1)}$	$a \in 2\mathbb{N}$	$a \in \mathbb{N} \setminus \{1\}$	$a \in \mathbb{N}$

Table 7: Parameters for pure apeirohedra

Long story short

$\mathcal{P}$	$a\Lambda_{(1,0,0)}$ α_0	α_1	$(\alpha_0, \alpha_1]$	$a\Lambda_{(1,1,0)}$ α_0	α_1	$(\alpha_0, \alpha_1]$	$a\Lambda_{(1,1,1)}$ α_0	α_1	$(\alpha_0$
$\{3, 3\}$	2	2	–	2	2	–	1	2	(
$\{3, 4\}$	1	2	$(1, 2)$	1	1	–	$\frac{1}{2}$	1	(

Polyhedron	Possible values of a	
$\{4, 4\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,0,0)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1, 2\})$	if $\Lambda = \Lambda_{(1,1,0)}{}^D$
	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \Lambda_{(1,1,1)}{}^D$
	$a \in \frac{1}{2}(\mathbb{N} \setminus \{1, 2\})$	if $\Lambda = \langle \Lambda_{(1,1)}, t_3 \rangle^D$
$\{3, 6\}$	$a \in \mathbb{N} \setminus \{1\}$	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,0)}, t_3 \rangle^D$
	1\})	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,1)}, t_3 \rangle^D$
	1, 2\})	if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,0)}, t_3 \rangle^D$
		if $\Lambda = \langle \Lambda^{\{3,6\}}_{(1,1)}, t_3 \rangle^D$

$\mathcal{P}$	$T(\mathcal{P})$	Possible values of a: $\Lambda = a\Lambda_{(1,0,0)}$	$\Lambda = a\Lambda_{(1,1,0)}$	$\Lambda = a\Lambda_{(1,1,1)}$
$\{4, 6 \mid 4\}$	$\Lambda_{(1,1,1)}$			
$\{6, 4 \mid 4\}$	$\Lambda_{(1,1,1)}$			
$\{6, 6 \mid 3\}$	$2\Lambda_{(1,1,0)}$			
$\{\infty, 6\}_{4,4}$	$\Lambda_{(1,1,1)}$			
$\{\infty, 4\}_{6,4}$	$\Lambda_{(1,1,1)}$			
$\{\infty, 6\}_{6,3}$	$2\Lambda_{(1,1,0)}$			
$\{6, 6\}_4$	$2\Lambda_{(1,0,0)}$			
$\{4, 6\}_6$	$2\Lambda_{(1,0,0)}$			
$\{\infty, 3\}^{(b)}$	$2\Lambda_{(1,1,1)}$			
$\{\infty, 3\}^{(a)}$	$2\Lambda_{(1,1,1)}$			
$\{6, 4\}_6$	$2\Lambda_{(1,0,0)}$			
$\{\infty, 4\}_{\cdot,*3}$	$\Lambda_{(1,1,1)}$			

