Regular Polyhedra in the 3-Torus.

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Workshop on Symmetries In Graphs, Maps and Polytopes

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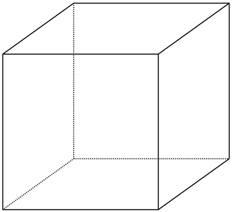

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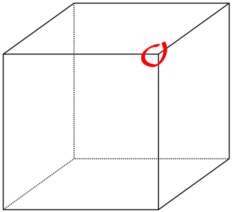

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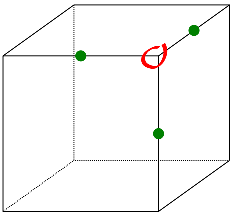

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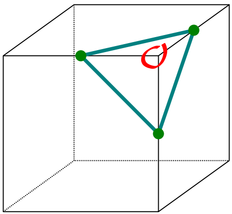

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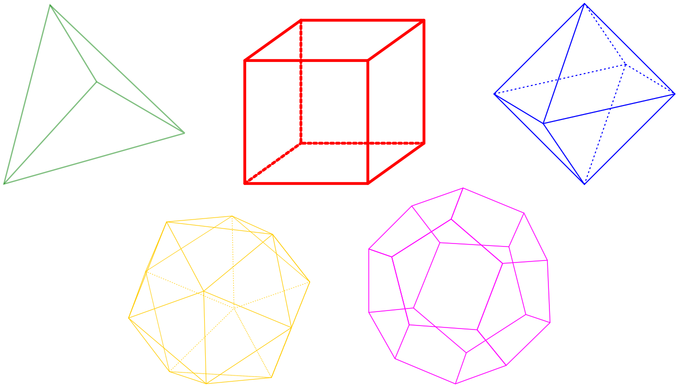
- $\mathcal{P}$ is regular if its group of symmetries acts transitively on flags.

