Research Statement

Antonio Montero [*]
Faculty of Mathematics and Physics
University of Ljubljana,
Jadranska 19 1000 Ljubljana, Slovenia

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Introduction

Many of my research interests are about symmetry properties of discrete objects. In particular, most of my work involve problems in highly symmetric abstract polytopes. In the following paragraphs I introduce the research topics that have kept my attention in the last few year. I decide to sacrifice detail and formality in order to keep the language as accessible as possible, but a large list of references is also provided.

An *abstract polytope of rank n* (*n-polytope*, for short) is a ranked poset satisfying some properties that resembles to the face-lattice of an n-dimensional convex polytope. Examples of abstract polytopes include the lattices of convex polytopes, tilings of Euclidean and Hyperbolic spaces, maps on surfaces and some other similar structures. The study of abstract polytopes has been historically related to the study of their symmetries, this offers a meeting point for techniques from (at least) geometry, combinatorics, group theory and topology. A natural consequence of this multiarea interplay is that the study of abstract polytopes branches into connections with Coxeter groups [11, 51, 108], Riemann surfaces [6] and even Grothendieck's *Dessins d'enfant* [54], just to mention some.

Grünbaum was one of the firsts that formally treated geometrical polyhedra-like objects as purely combinatorial objects by introducing the notion of the notion *polystroma* [38]. This notion eventually evolved to what we know today as *abstract polytopes*, introduced in the early 80's by Schulte [97]

A polytope is *regular* if it admits maximal (reflectional) symmetry. Regular polytopes are the most studied class of highly symmetric abstract polytopes. They generalise the platonic solids, the regular convex polytopes in higher dimensions, the regular tilings of Euclidean and Hyperbolic spaces and the reflexive maps on surfaces [66, Section 1A]. Most of the wealth theory developed on abstract regular polytopes is condensed in [66], but the extension of the results comes from constructions with prescribed combinatorics [20, 31, 50, 63, 82, 83, 85, 98–100], or with interesting families of groups as automorphism groups [7, 8, 34, 43, 57, 69–71, 104], to analysing their possible geometrical realisations [1, 2, 4, 36, 59–61, 65, 68, 76].

The problems on highly symmetric polytopes can be attacked from different approaches and involve techniques of many mathematical disciplines (see [86, 106]). Of my particular interest are those that involve constructions of abstract regular polytopes and related objects with prescribed symmetry conditions (Section 1) and/or prescribed local combinatorics (Section 2). As a natural research direction of those theoretical problems arise computational problem of creating, managing and publishing datasets of the constructed objects (Section 3). However, given the intrinsic geometric nature of the theory, I am also interested in a series of geometrical problems related to these objects (Section 4).

[*]antonio.montero@fmf.uni-lj.si

1 Symmetry type of polytopes and maniplexes

The degree of symmetry of a polytope can be measured by the number of orbits of its automorphism group on certain substructures called *flags*. Regular polytopes are transitive on flags.

Besides regular polytopes, the second most studied *symmetry type* of polytopes is that of *chiral polytopes*. Chiral polytopes have two orbits on flags and informally speaking, they have full (combinatorial) rotational symmetry but do not admit reflections.

They were formally introduced by Schulte and Weiss in [103] as a combinatorial generalisation of chiral maps and Coxeter's twisted honeycombs [24]. Examples of chiral polytopes in rank (dimension) 3 and 4 exist in abundance (see [14, 21, 24, 25, 104]), the problem of finding examples of chiral polytopes in ranks higher than 5 has proved to be more difficult. The first example in rank 5 was build by Schulte and Weiss in [105] and it is infinite. The first finite example was constructed by Conder, Hubard and Pisanski in [22]. In 2010 Pellicer [84] showed that chiral polytopes exist in every rank, but this construction is impractical for finding explicit examples in ranks higher than 5. In Section 2 we will explore a technique to construct chiral polytopes with prescribed local combinatorics. These problems are of my deep interest and that is why we prefer to have a separate section to discuss them.

Chiral n-polytopes are just one of the 2^n-1 classes of possible 2-orbit polytopes. Very little is known about the existance of non-chiral 2-orbit n-polytopes. In [90] Pellicer, Potočnik and Toledo proved the existance of 2-orbit n-maniplexes of arbitrary symmetry type. Maniplexes are a graph-theoretical generalisation of abstract polytopes. It is not known whether or not the maniplexes constructed in [90] are polytopal, although it is strongly believed that they are.

If $k \geqslant 3$ the problem of determining the symmetry type of an n-polytope is controlled by the *symmetry type graph (STG)* The STG of a polytopes is a coloured graphs having one vertex per orbit of the automorphism group and whose connections are determined by the local configuration of flag orbits. In [27], Cunningham et. al. study the possibilities for these graphs for $k = 3$ and $k = 4$. In a recent paper [47] Hubard and Mochán build 3-polytopes with 3-flag orbits for all the possible symmetry type graphs. In [42] Helfand develops a technique to build k-orbit n-polytopes for every possible pair (k, n), however this construction yields polytopes with digonal sections, which are often considered degenerate. If we forbid those polytopes the problem remains mostly open and it is of my deep interest.

