

# Highly symmetric tessellations of the $n$ -torus

Antonio Montero

University of Ljubljana

Matematični pogovori/ Mathematical conversations  
June 2026, Ljubljana, Slovenia.

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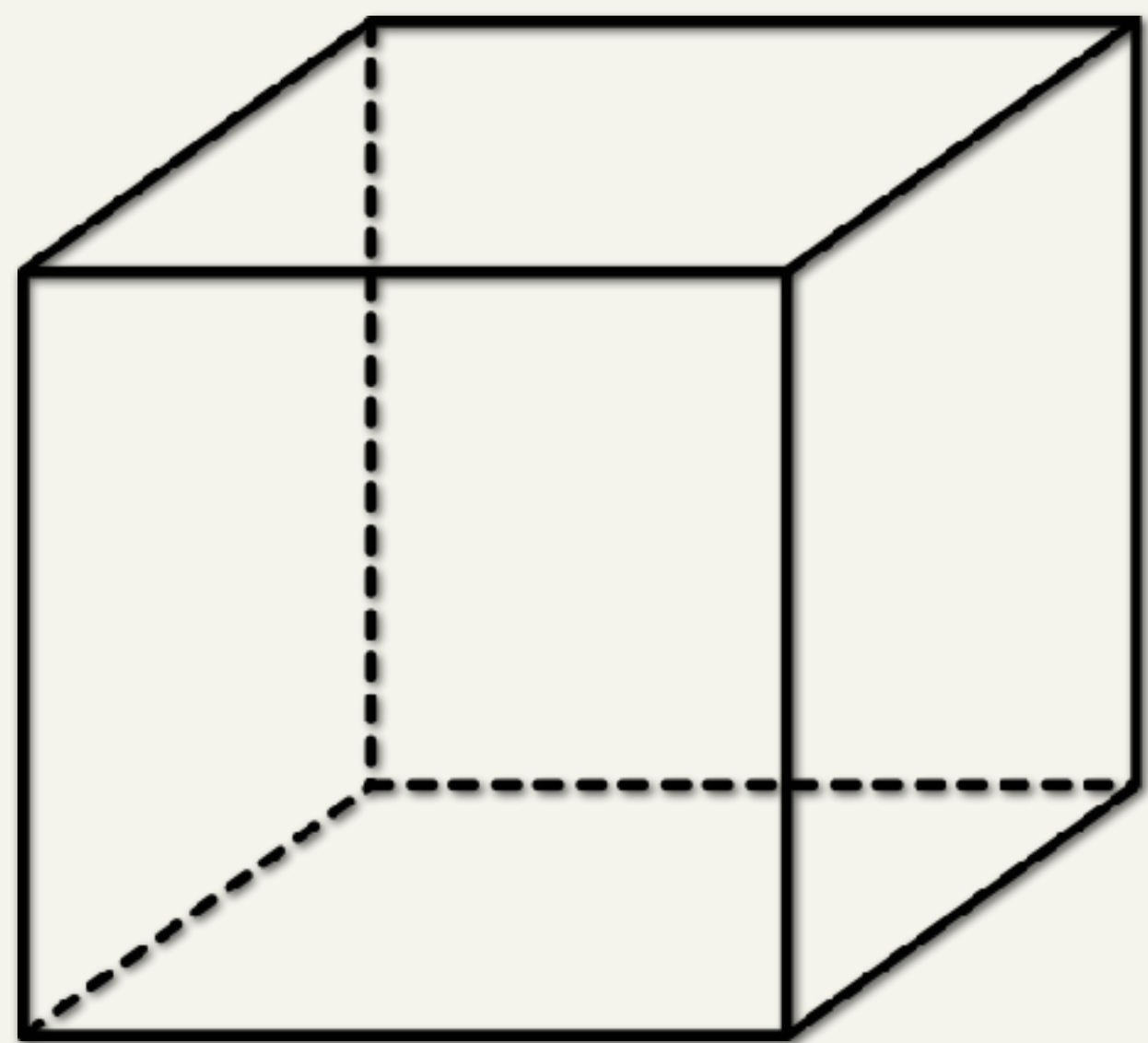
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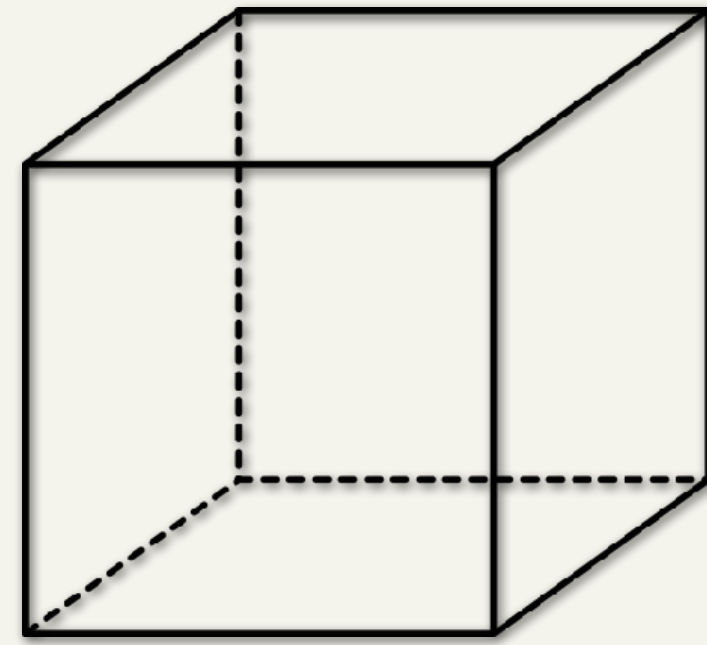
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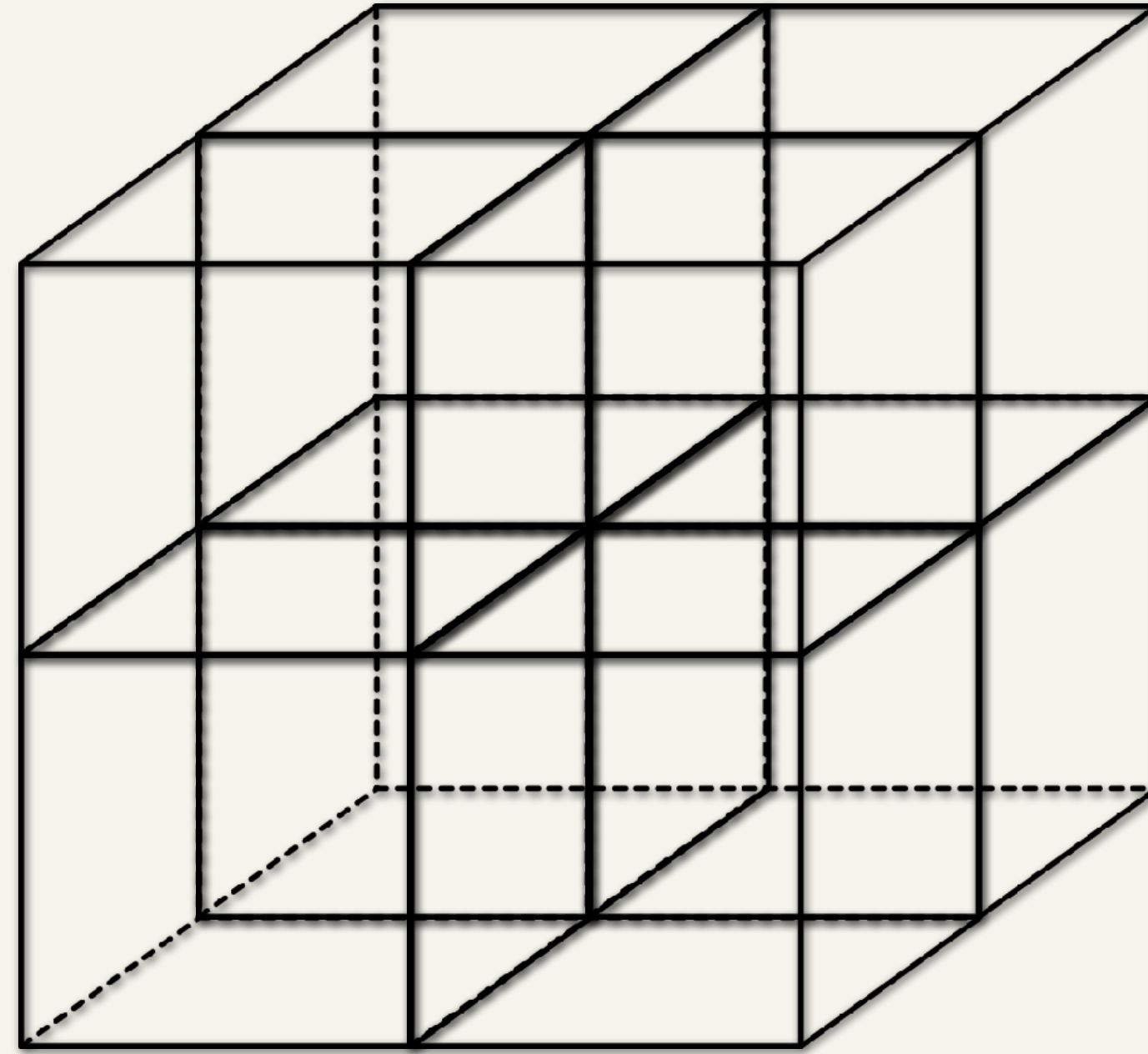