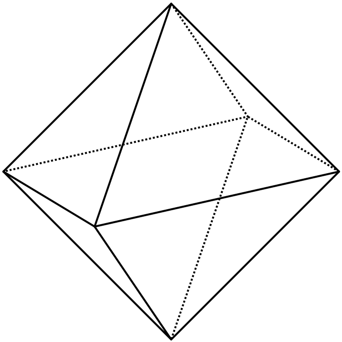
Table 7: Parameters fo

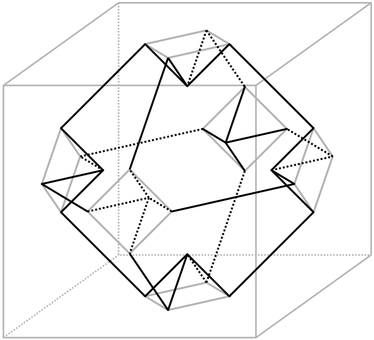
$\mathcal{P}$	$\Lambda_{(1,0,0)}$	$\Lambda_{(1,1,0)}$	$\Lambda_{(1,1,1)}$	$\langle \Lambda_{(1,1)}, t_3 \rangle$
$\{4, 4\}\#\{\,\}$	$2 \nmid p$, $2 \nmid r$ and $r \leqslant g$. $2 \nmid p$, $2 \mid r$ and $r < g$. $2 \mid p$, $2 \nmid r$ and $2r < g$. $2 \mid p$, $2 \mid r$ and $r < g$.	$2 \nmid p$, $2 \nmid r$, $2 \mid \frac{q}{g}$ and $2r \leqslant g$. $2 \nmid p$, $2 \nmid r$, $2 \nmid \frac{q}{g}$, $2 \nmid \frac{s}{g}$ and $r \leqslant g$. $2 \nmid p$, $2 \nmid r$, $2 \nmid \frac{q}{g}$, $2 \mid \frac{s}{g}$, $2 \nmid g$ and $r \leqslant g$. $2 \nmid p$, $2 \nmid r$, $2 \nmid \frac{q}{g}$, $2 \mid \frac{s}{g}$, $2 \mid g$ and $2r \leqslant g$. $2 \nmid p$, $2 \mid r$ and $2r < g$. $2 \mid p$, $2 \nmid r$ and $2r < g$. $2 \mid p$, $2 \mid r$ and $r < g$.	$2 \nmid p$, $2 \nmid r$ and $2r \leqslant g$. $2 \nmid p$, $2 \mid r$, $2 \mid q$ and $2r < g$. $2 \nmid p$, $2 \mid r$, $2 \nmid q$ and $r < g$. $2 \mid p$, $2 \nmid r$ and $2r < g$. $2 \mid p$, $2 \mid r$ and $2r < g$.	$2 \nmid p$, $2 \nmid r$, $2 \nmid \frac{q}{g}$ and $2r \leqslant g$. $2 \nmid p$, $2 \nmid r$, $2 \mid \frac{q}{g}$ and $r \leqslant g$. $2 \nmid p$, $2 \mid r$ and $r < g$. $2 \mid p$, $2 \nmid r$ and $2r < g$. $2 \mid p$, $2 \mid r$ and $r < g$.
$\{4, 4\}\#\{\infty\}$	$g = 1$, $2 \nmid p$ and $2 \nmid r$. $g = 3$ and do not occur that $r \equiv \pm 4 \pmod{12}$ and $2 \mid p$. $g \notin \{1, 3\}$.	$g = 1$, $2 \nmid p$, $2 \nmid r$, $2 \nmid q$, $2 \nmid s$. $g = 3$ and none of the following occur: $r \equiv \pm 4 \pmod{12}$ and $2 \mid p$. $r \equiv \pm 2 \pmod{12}$ and $2 \nmid pq$. $g \notin \{1, 3\}$.	$g = 3$ and do not occur that $r \equiv \pm 4 \pmod{12}$ and $2 \mid p$. $g \notin \{1, 3\}$.	$g = 1$, $2 \nmid p$, $2 \nmid r$ and $2 \mid q$. $g = 3$ and do not occur that $r \equiv \pm 4 \pmod{12}$ and $2 \mid p$. $g \notin \{1, 3\}$.

Table 9: Parameters for $\{4, 4\}\#\{\,\}$, $\{4, 4\}\#\{\infty\}$ and their Petrials

I wanted to see them...

I wanted to see them...





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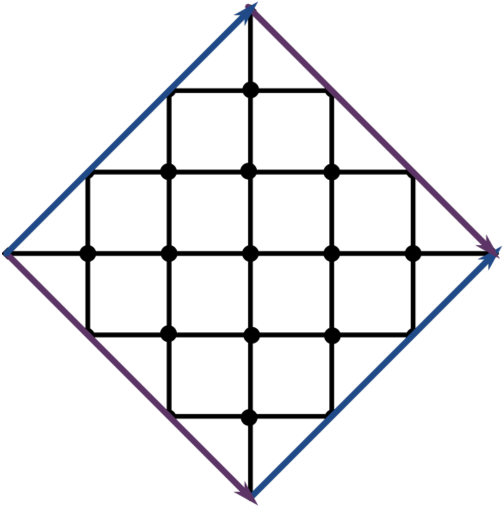

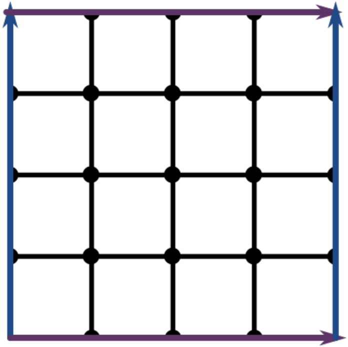

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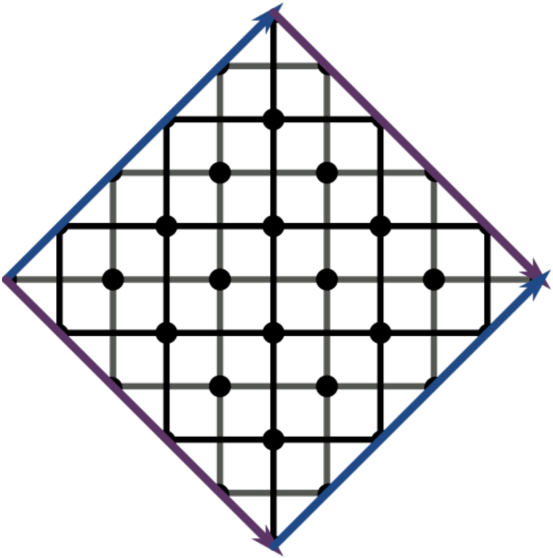