Problem 1 (Problem 10 in [30]). Are there k-orbit n-polytopes for every $k \geqslant 1$ and every $n \geqslant 3$ such that no section of rank 2 is a digon?

Informally speaking, a *premaniplex* is a n-edge-coloured graph that, when connected, is a candidate to be the STG of a polytope in the sense that every STG of a polytope is a connected premaniplex.

A refined version of Problem 1 is

Problem 2 (Problem 12 in [30]). Given a connected premaniplex $\mathcal{T}$ with k-vertices, is there an n-polytope (maniplex) $\mathcal{P}$ such that the STG of $\mathcal{P}$ is isomorphic to $\mathcal{T}$.

In [49] we introduce the notion of *voltage operation* as a potential technique to attack Problem 2. We only obtained partial result but we strongly believe that our techniques can be used to solve Problem 2 and we are preparing a second manuscript on the topic (see [48]).

2 Extensions of highly symmetric polytopes

An $(n+1)$-polytope $\mathcal{P}$ is an *extension* of an n-polytope $\mathcal{K}$ if all the maximal faces (*facets*) of $\mathcal{P}$ are isomorphic to $\mathcal{K}$. An extension of $\mathcal{K}$ is of type $\{\mathcal{K}, q\}$ for some $q \in \mathbb{N}$ if q facets meet around every $(n-2)$-face of the extension. Determining whether or not an abstract polytope has an extension was implicitly proposed by Grünbaum in [38] as an approach to classify objects which are known today as *locally toroidal polytopes*. Grünbaum also considered the possibility of imposing symmetry on the resulting extension.

In [98, 100] Schulte shows that every regular n-polytope $\mathcal{K}$ admits a regular extension of type $\{\mathcal{K}, 6\}$. Moreover, the extension is finite if the polytope is finite. Independently, in [31] Danzer introduces a construction of a regular extension of type $\{\mathcal{K}, 4\}$ for every abstract regular polytope $\mathcal{K}$. In [85] Pellicer generalises Schulte's construction for a particular class of regular polytopes, the so-called dually bipartite polytopes. In [83] Pellicer introduces the construction $2s^{\mathcal{K}-1}$ which generalizes Danzer's construction. In both cases, the extensions constructed by Pellicer are of type $\{\mathcal{K}, 2s\}$ for any preassigned even number $2s \geqslant 4$.

Many of my research interest have been around the idea of using extensions to build new examples of polytopes with prescribed symmetry.

As a consequence of [103, Proposition 9], if $\mathcal{P}$ is a chiral extension of an abstract polytope $\mathcal{K}$ then $\mathcal{K}$ is either an orientably regular polytope or a chiral polytope with regular facets. In the latter situation there have been some advances that include universal extensions [105] and finite extensions of finite polytopes [29]. However the construction in [29] has little control over the type of the extension (cf. [83]). This fact motivates some problems that have been of my interest during the last few years:

Problem 3 (Problem 25 of [86])**.** Does every chiral polytope $\mathcal{K}$ with regular facets admit a chiral extension with prescribed type?

Problem 4 (Problem 26 of [86])**.** Does every finite polytope $\mathcal{K}$ with regular facets admit a finite chiral extension with prescribed type?

I have some partial results on Problem 4. In [78], I show that every dually bipartite chiral polytope with regular facets admits a infinite number of chiral extensions with different type. It is also proved that if $\mathcal{P}$ is a chiral extension of a chiral polytope $\mathcal{K}$ with regular facets of type $\{\mathcal{K}, q\}$ and $\mathcal{K}$ has a regular quotient (not necessarily polytopal) with at least two facets, then $\mathcal{K}$ admits a chiral extension of type $\{\mathcal{K}, (2s)q\}$ for every $s \in \mathbb{N}$. See also [77].

The problem of constructing chiral extensions of orientably regular polytopes has still several questions open. In particular the following problems are still open and are greatly of my interest.

Problem 5 (Problem 27 of [86])**.** Does every orientably regular n-polytope admit a chiral extension?

Problem 6 (Problem 28 of [86])**.** Does every finite orientably regular n-polytope admit a finite chiral extension?

And of course, the analogues to Problems 3 and 4 for orientably regular polytopes:

Problem 7. Does every orientably regular polytope admit a chiral extension with prescribed type?

Problem 8. Does every orientably regular polytope admit a finite chiral extension with prescribed type?