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- **Research: Symmetries of abstract polytopes.**

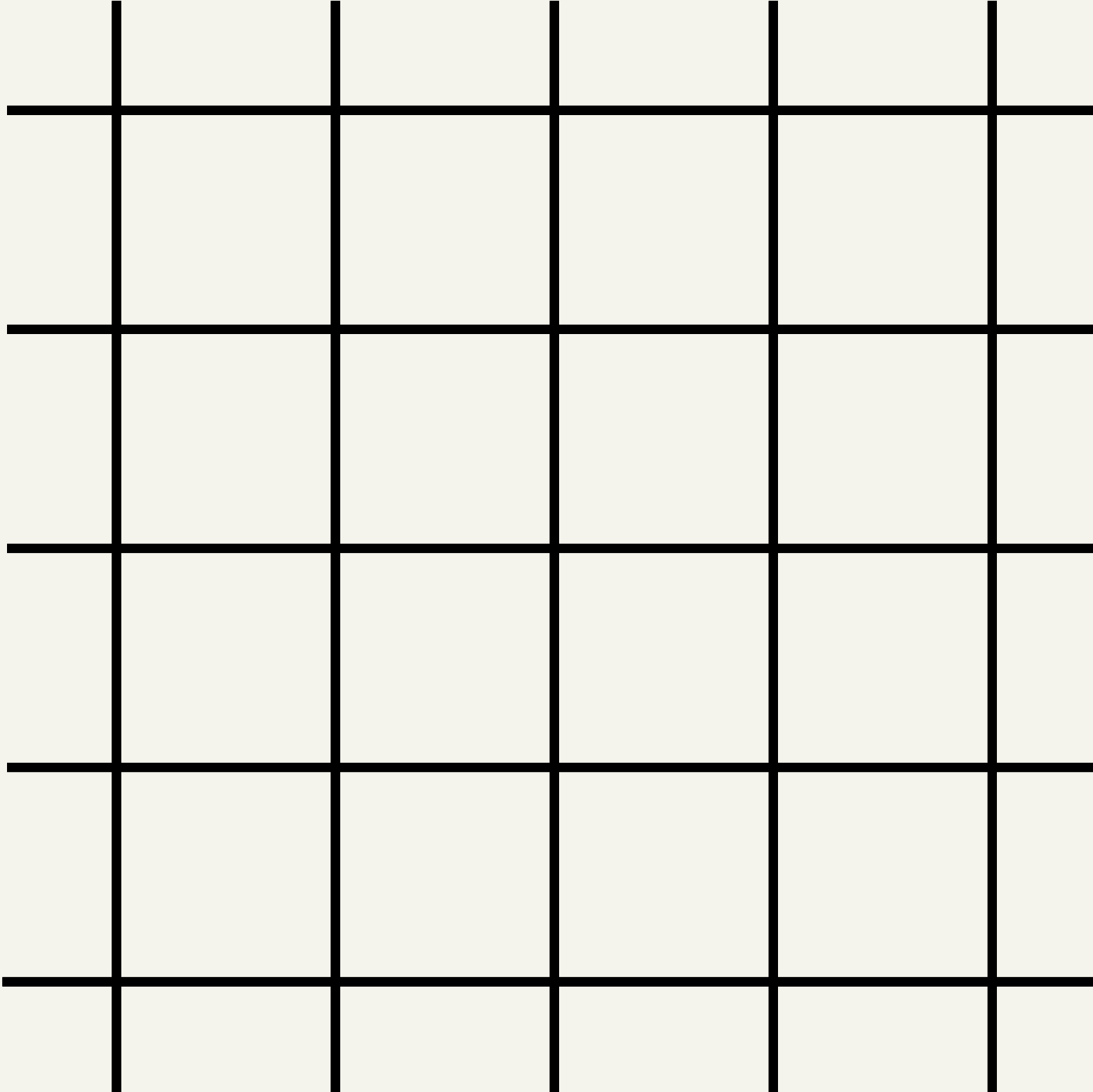


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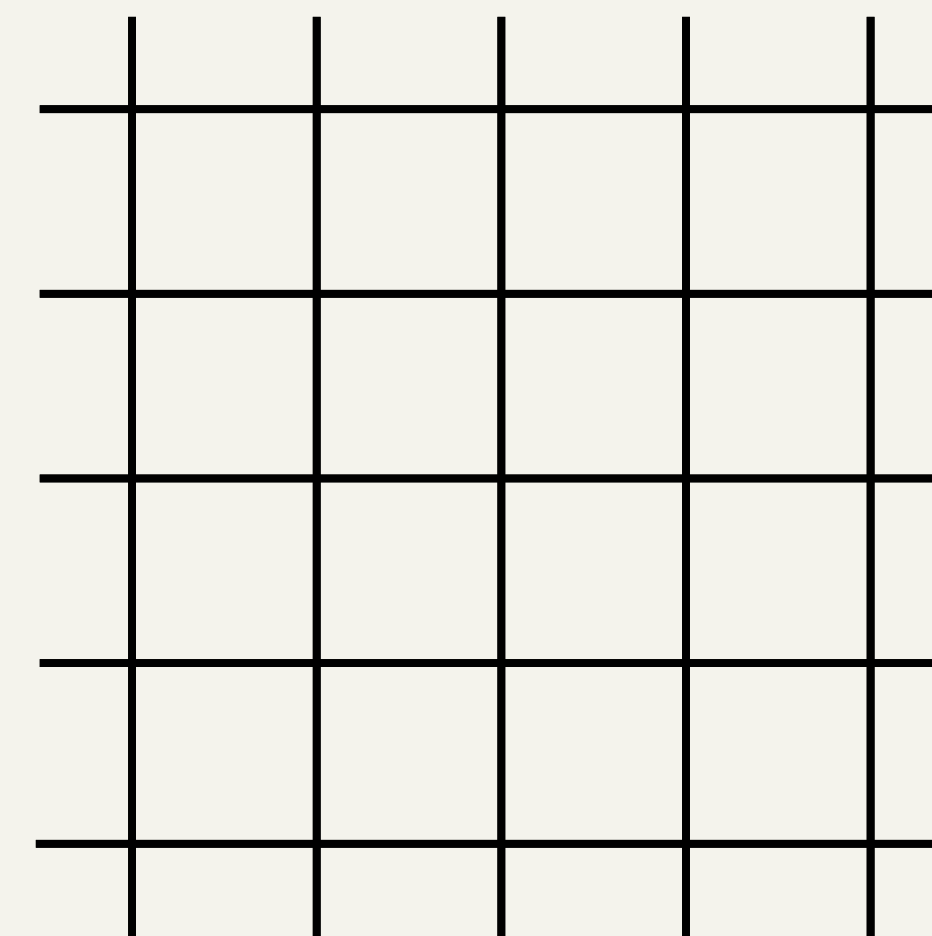








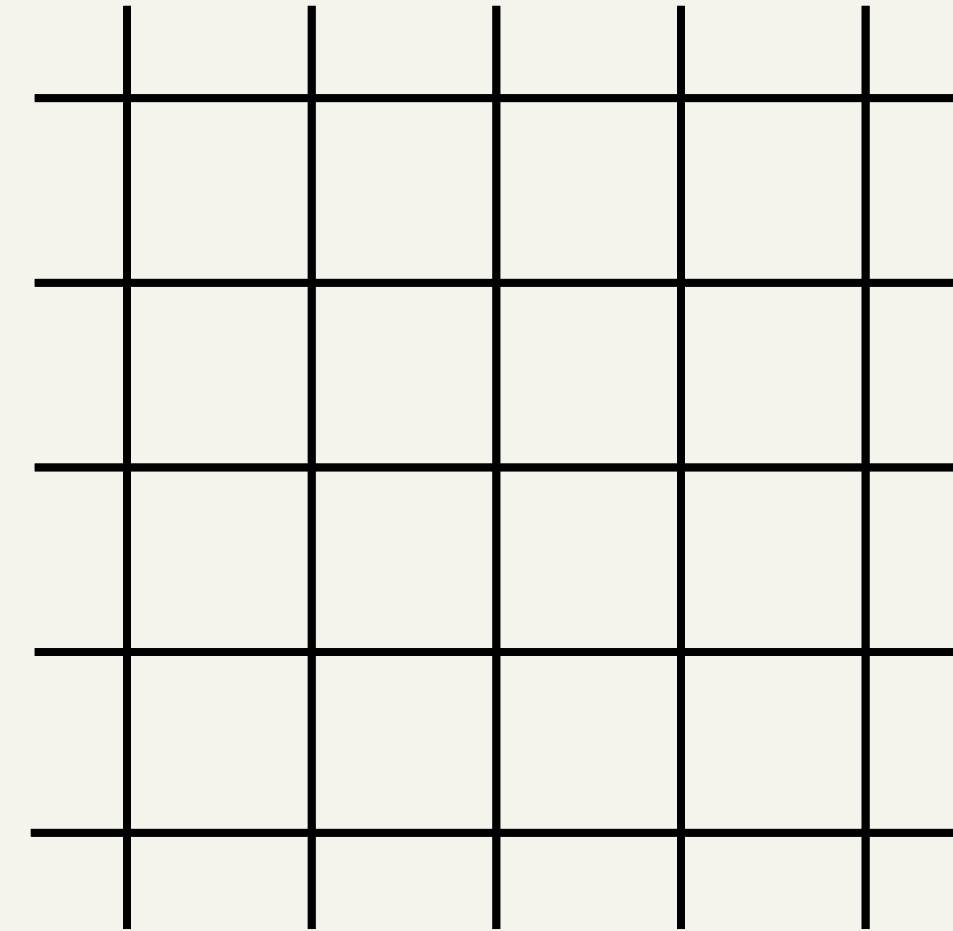
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Collection  $\mathcal{U}$  of convex finite polytopes such that:

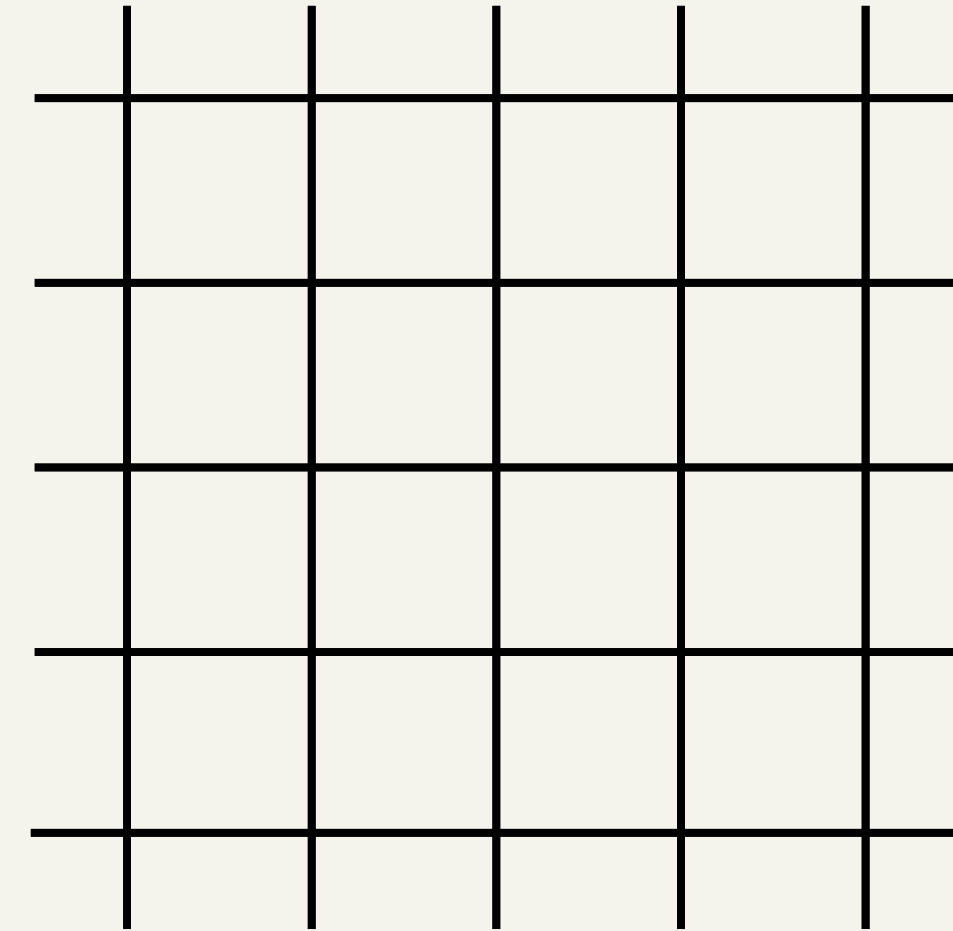