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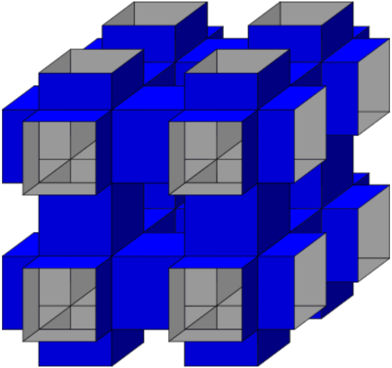

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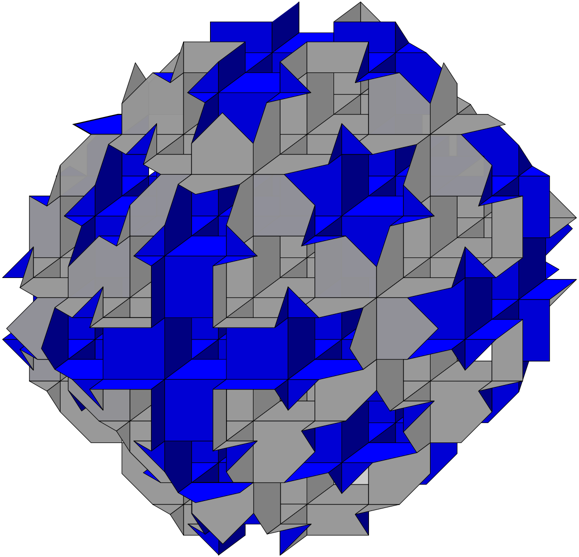

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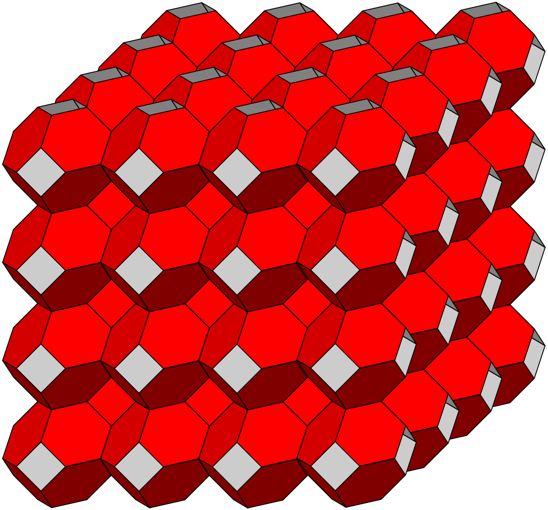

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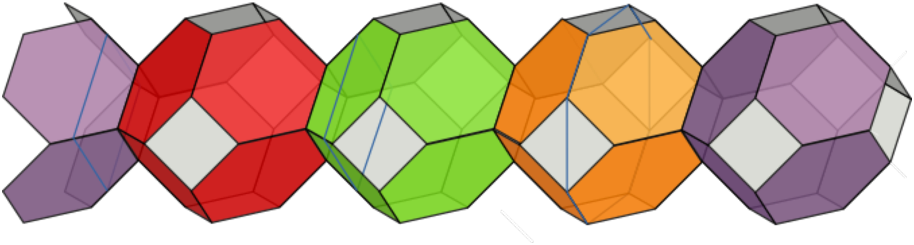

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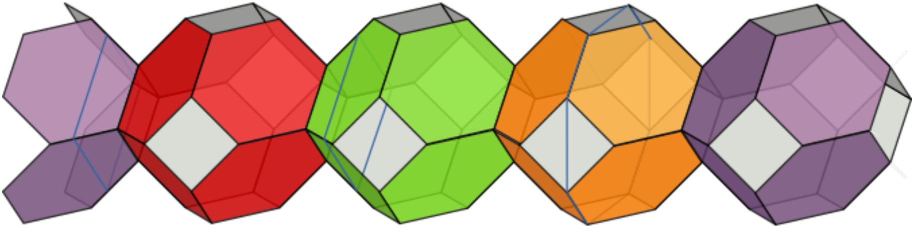

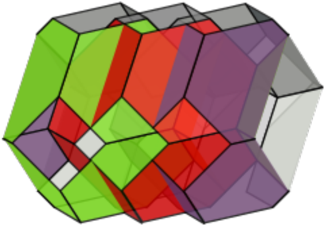

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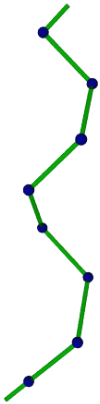