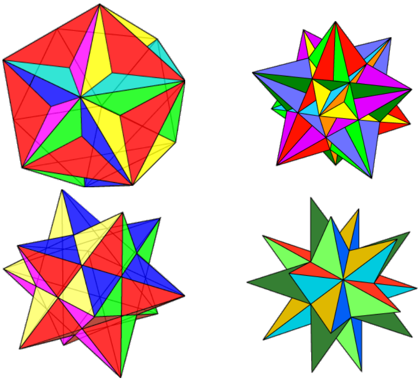
Regular Polyhedra

Platonic Solids



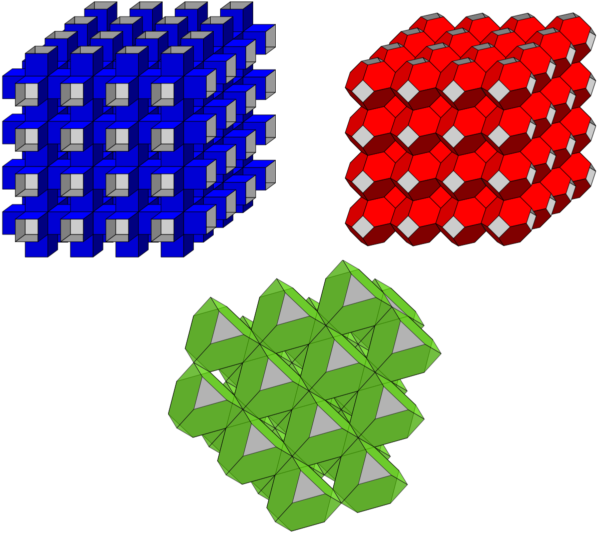
Regular Polyhedra

Kepler-Poinsot Polyhedra



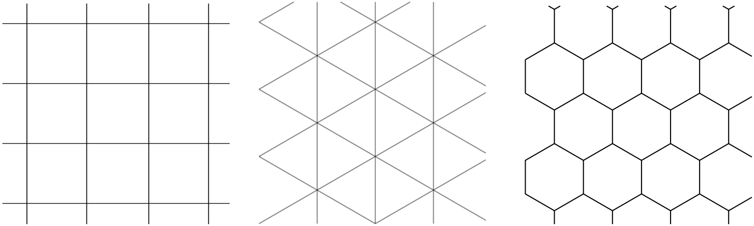
Regular Polyhedra

Petrie-Coxeter Polyhedra



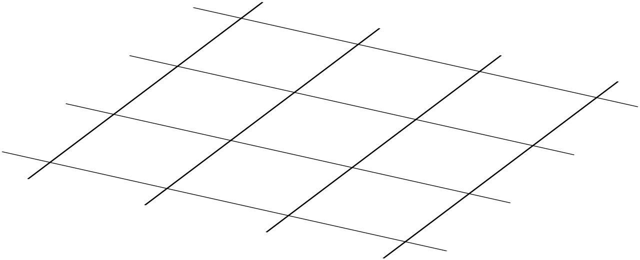
Regular Polyhedra

Plane Tessellations



Regular Polyhedra

Blended Polyhedra



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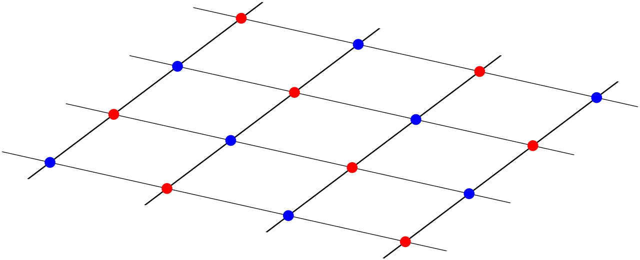

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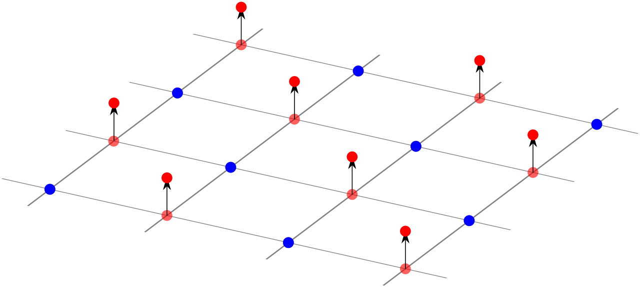

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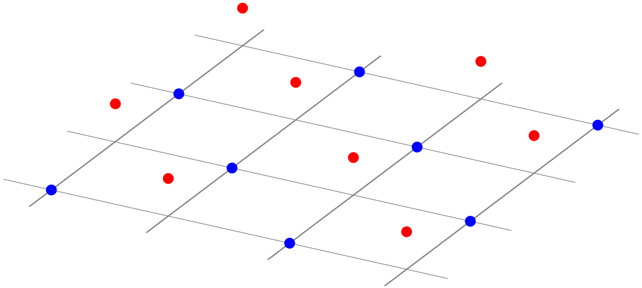

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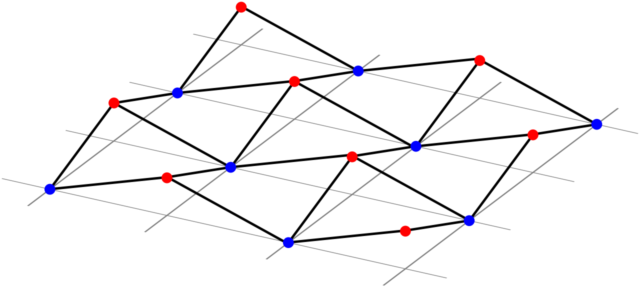

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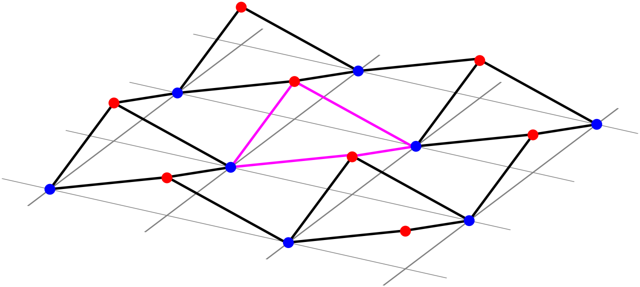

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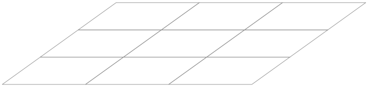

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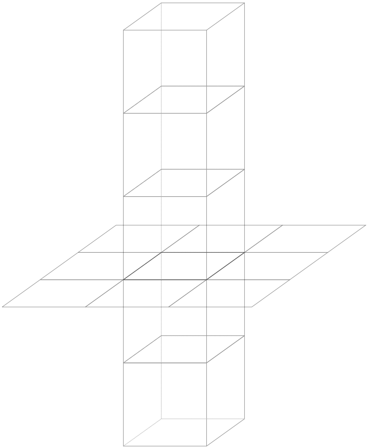