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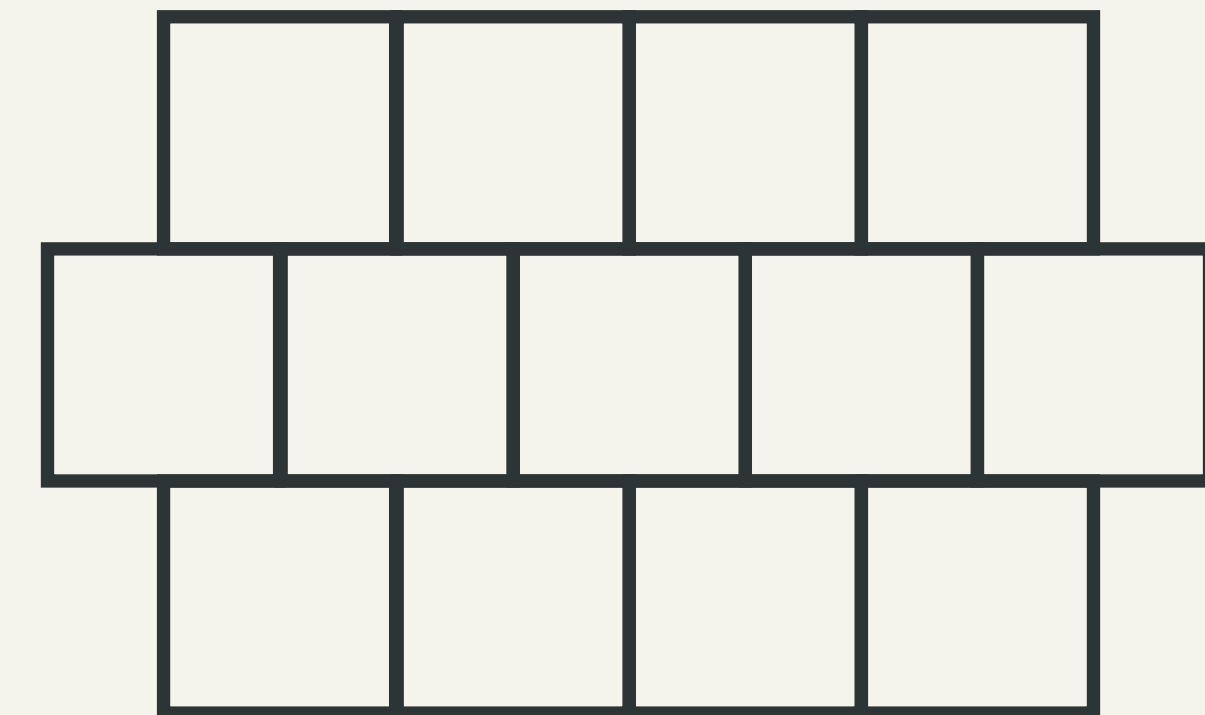
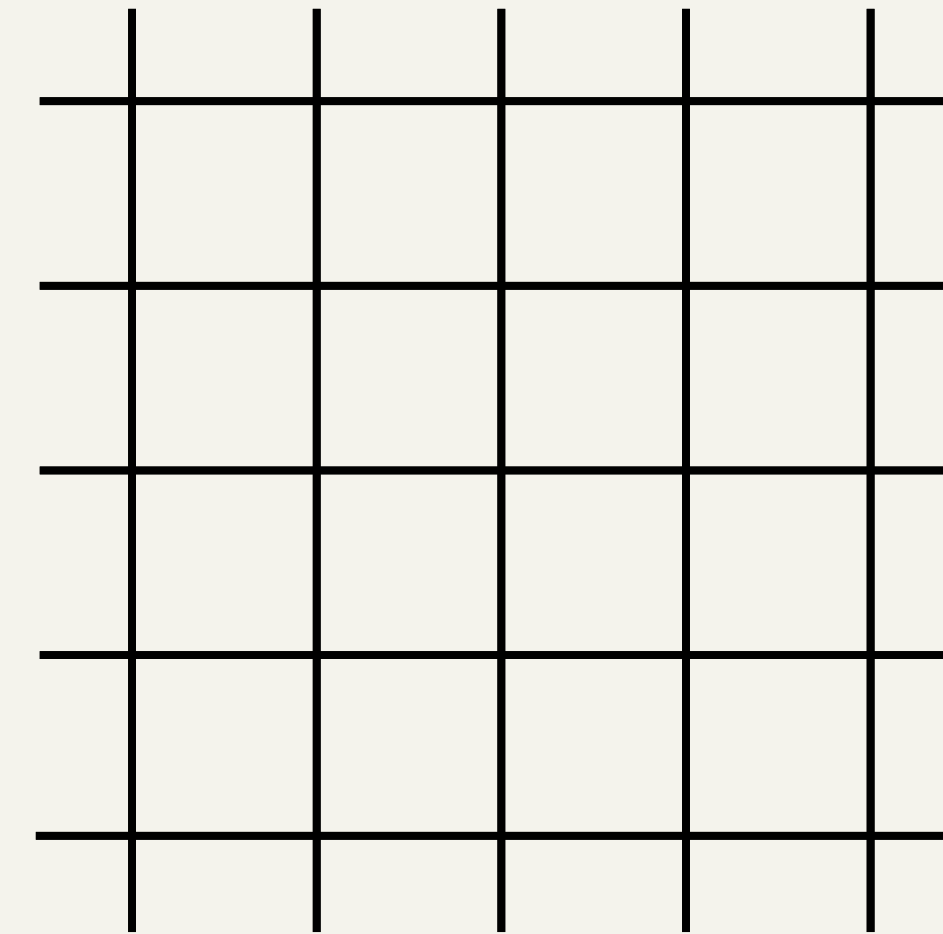
- $\mathcal{U}$  is a **locally finite cover** of  $\mathbb{E}^n$
- For every two elements  $P_1, P_2 \in \mathcal{U}$  (**tiles**) the following hold:
  - $\text{int}(P_1) \cap \text{int}(P_2) = \emptyset$ ;
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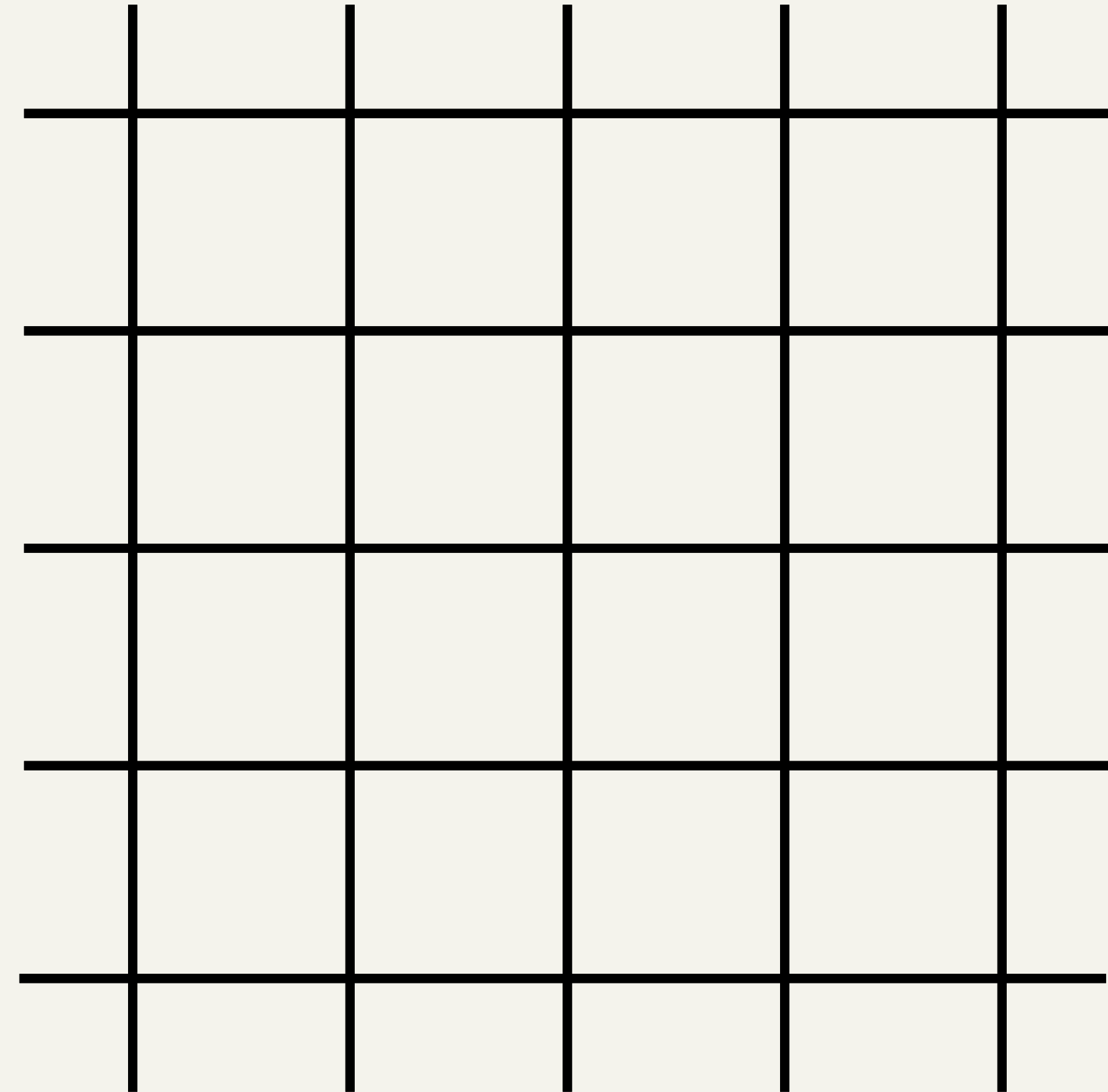