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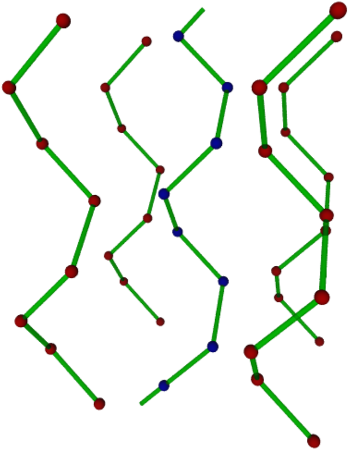

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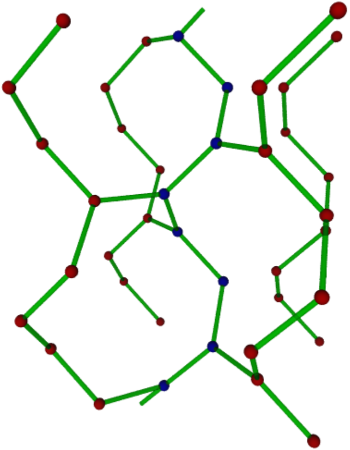

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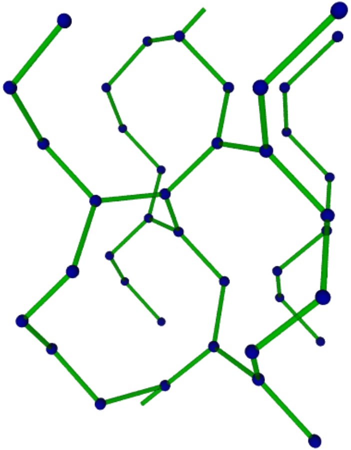

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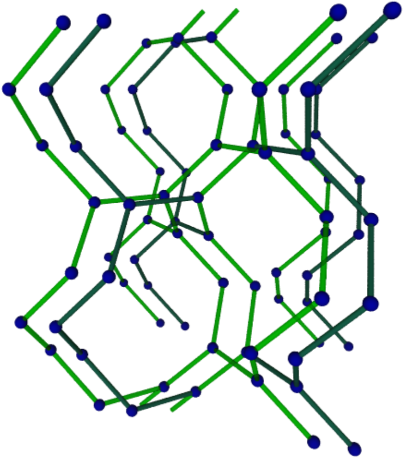

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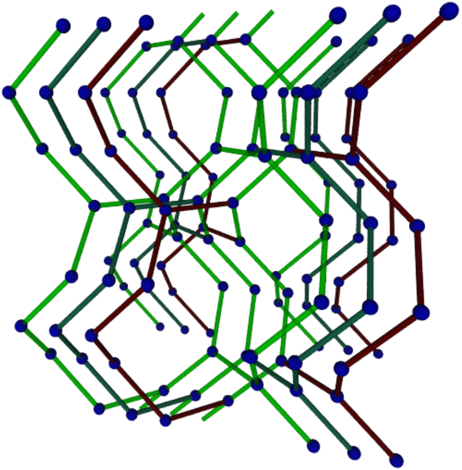

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  - $\{3,3\}$, $\{3,4\}$, $\{4,3\}$ and their petrials: regular with boring drawings.

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- Planar polyhedra: Regular with boring and no-that-boring drawings.

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  - $\{3,5\}$, $\{5,3\}$, their petrials and stellations: non-regular.
  - $\{3,3\}$, $\{3,4\}$, $\{4,3\}$ and their petrials: regular with boring drawings.
- Planar polyhedra: Regular with boring and no-that-boring drawings.
- Pure apeirohedra: Regular, almost all with only boring drawings.

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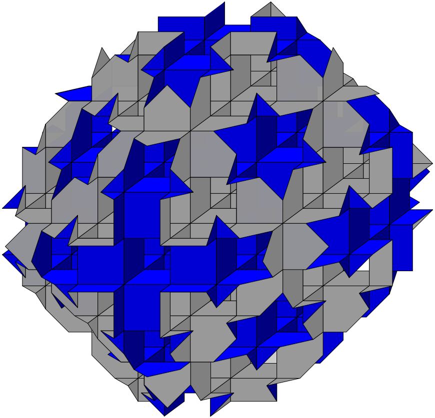
- Finite polyhedra
  - $\{3, 5\}$, $\{5, 3\}$, their petrials and stellations: non-regular.
  - $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$ and their petrials: regular with boring drawings.
- Planar polyhedra: Regular with boring and no-that-boring drawings.
- Pure apeirohedra: Regular, almost all with only boring drawings.
- Blended apeirohedra : Regular with crazy drawings.



Thank you!