Some partial answers for Problems 5 and 6 for specific families of polytopes have been found recently. In the conference SIGMAP 2018 Conder announced some result that imply that the simplex of every rank admits a chiral extension. In [26] Cunningham gives a series of restrictions on chiral polytopes that imply that the regular *(1,n-1)-flat polytopes* do not admit a chiral extension.

The author, Pellicer and Toledo have shown that almost every regular toroid admits a chiral extension (see [77, 80]). However, Problems 7 and 8 have proved to be very difficult and any partial result would be of my interest.

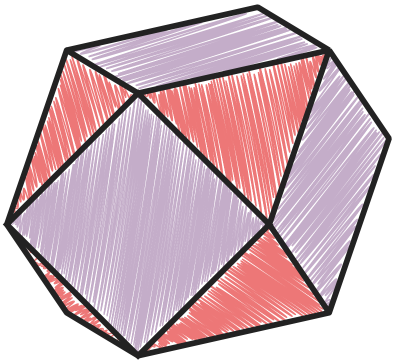

Figure 1: The cuboctahedron

A particular symmetry type that has gained some interest in the recent years is that of *alternating semiregular polytopes*. Those are 2-orbit polytopes with 2 (usually different) types of regular *facets* (maximal faces) arranged in an alternating way around each $(n-3)$-face. A convex example of such polytope (with $n = 3$) is the cuboctahedron (Figure 1). The facets are triangles and squares and two of each arrange in an alternate way around a vertex.

The automorphism group of an alternating semiregular polytope has been fully described [72] and some universal constructions have been exhibit [74].

Alternating semiregular polytopes have non-isomorphic facets, hence the extension problem makes no sense directly. In [73, 75] Monson and Schulte study the problem of constructing alternating semiregular polytopes and propose the following problem, which is the analogous to the extension problem and it is of my deep interest.

Problem 9 (The Assembly Problem in [73])**.** Let $\mathcal{P}$ and $\mathcal{Q}$ regular n-polytopes with isomorphic facets and fix $k \geqslant 2$. Does there exist an alternating semiregular $(n+1)$-polytope with facets $\mathcal{P}$ and $\mathcal{Q}$, $2k$ around each $(n-2)$-face in an alternating way.

The cuboctahedron is a positive answer to Problem 9 with $\mathcal{P}$ being a square, $\mathcal{Q}$ being a triangle and $k = 2$. In [75] the authors give negative answer to Problem 9 for very specific values of $(\mathcal{P}, \mathcal{Q}, k)$. Therefore a refined version of Problem 9:

Problem 10. Classify the triplets $(\mathcal{P}, \mathcal{Q}, k)$ or at least, describe interesting families of such triplets that satisfy that Problem 9 has a positive answer.

With the emerge of recent results on k-orbit polytopes [46] the interest on polytopes with more than 2 orbits has increase. In a joint work with Cunningham and Mochán we have explored a technique using voltage graphs to study extensions of polytopes. As a result we have submitted a manuscript [28] where we generalise classical notions on the theory of maps on surfaces to that of abstract polytopes. We use the techniques to build some universal extensions as well as some particular finite constructions. Finally we explore the possible symmetry-type-graphs arising from our constructions.

A fairly new generalisation of an abstract polytope is the concept of *hypertope*. A hypertope is a common generalisation of both abstract polytopes and hypermaps. Hypertopes were introduced in [35] where some of their basic properties are established. In particular, there are several results that are natural generalisations of well-known results for abstract polytopes. The results regarding regular extensions of regular polytopes in [31, 83, 100] can be stated in terms of the *Coxeter diagram* of the abstract polytope. The Coxeter diagram is a labelled graph that encodes some of the properties of a group generated by involutions. It is known that for regular polytopes, the Coxeter diagram is a string. For regular hypertopes the Coxeter diagram is in general a labelled complete graph. This motivates my interest in the following problem as a generalisation of some of the extension problems for abstract polytopes.

Problem 11. Given a labelling D of the complete graph on n nodes, is there a finite regular n-hypertope whose Coxeter diagram is precisely D?

As a consequence of our results in [79, 81], we give a positive answer for a very particular family of labellings. In [9] Catalano, Fernandes, Hubard and Leemans explore Problem 11 when the underlying complete graph is K_4.

3 Datasets of symmetrical polytopes and related objects

The problem of enumerating and classifying regular polyhedra in the Euclidean space is as old as formal mathematics themselves. The enumeration and classification of the five Platonic Solids is one the most antique classification problems. For many years it was considered a complete classification problem (and it was) but later on it was shown that by relaxing geometrical constraints we could generalise platonic solids to stellated polyhedra [10, 55, 94], *infinite skew polyhedra* [23] and finally to *Grünbaum-Dress polyhedra* [32, 33, 37], which is what today is accepted as the complete classification of regular polyhedra in the Euclidean space [65].