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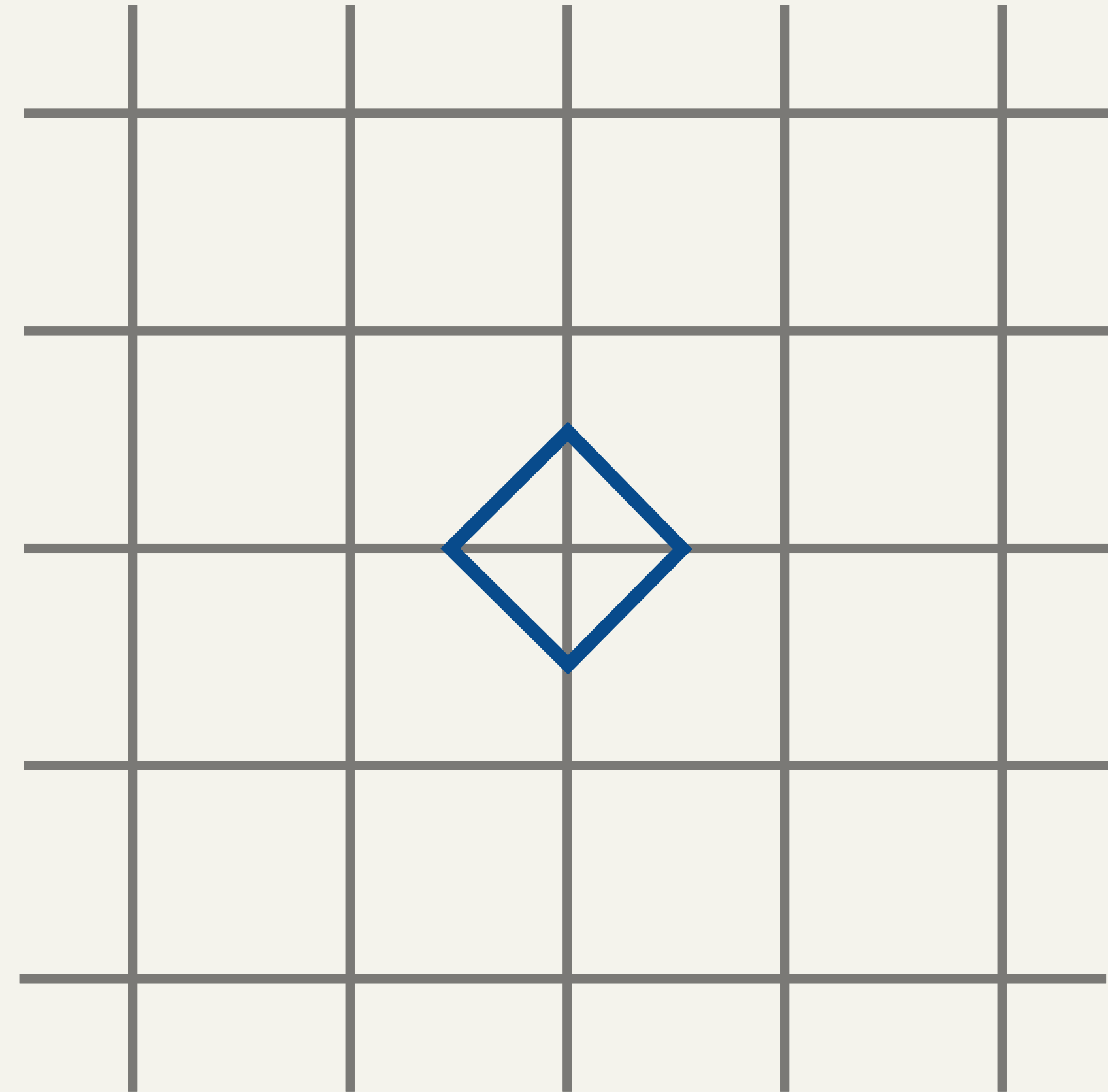
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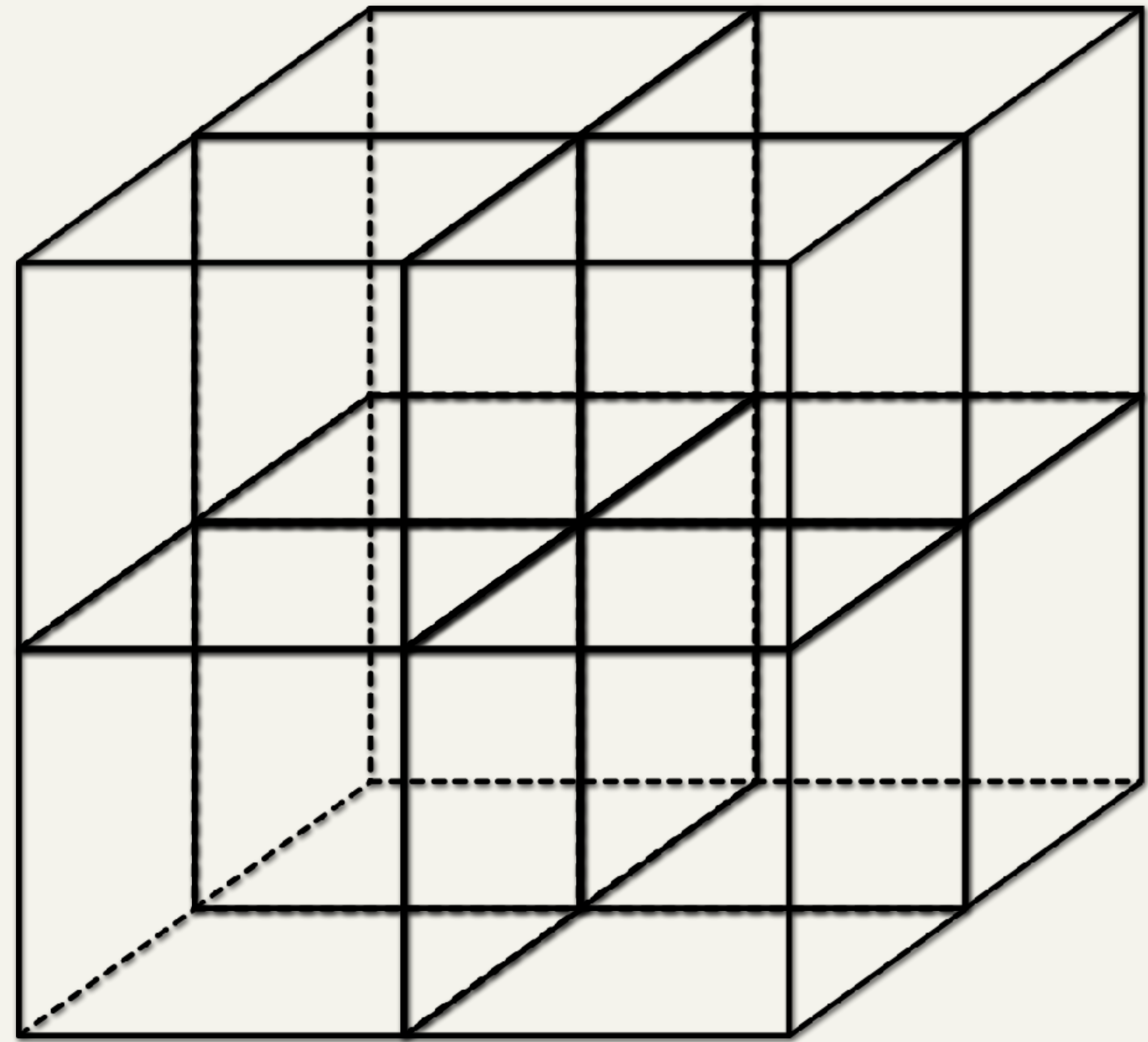
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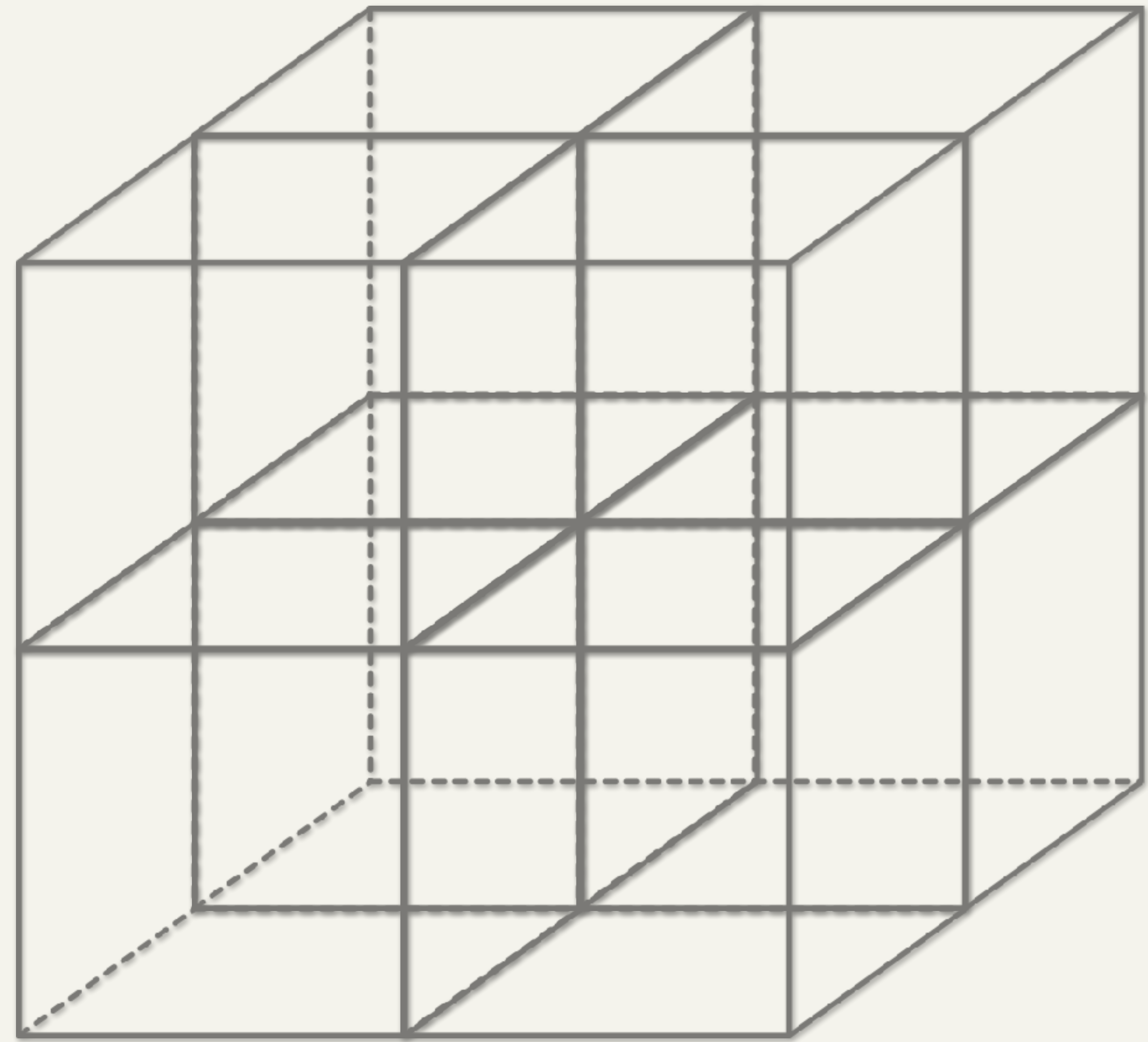
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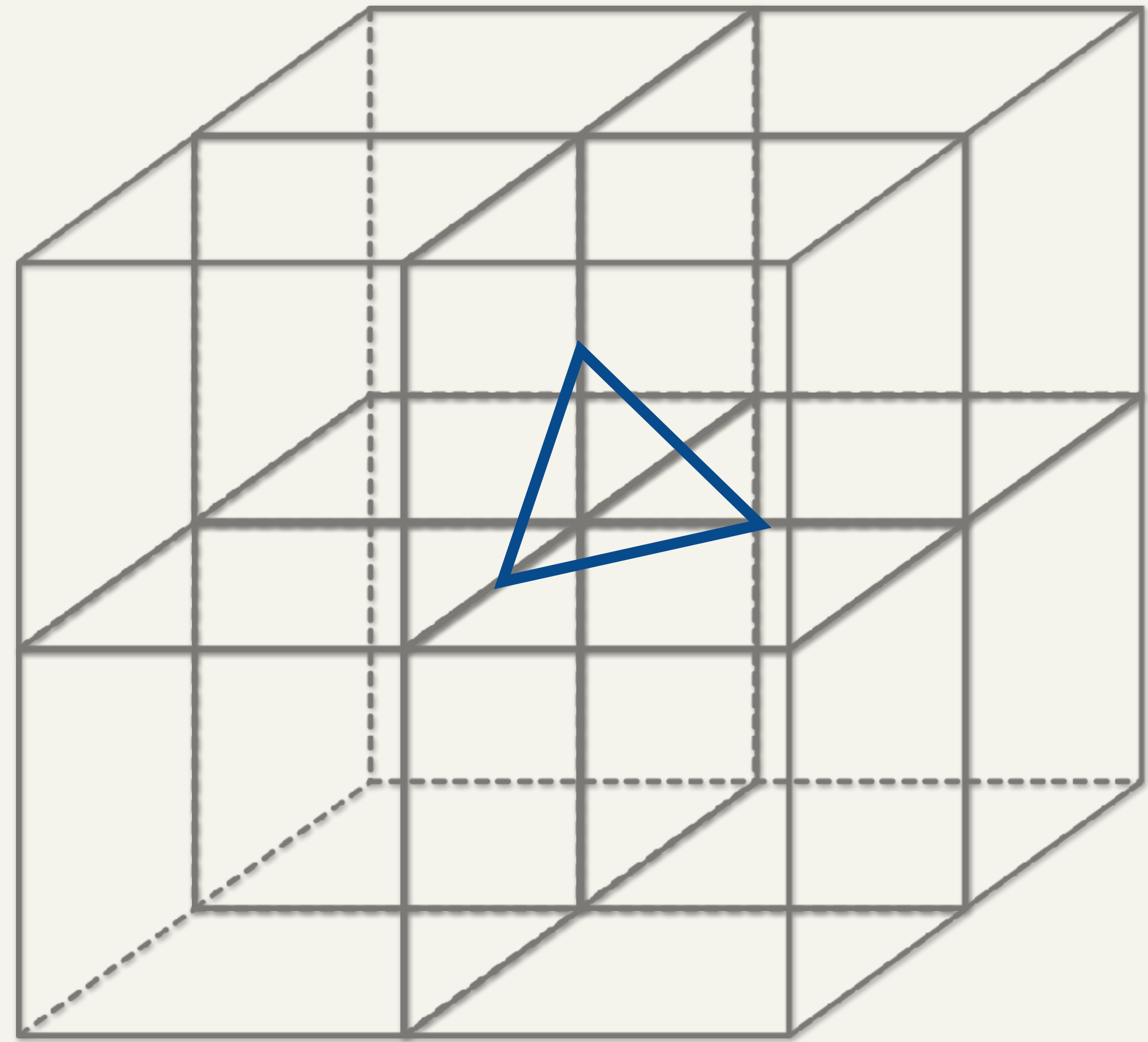
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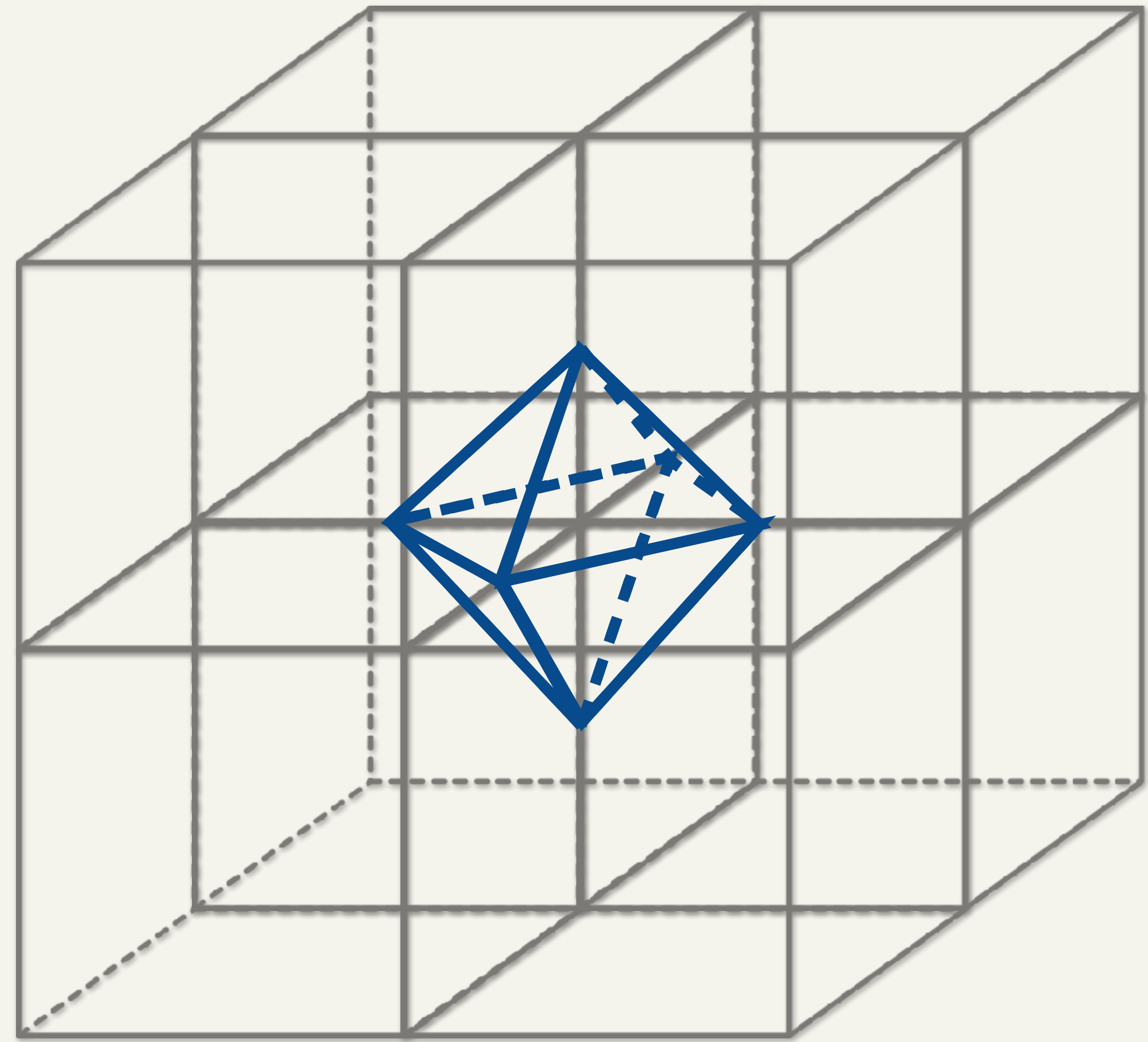