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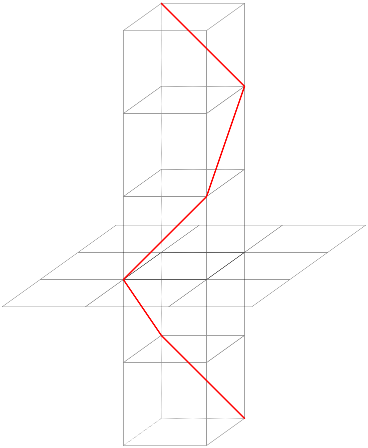

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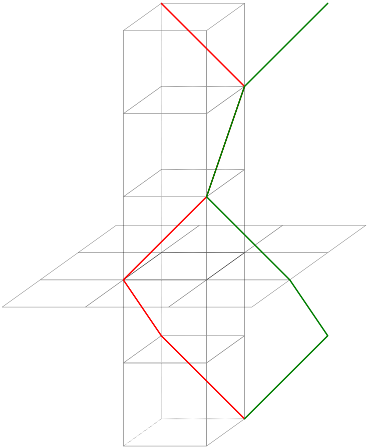

Regular Polyhedra

Classification Theorem

Theorem (Grünbaum-Dress (70's - 80's); McMullen-Schulte (1997))

There exists 48 *regular polyhedra in euclidean space* $\mathbb{E}^3$*.*

- 18 *finite polyhedra*

  2 *with tetrahedral symmetry.*
  4 *with octahedral symmetry.*
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What is next?

- Higher dimension (rank).

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- Change the ambient space.

The 3-torus

Consider a group $\mathbf{\Lambda}$ generated by 3 linearly independent translations of $\mathbb{E}^3$.

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Consider a group $\mathbf{\Lambda}$ generated by 3 linearly independent translations of $\mathbb{E}^3$. The 3-torus associated to $\mathbf{\Lambda}$ (denoted by $\mathbb{T}^3(\mathbf{\Lambda})$) is the quotient space $\mathbb{E}^3/\mathbf{\Lambda}$.

The problem

Determine the groups $\mathbf{\Lambda}$ such that a regular polyhedron in $\mathbb{E}^3$ induces a regular polyhedron in $\mathbb{T}^3(\mathbf{\Lambda})$.

Let $\mathbf{\Lambda}$ be a group generated by 3 linearly independent translations by the vectors v_1, v_2 and v_3. Let o the origin of $\mathbb{E}^3$, we define de lattice Λ associated to $\mathbf{\Lambda}$ as the set

$$\Lambda = o\mathbf{\Lambda} = \{n_1 v_1 + n_2 v_2 + n_3 v_3 : n_i \in \mathbb{Z}\}$$

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- Face-centred Lattice: $\mathbf{\Lambda_{(1,1,0)}}$.

Lemma

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- g *induces an isometry* $\bar{g} : \mathbb{T}^3(\mathbf{\Lambda}) \to \mathbb{T}^3(\mathbf{\Lambda})$.

$$\begin{array}{ccc} \mathbb{E}^3 & \xrightarrow{\;g\;} & \mathbb{E}^3 \\ \downarrow & & \downarrow \\ \mathbb{T}^3(\mathbf{\Lambda}) & \overset{\bar{g}}{\dashrightarrow} & \mathbb{T}^3(\mathbf{\Lambda}) \end{array}$$

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$$

- s *preserves* Λ.

The results

Tetrahedral Symmetry

Theorem

Let $\mathbf{\Lambda}$ be a group generated by 3 linearly independent translations. If $\mathcal{P}$ is a regular polyhedron in $\mathbb{E}^3$ with tetrahedral or octahedral symmetry, then $\mathcal{P}$ induces a regular polyhedron in $\mathbb{T}^3(\mathbf{\Lambda})$ if and only if

$$\mathbf{\Lambda} \in \{a\mathbf{\Lambda}_{(\mathbf{1},\mathbf{0},\mathbf{0})}, b\mathbf{\Lambda}_{(\mathbf{1},\mathbf{1},\mathbf{0})}, c\mathbf{\Lambda}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\}.$$

for some parameters a, b, c.

Icosahedral Symmetry

Theorem (Crystallographic Restriction)

If G is a group of isometries in $\mathbb{E}^3$ that preserves a lattice, then G does not contain rotations of periods other than 2, 3, 4 and 6.

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Theorem

Let $\mathcal{P}$ be a regular polyhedron in $\mathbb{E}^3$ with icosahedral symmetry. There is not group $\mathbf{\Lambda}$ generated by 3 linearly independent translations such that $\mathcal{P}$ is a regular polyhedron in $\mathbb{T}^3(\mathbf{\Lambda})$.