However, the combinatorial nature of abstract poltypes open the possibilities to, in principle, have numerous examples of abstract polytopes. These has lead to the construction of some datasets of highly symmetric polytopes (see [13, 15–19, 39, 40, 56, 95, 96]).

The existing datasets of polytopes suffer of the following restrictions

(i) They are mainly focused on regular or chiral polytopes.

(ii) The size of the examples is very restrictive.

(iii) They often exhibit numerous examples of rank 3 but the amount of examples of rank higher than 4 drops dramatically.

(iv) They are not very user-friendly, either because they exist only as raw data or because the are specific-programming language oriented.

The problematic mentioned above lead us to a problem that has been of my interest recently and that could offer several ramifications.

Problem 12. Extend the existing and build new datasets of abstract polytopes and related structures with particular focus on

(i) Building examples on ranks higher than 3

(ii) Exploring different symmetry types.

4 Highly symmetric geometric polytopes

The modern treatment of highly symmetric polytopal structures in goemetric space is through the concept of *geometric realisation* (see [67] for a nice historical review on the topic).

This notion was first used to classify regular polyhedra in the Euclidean space [65] and since then there have been several extensions. In [101, 102], Schulte relaxes the symmetry condition and classifies all chiral polyhedra in 3-space. In [3, 87–89] extended Schulte's results to chiral polytopes in 4 and 5 dimensions. Other symmetry conditions have been explored [58, 62, 107] Other skeletal structures in the ordinary space have been also classified (see [91–93]).

In [1, 2], Bracho et. al. considered the problem of changing the ambient space and classified the regular polyhedra in the projective space with planar faces.

As an analogue to the work of [1, 2] for the projective space, in [76] I have classified the regular polyhedra in the 3-dimensional torus. The obtained results are closely related to those proved in [36, 65].

It is clear that there are several possible paths to follow in order to classify skeletal objects in different ambient spaces. In particular, I am confident that most of the techniques developed in [76] can be applied to relate classifications of Euclidean objects with their corresponding toroidal analogues.

A slightly different approach is the problem of finding and classifying symmetric tillings of manifolds. Of course these area arises from the theory of maps on surfaces. The study of symmetric maps on surfaces is wide and it would be impossible to give a complete panorama of it in this document, but basic references are [52, 53]. In particular, in [52] Jones and Singerman introduce a technique that relates a map on a surface with a tessellation of the corresponding universal cover. This technique has been regarded as the combinatorial analogue of Riemann Uniformiation Theorem.

Arguibly, the most studied tilings of compact manifolds are those of the n-dimensional torus. Regular an chiral maps on the torus (3-toroids) were known by Coxeter in the 70's [25]. Regular tessellations of the n-dimensional torus $(n+1)$-toroids were classified in [64] and in [41] the authors proved that the only Euclidean manifold admiting chiral tessellations is the 2-torus.

In [5] Brehm and Kühnel classified the equivelar maps on the torus. The same classification was obtained by Hubard, Orbanić, Pellicer and Weiss in [50] where they also classify equivelar 4-toroids.

A consequence of the classification of equivelar tessellations of the 3-dimensional torus in [50] is that there are no equivelar 4-toroids with 2 flag-orbits. This observation, together with the results of [41] about chiral toroids on higher dimensions, it was conjectured that there were no equivelar tessellations of the n-dimendional torus with 2 flag-orbits.

In [12] the author and Collins disproved this conjecture by classifing *equivelar toroids with few flag orbits*, that is equivelar tessellations of the n-dimensional torus with at most n flag orbits. In particular we exhibit an infinite family of 2-orbit cubic tessellations of the n dimensional torus for even $n \geqslant 4$ and prove that such tesellations do not exists if n is odd.

The few-orbit phenomenon makes sense not only in the torus but in other similar structures. Equivelar maps on the Klein bottle are classified in [109] and there are no maps with 2 flag-orbits. In [44, 45], Hubard, Mixer, Pellicer and Weiss classify cubic tessellations of the orientable euclidean manifolds of dimension 3 and the only examples with 3 or less orbits are those found in the torus in [50]. Considering that among the euclidean manifolds, only the n-dimensional torus admit regular tessellations, the following problem is of my interest.

Problem 13. Are there examples of tessellations of euclidean n-manifolds with at most n flag orbits?

The author conjectures that the answer to Problem 13 is negative. In such situation this will increase the strength of the results of [41].

As mentioned before, few-orbit tessellations make sense in some other objects. The understanding of highly symmetric tessellation of hyperbolic manifolds might be of the interest not only for people working on highly symmetric polytopes, but for those working on hyperbolic manifolds from some other approaches. That is the reason why I list the following problem as a problem of my interest.

Problem 14. Explore the few-orbit phenomenon on hyperbolic manifolds.

I believe that Problem 14 is extremely difficult and some results in concrete examples will be of deep interest.

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