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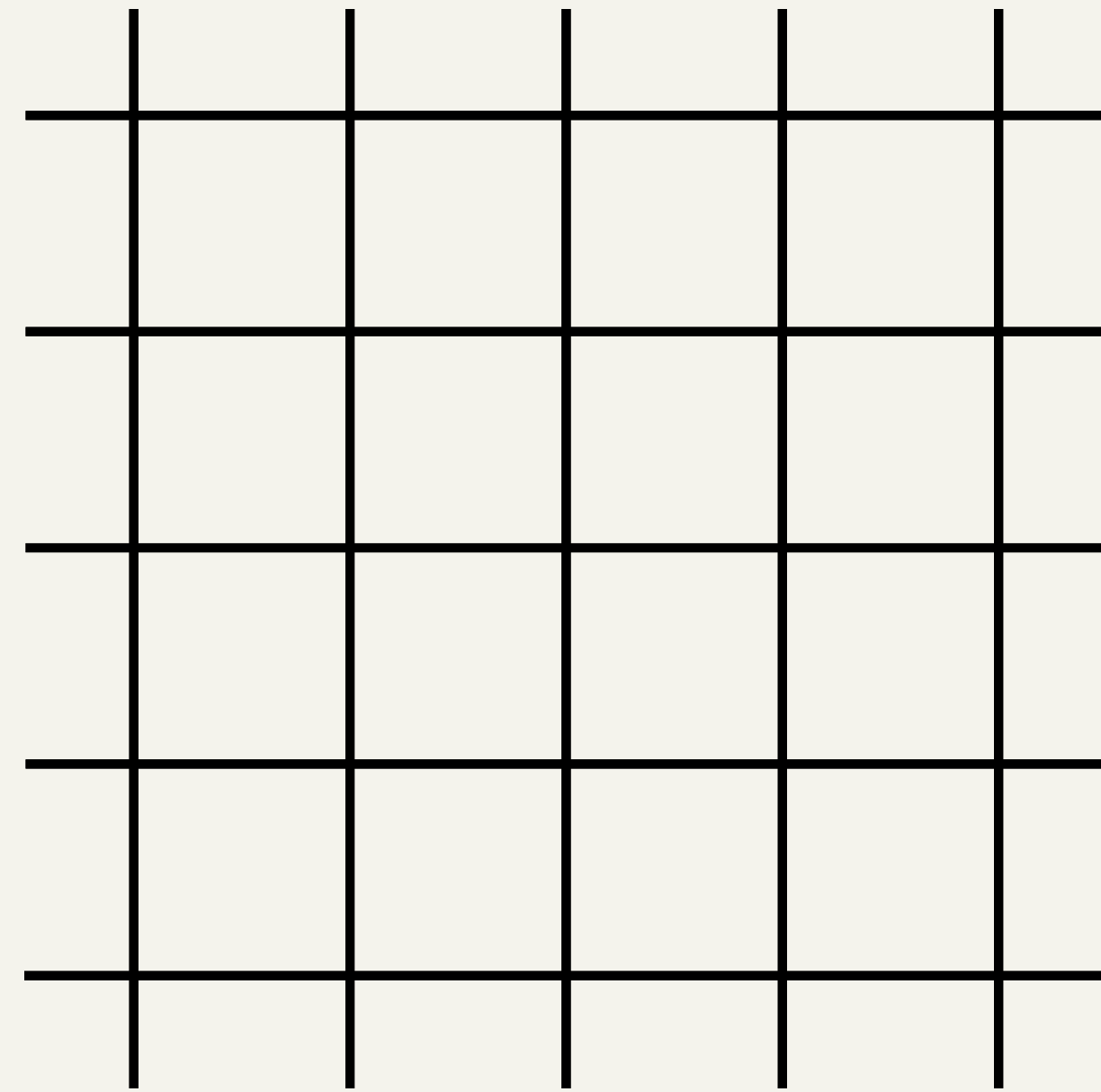
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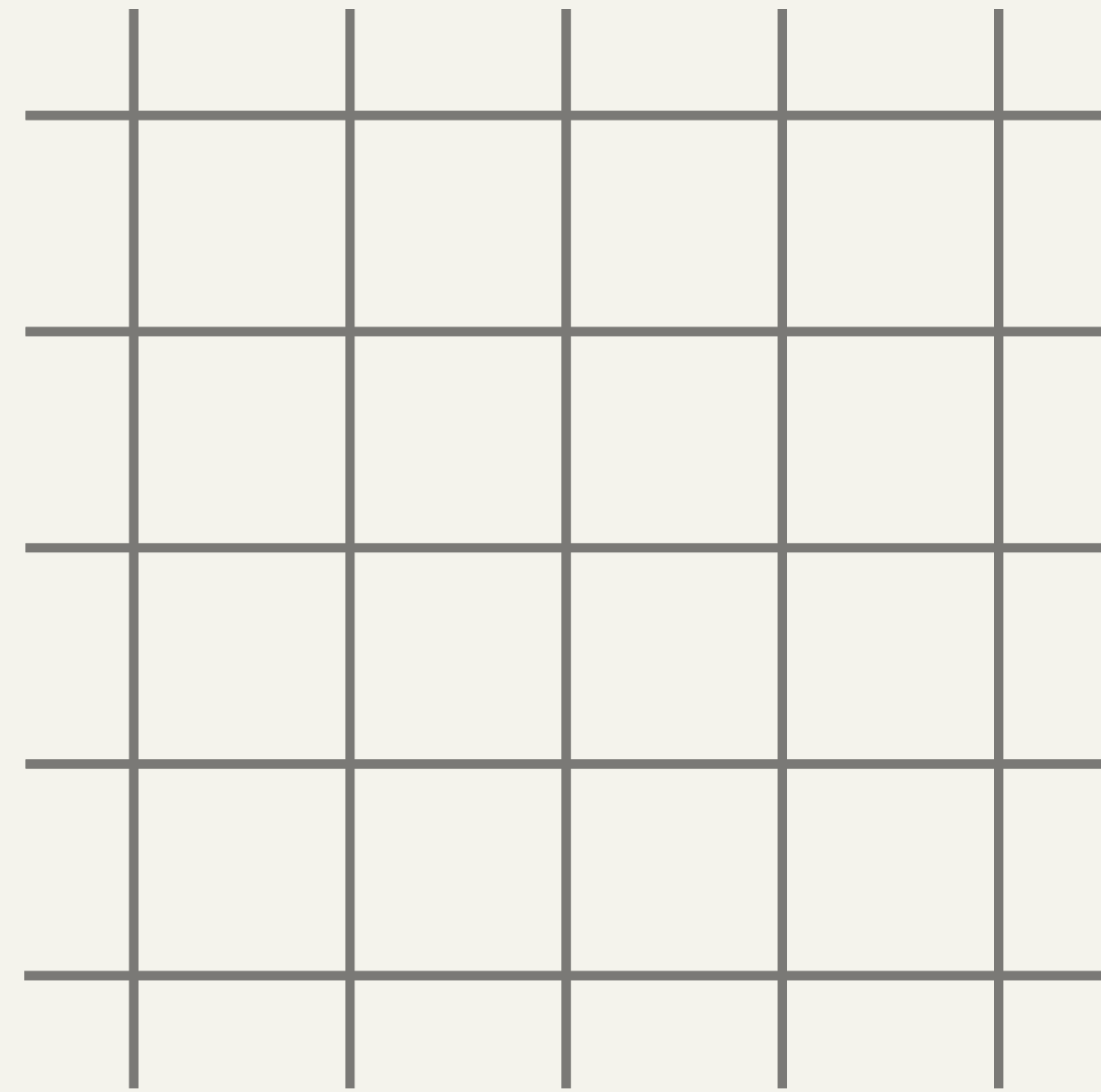
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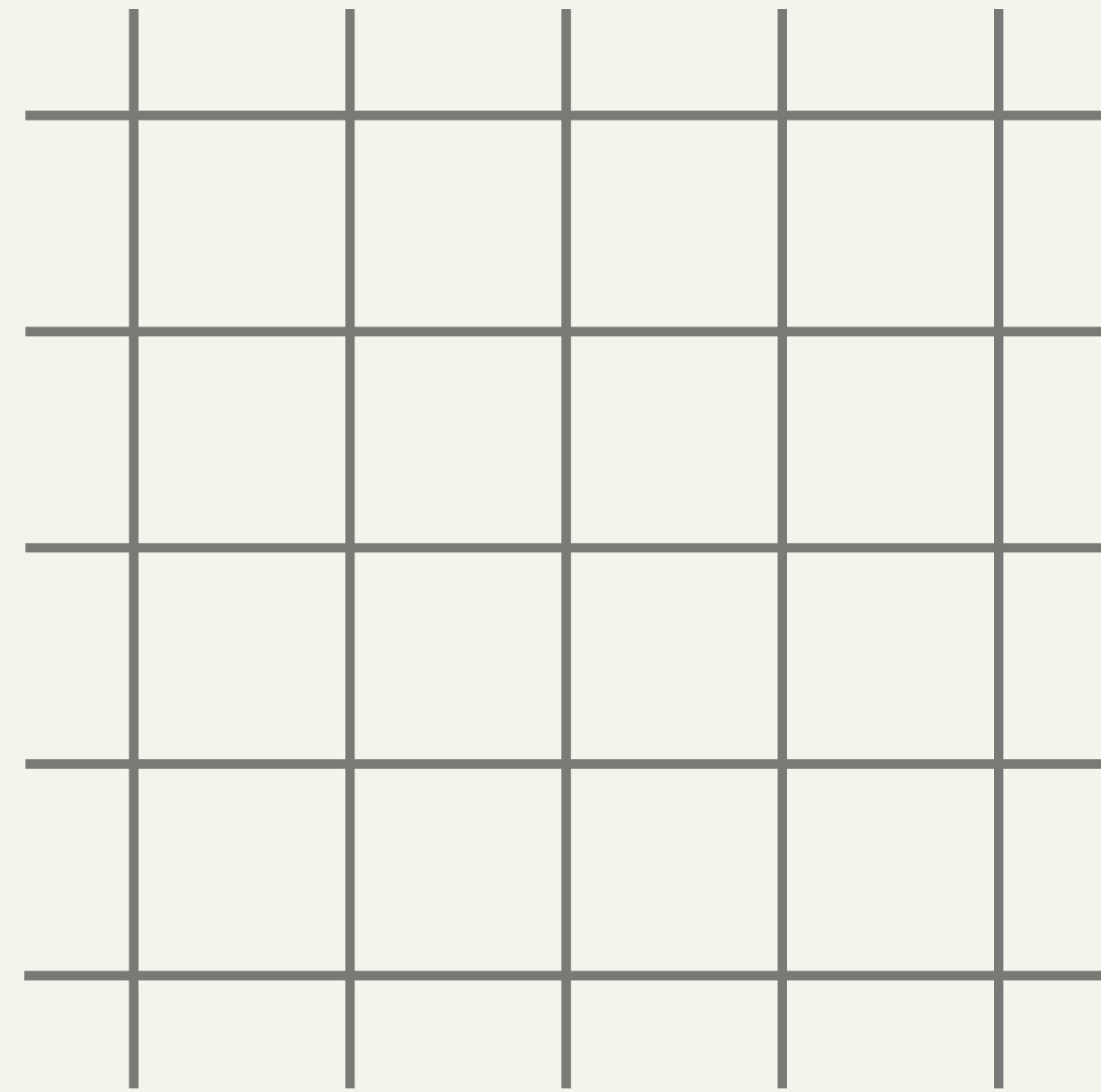
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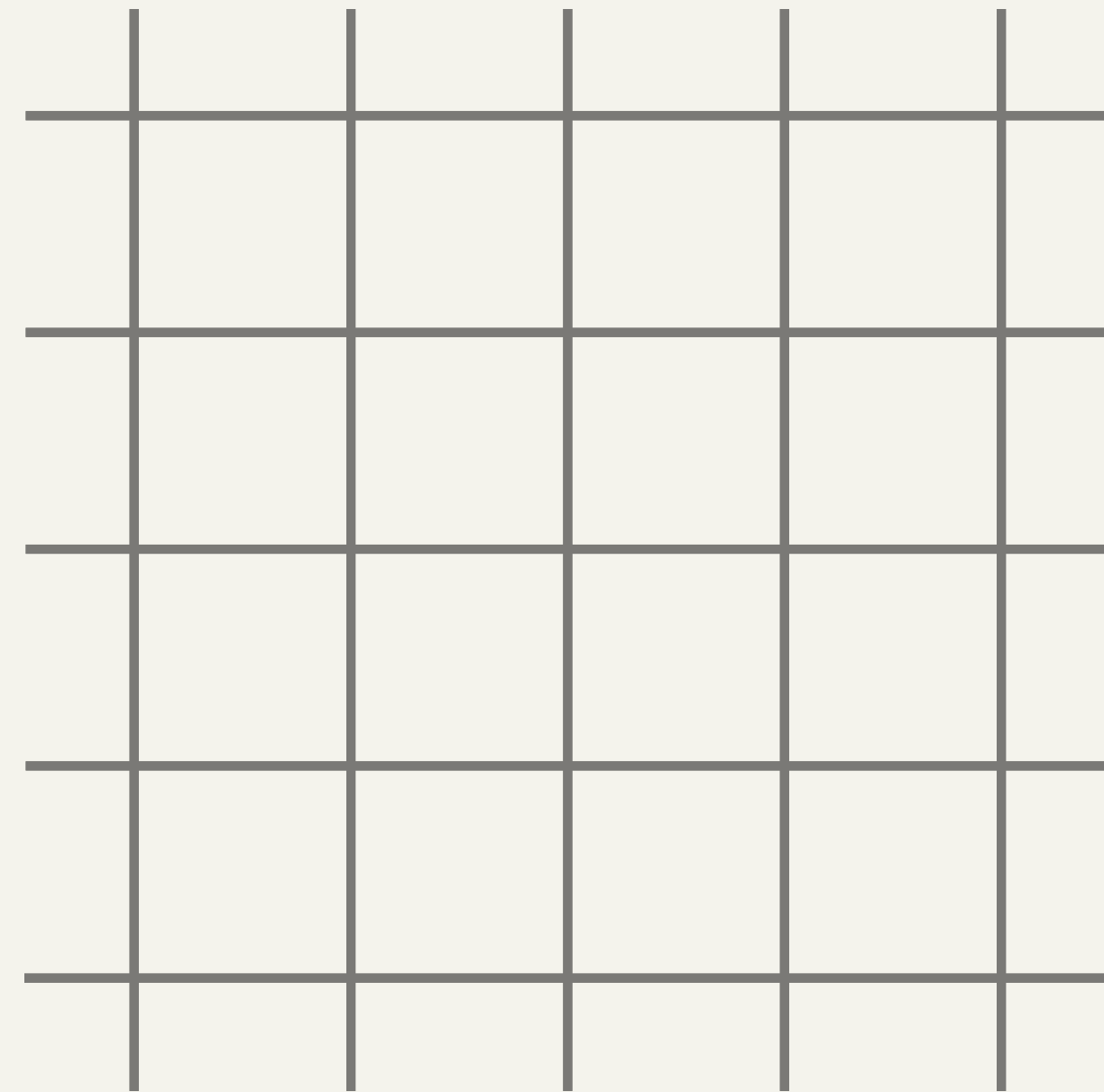
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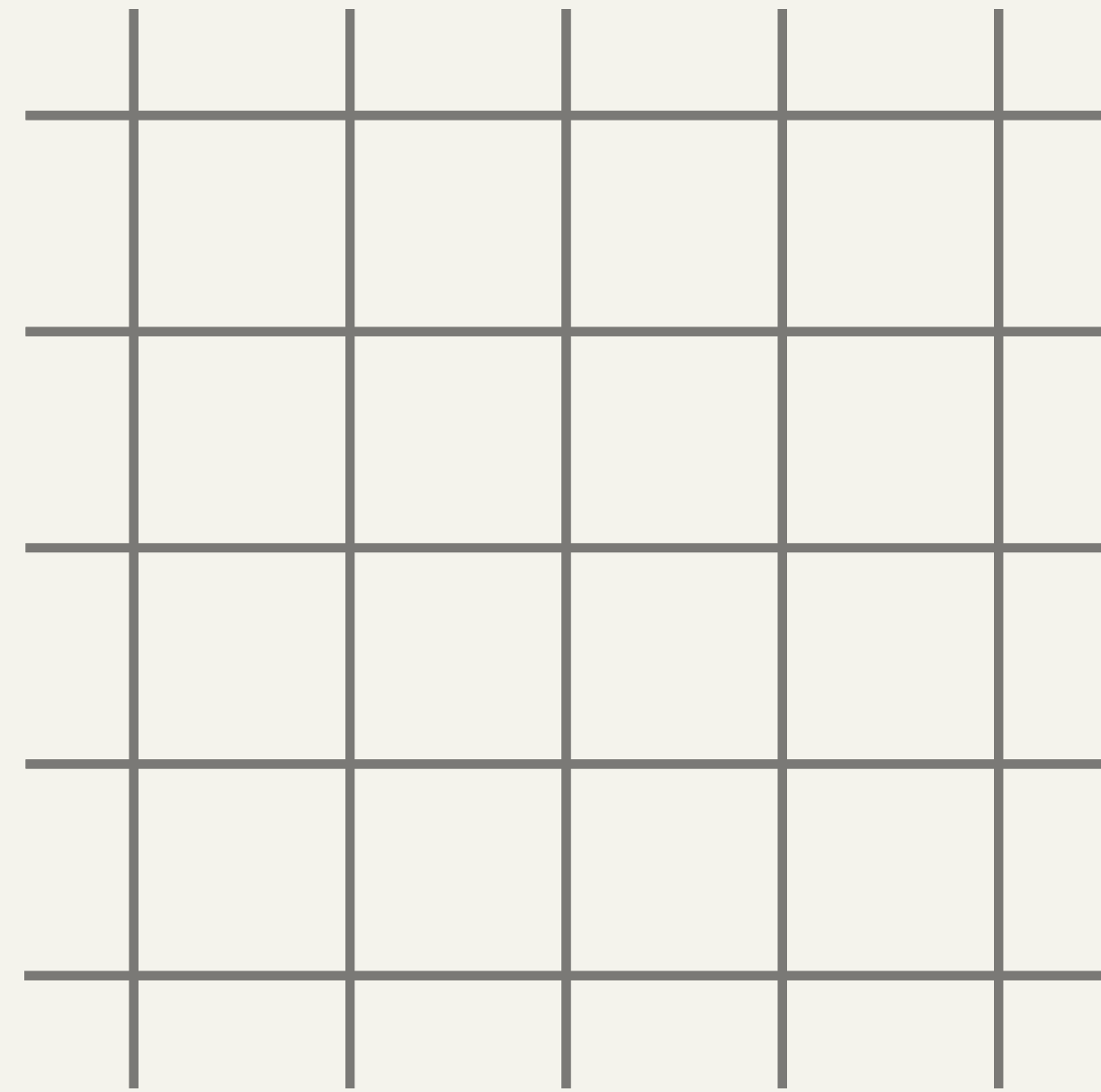