Infinite Polyhedra

. . . too many vertices . . .

Infinite Polyhedra

... too many vertices ...

$$\mathbf{\Lambda} \leqslant \mathbf{T}(\mathcal{P})$$

Pure Polyhedra

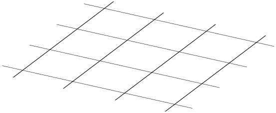
Theorem

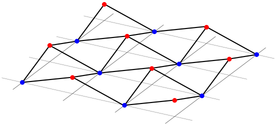
Let $\mathbf{\Lambda}$ be a group generated by 3 linearly independent translations. If $\mathcal{P}$ is an infinite pure regular polyhedron in $\mathbb{E}^3$, then $\mathcal{P}$ induces a regular polyhedron in $\mathbf{\Lambda}$ if and only if

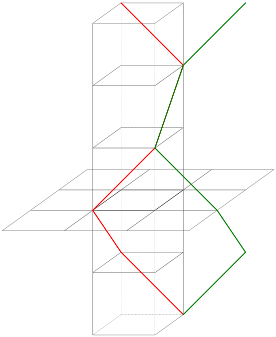
$$\mathbf{\Lambda} \in \{a\mathbf{\Lambda}_{(\mathbf{1},\mathbf{0},\mathbf{0})}, b\mathbf{\Lambda}_{(\mathbf{1},\mathbf{1},\mathbf{0})}, c\mathbf{\Lambda}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\}.$$

for some discrete parameters a, b, c.

Planar and Blended Polyhedra







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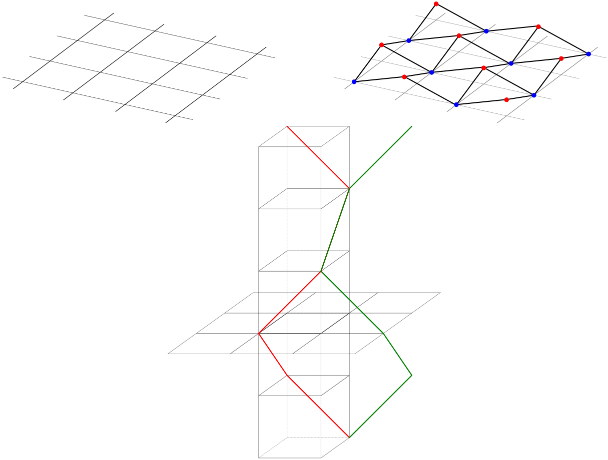
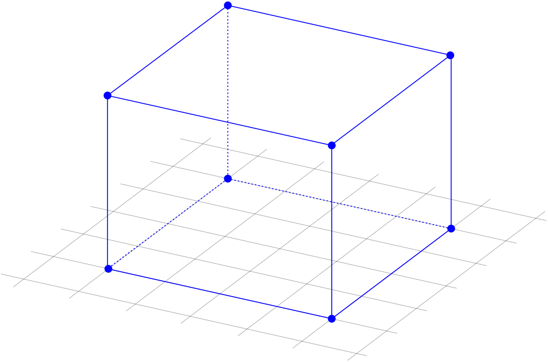


There is a distinguished plane with a planar tessellation associated.

Planar and Blended Polyhedra

$\{4,4\}$, $\{4,4\}\#\{\}$ and $\{4,4\}\#\{\infty\}$



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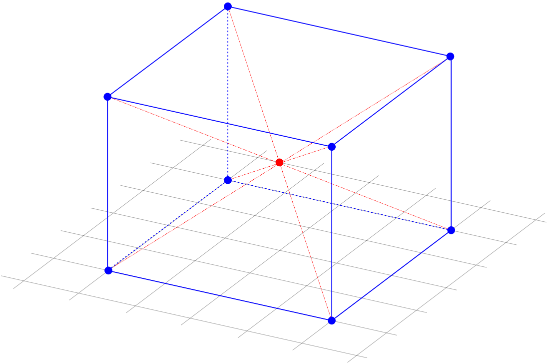

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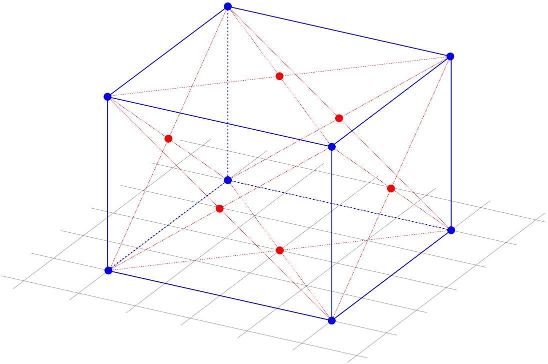