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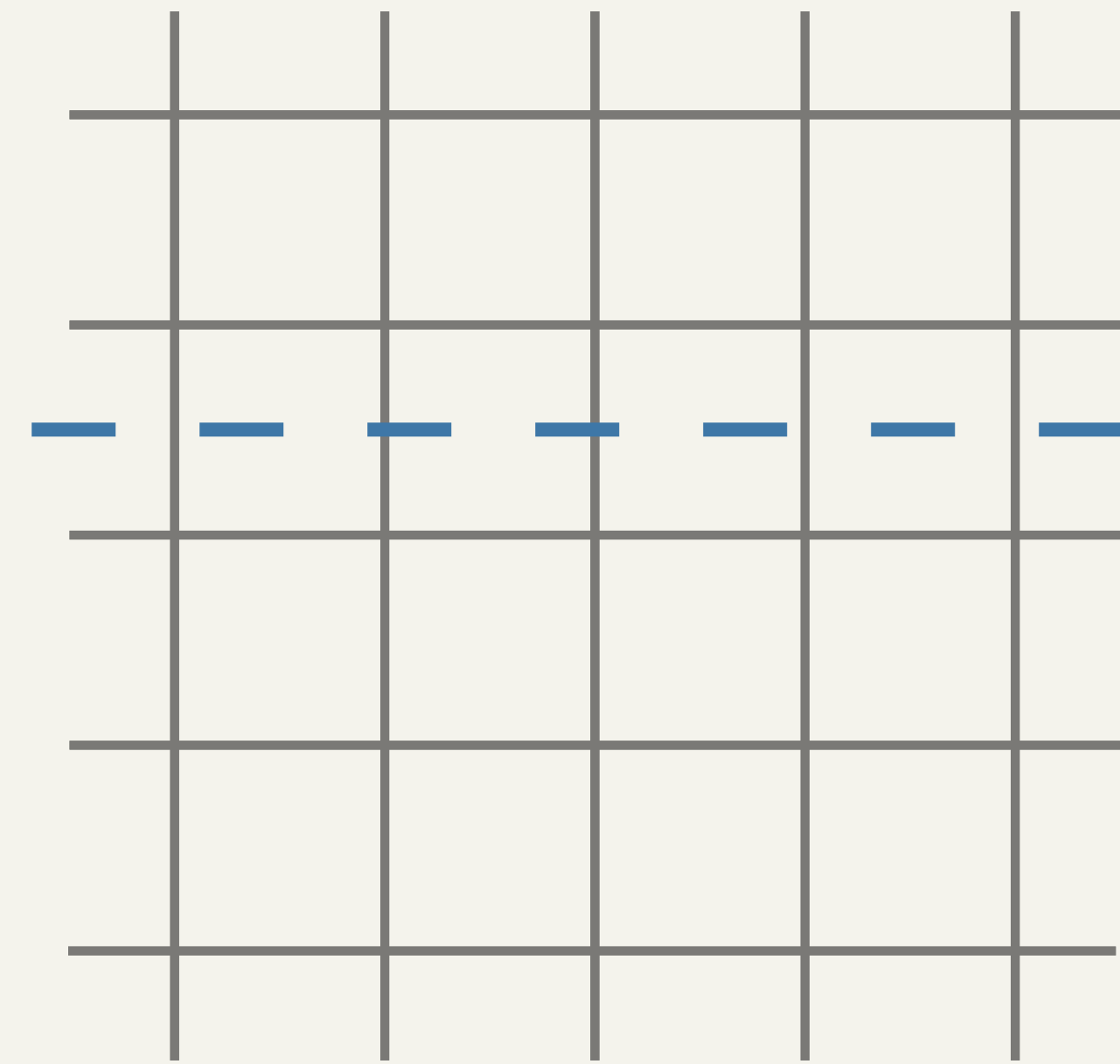
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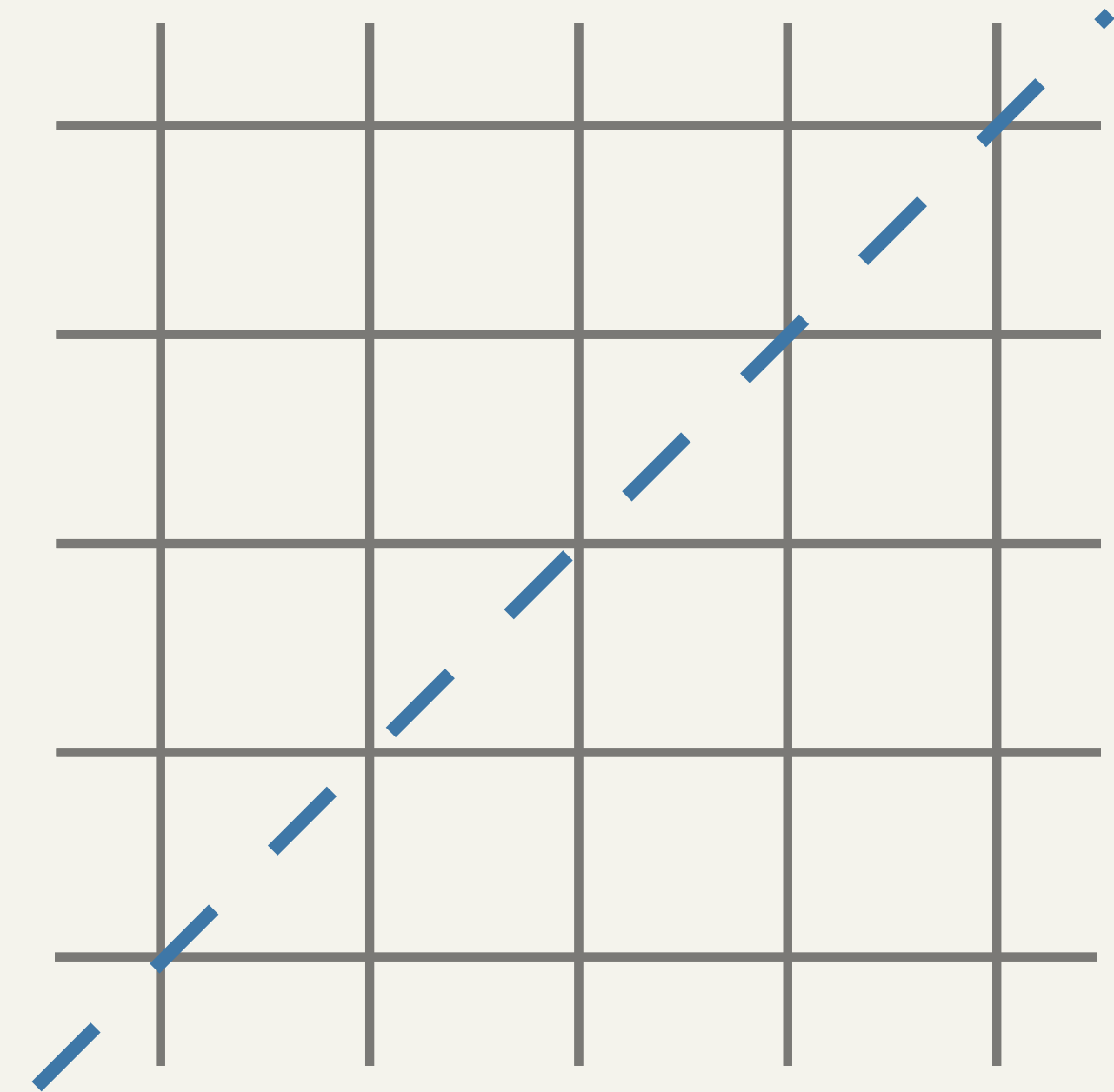
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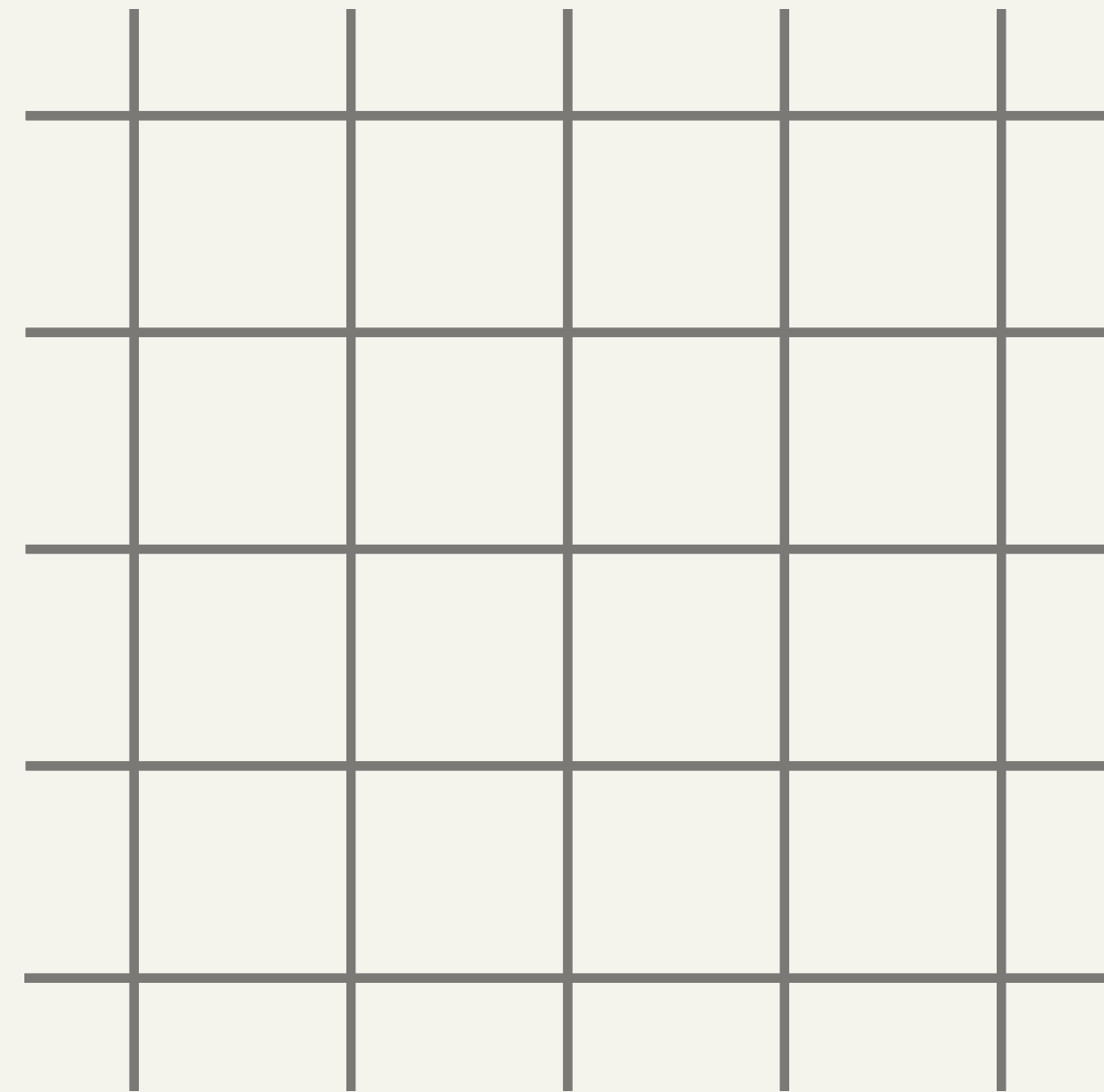
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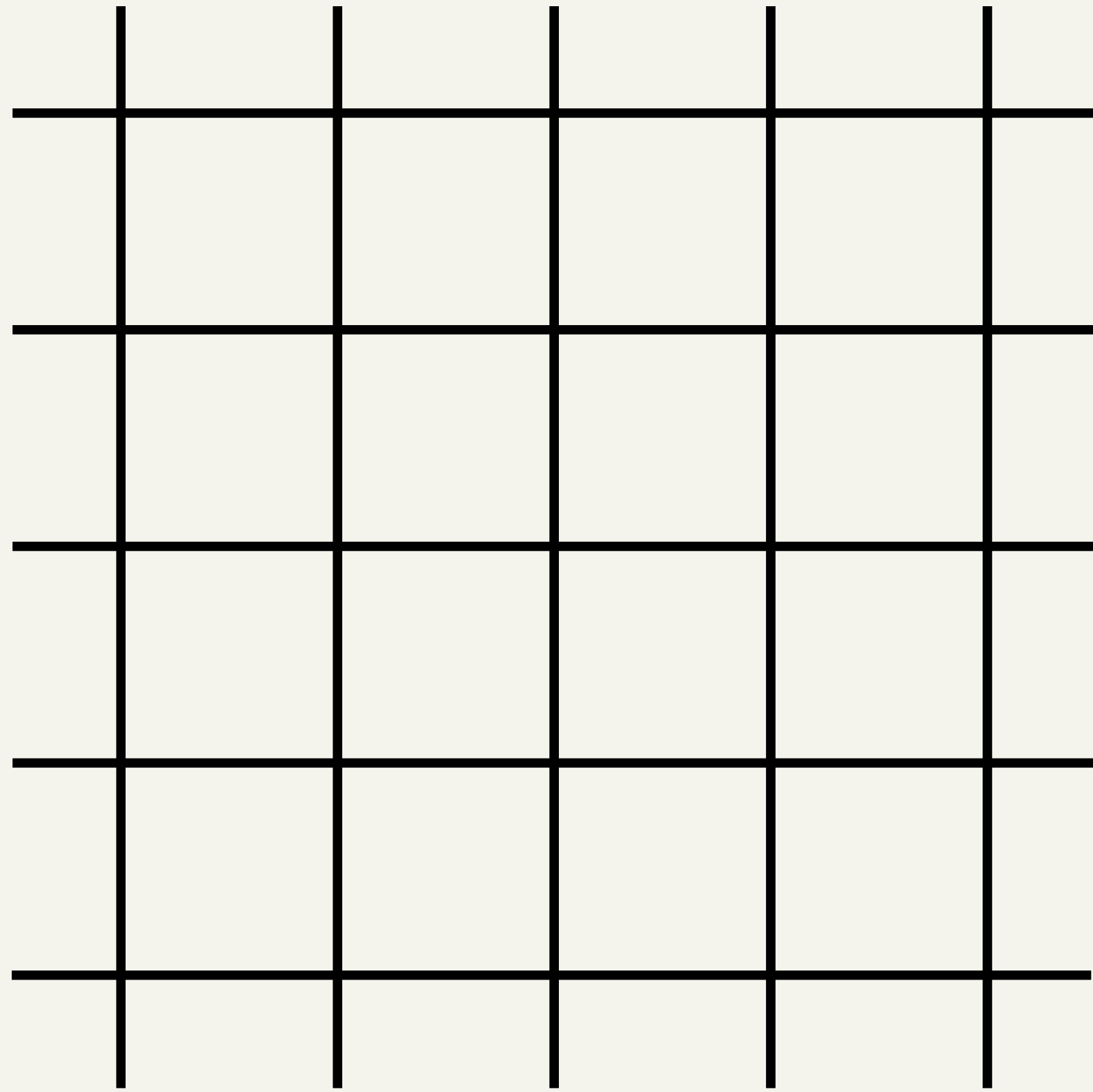


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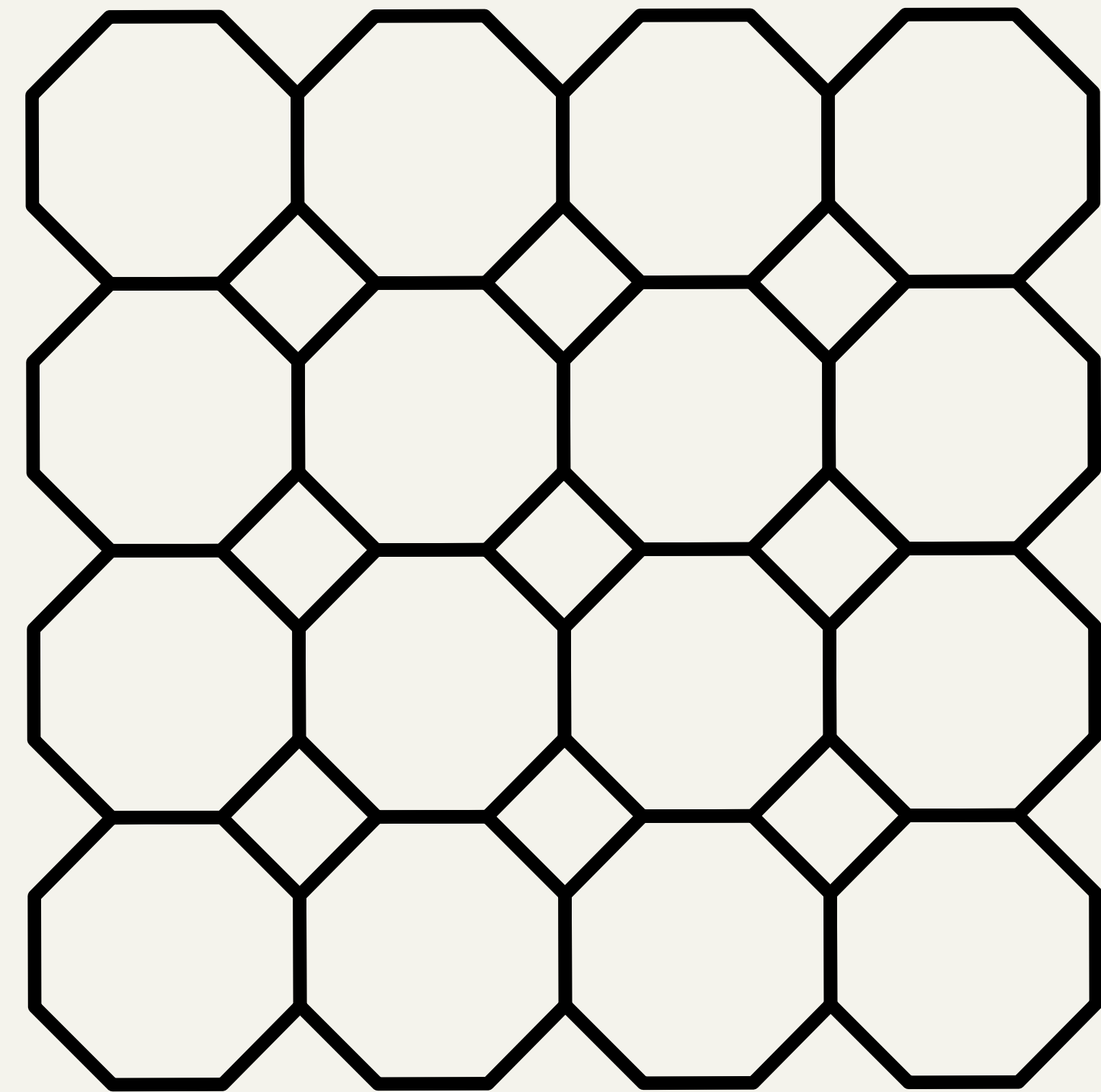
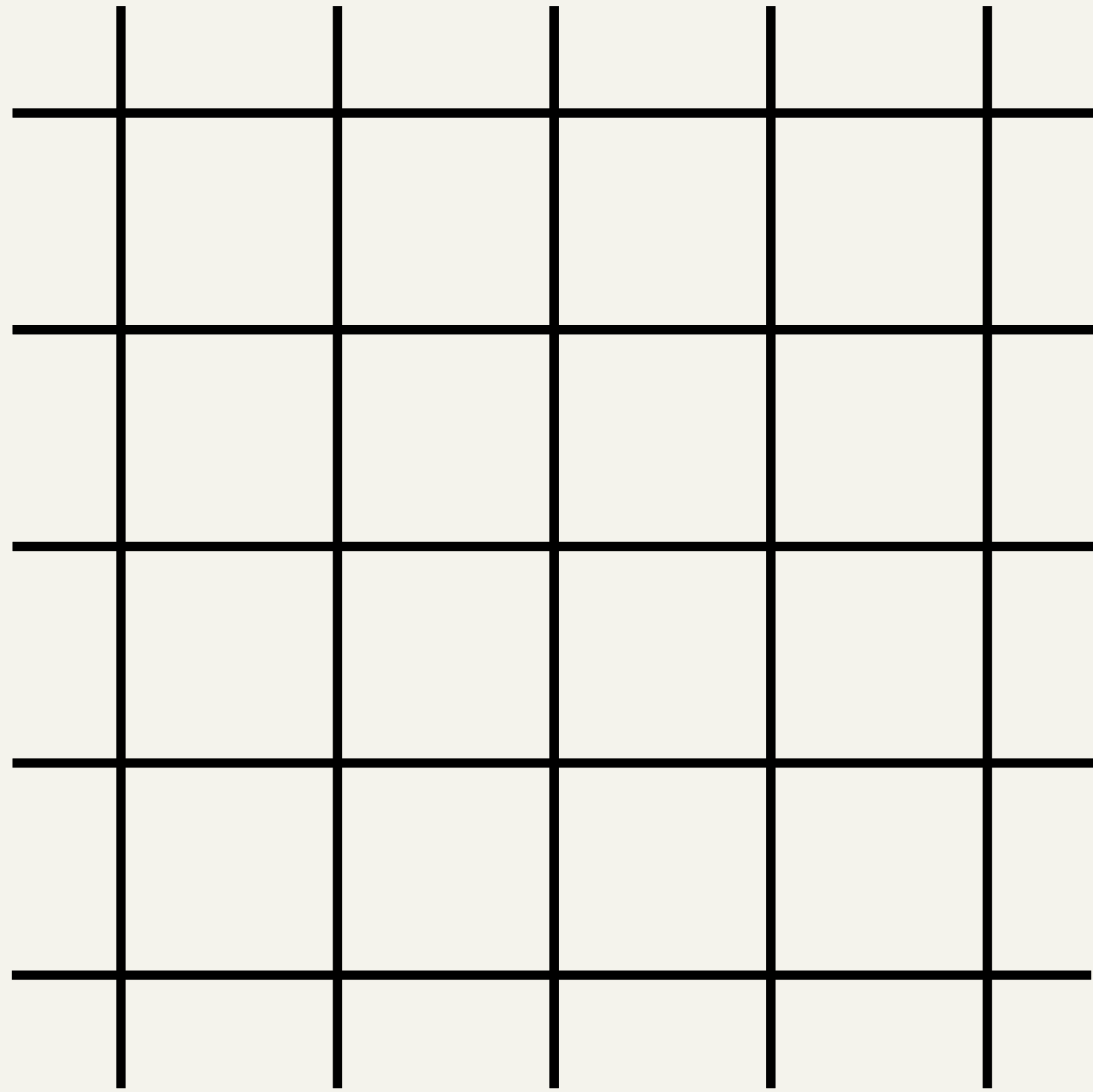
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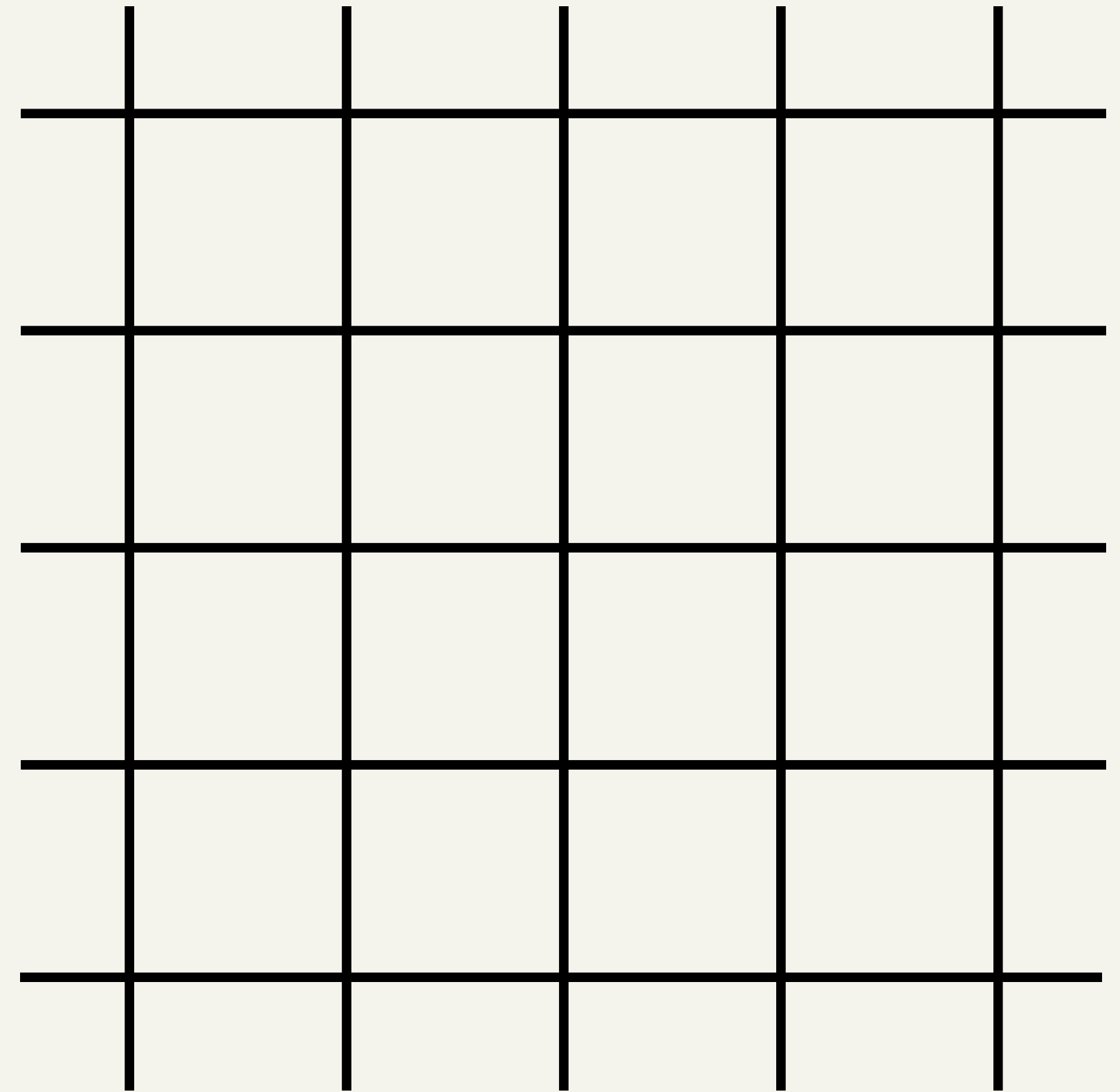
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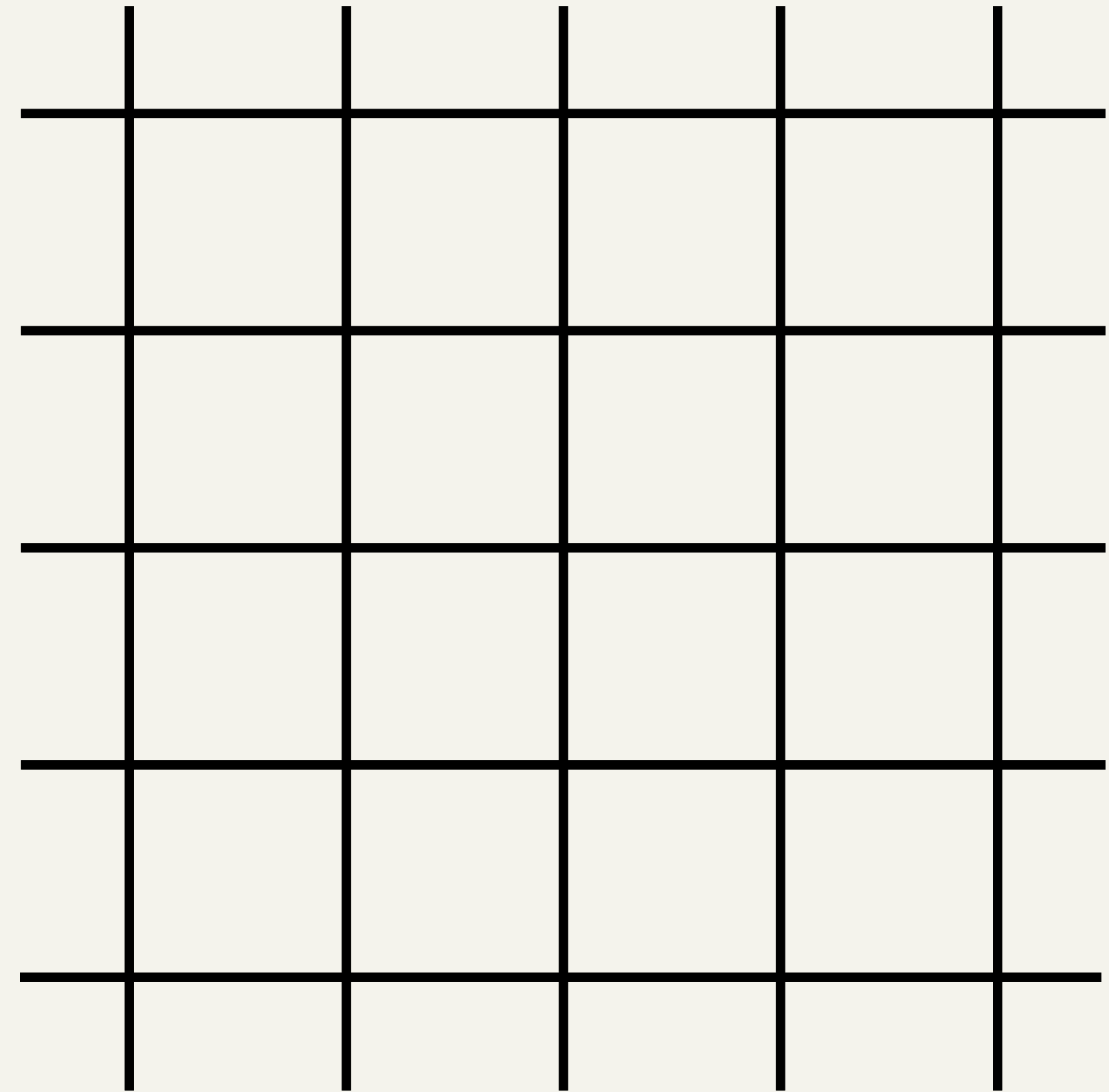
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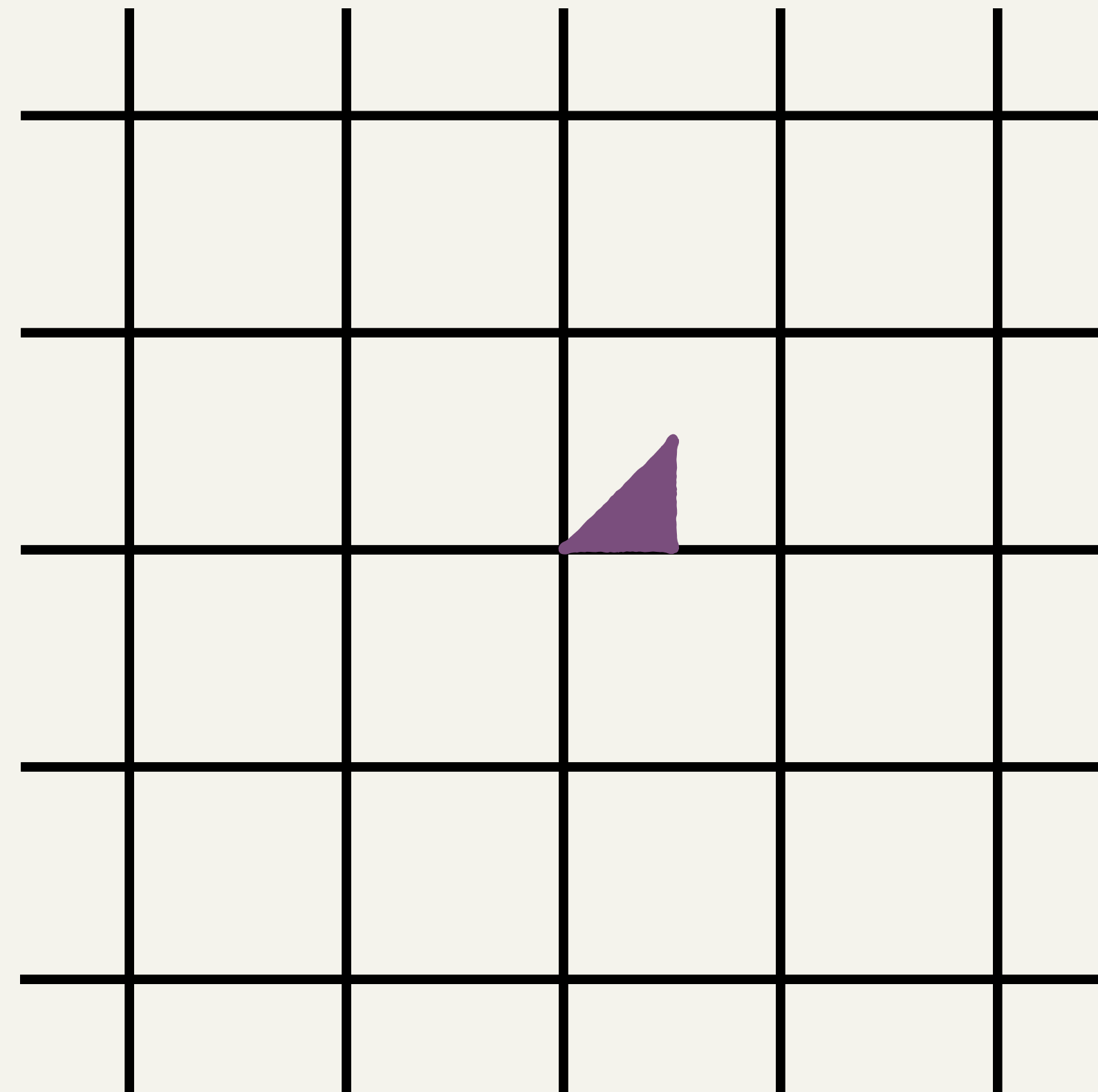
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The are often represented by **simplices**.