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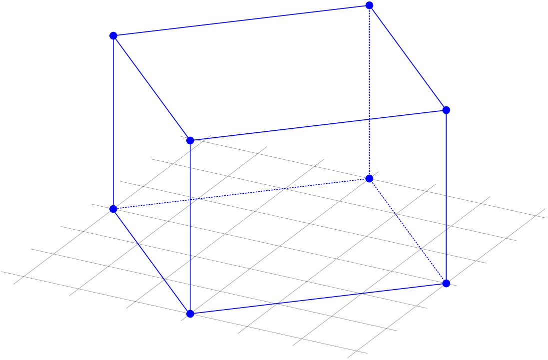

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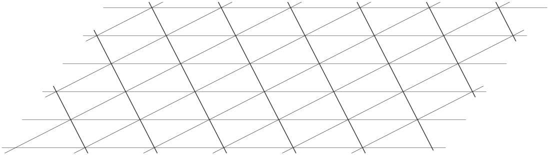

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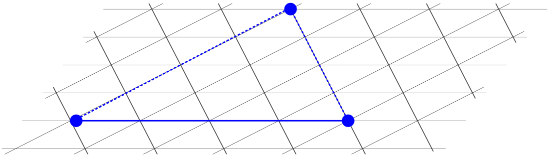

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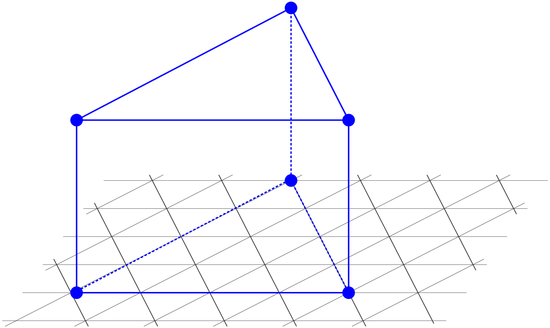

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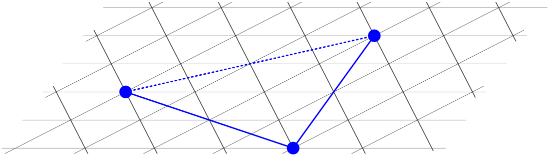

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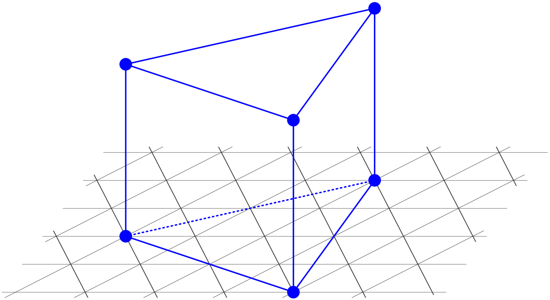

Planar Polyhedra

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Planar Polyhedra

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- Our results generalize Coxeter's, in the sense that every planar polyhedron in $\mathbb{T}^3$ is an embedding of a regular toroidal map.

What have we done and what's next?

- We classify the regular polyhedra in $\mathbb{T}^3$ that come from regular polyhedra in $\mathbb{E}^3$.

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- We classify the regular polyhedra in $\mathbb{T}^3$ that come from regular polyhedra in $\mathbb{E}^3$.
- Is the list complete?

Thank you!

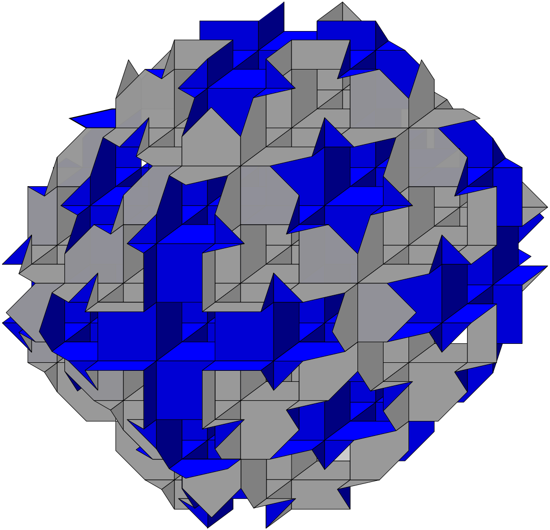

Figure: $\{4,6|4\}/\mathbf{\Lambda_{(1,1,1)}}$