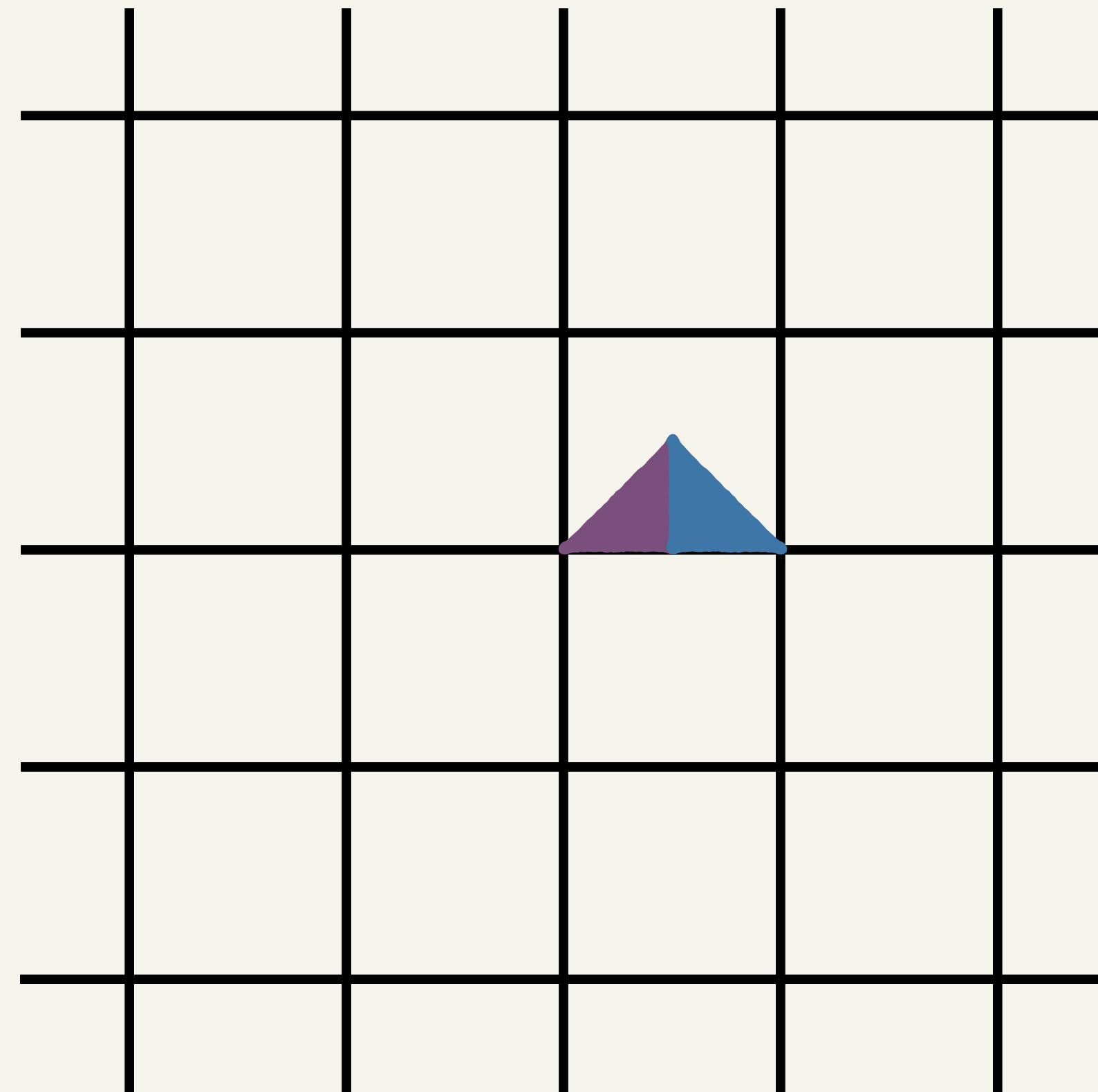
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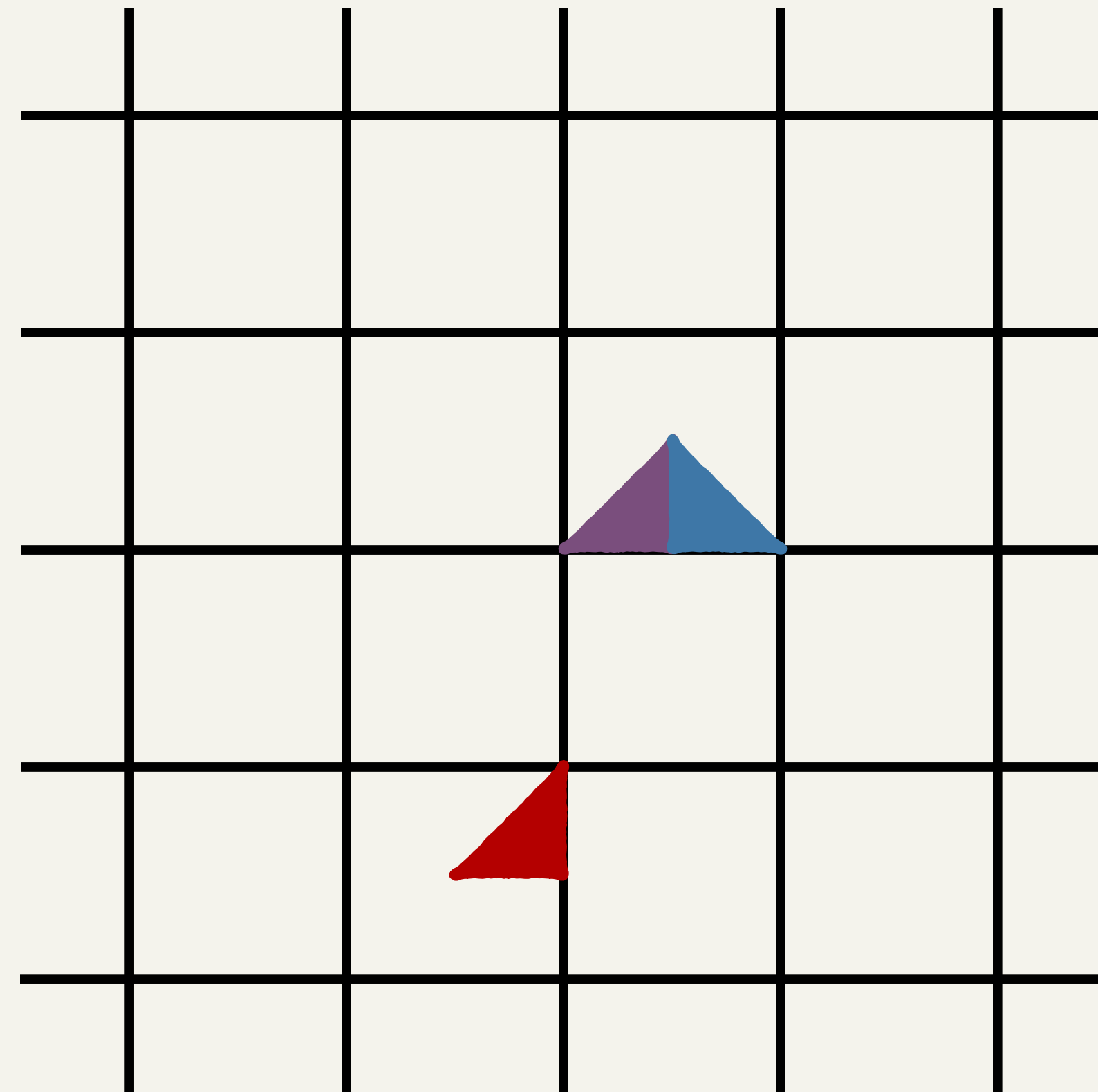
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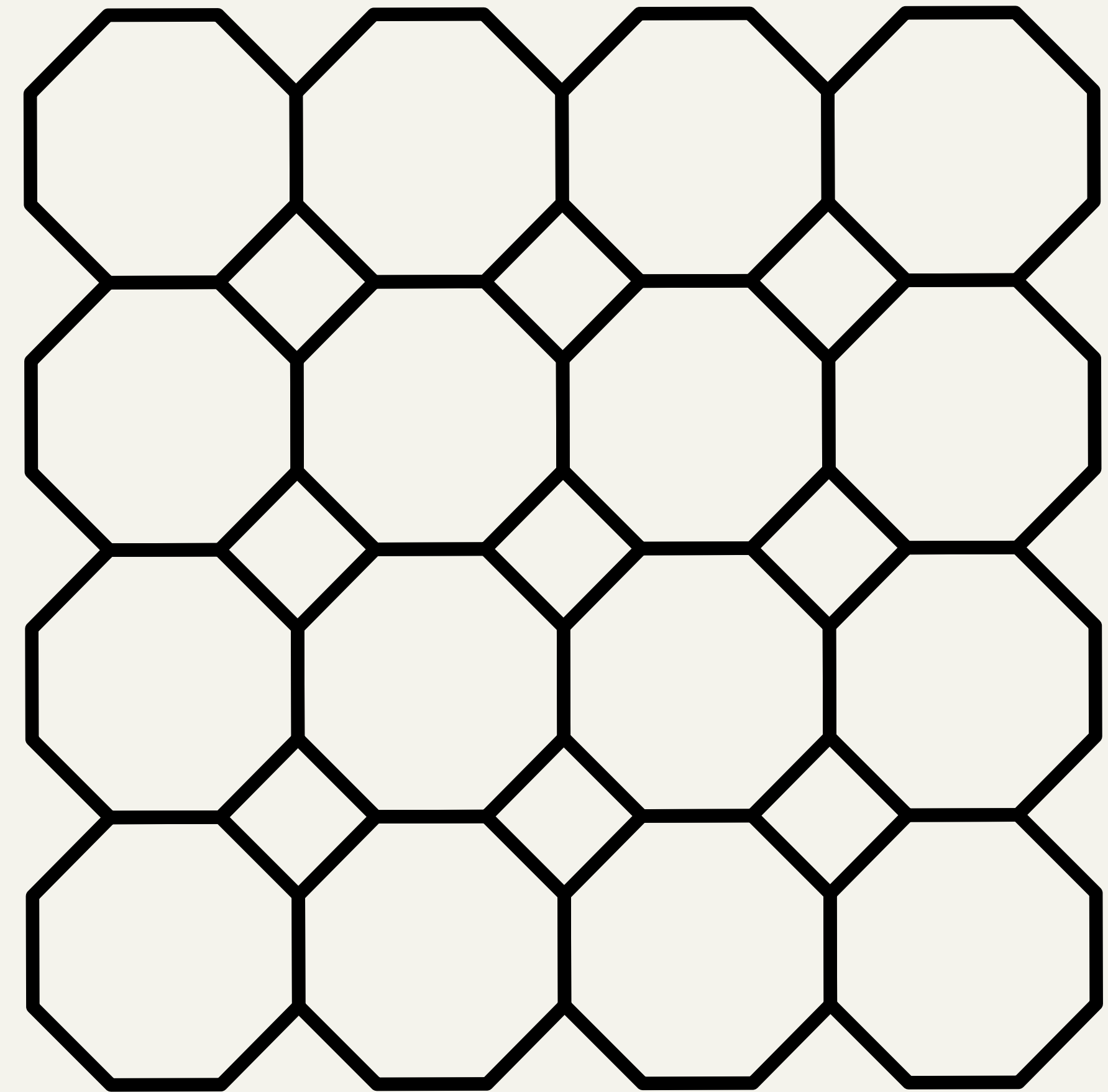
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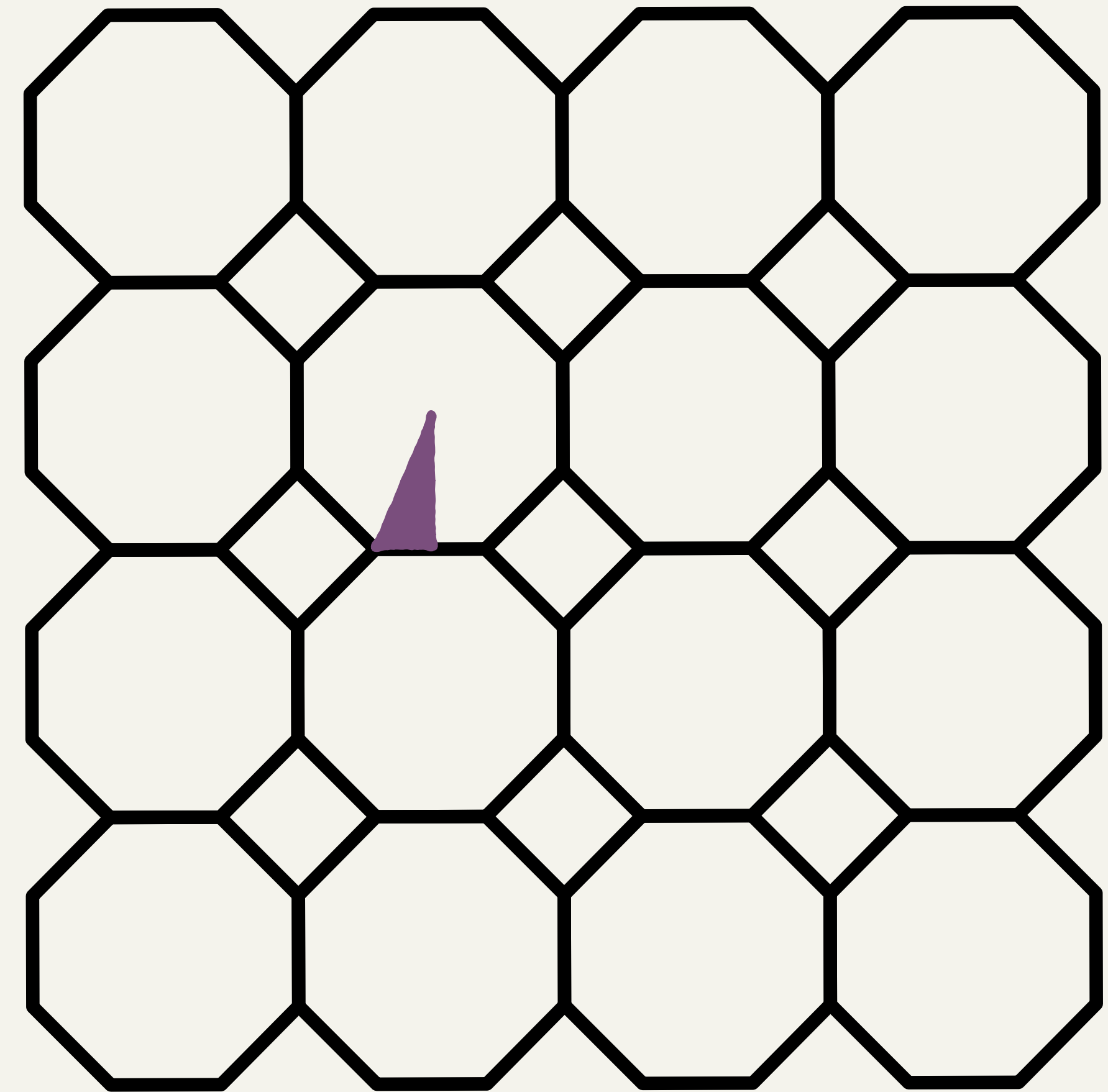
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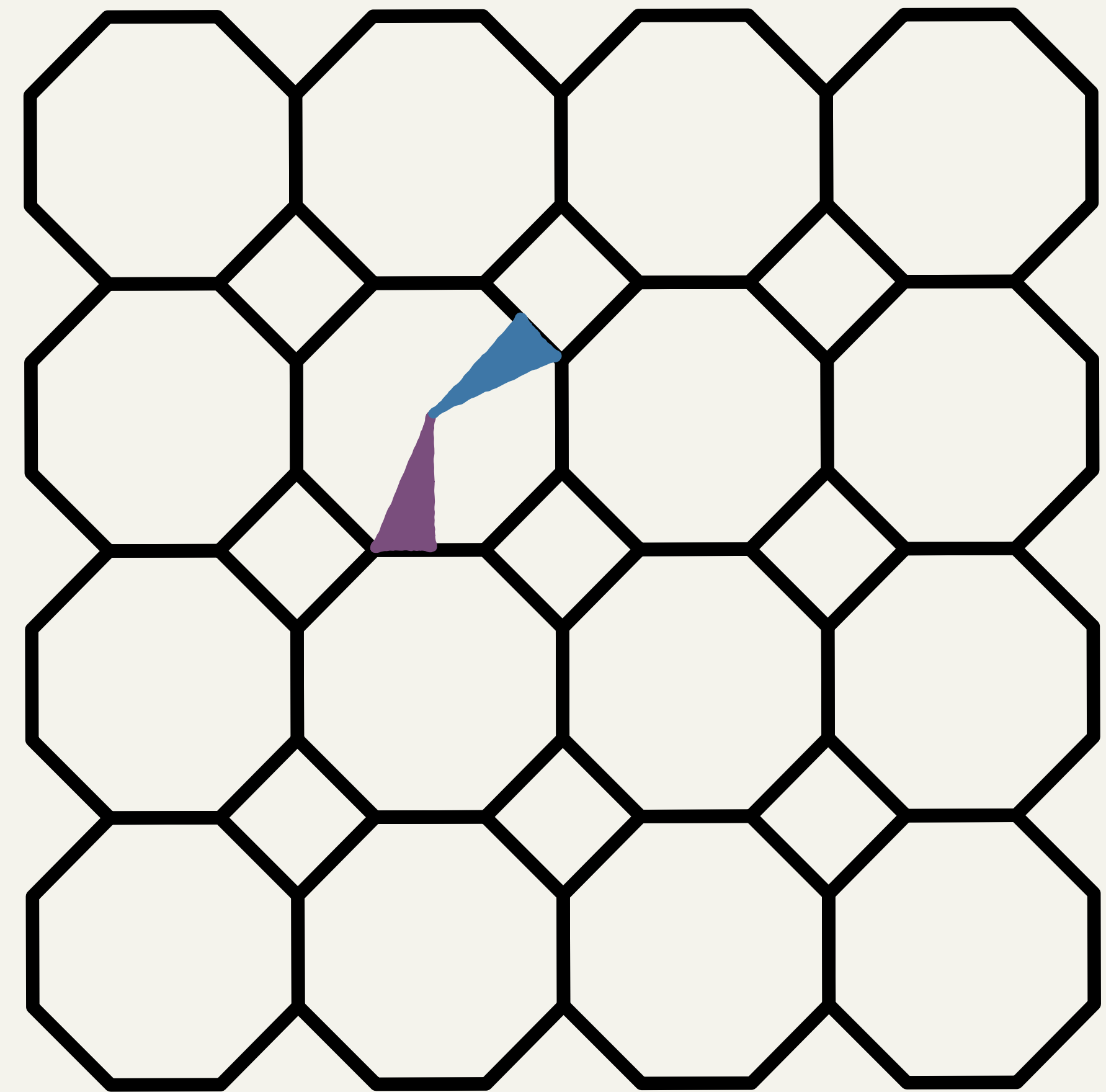
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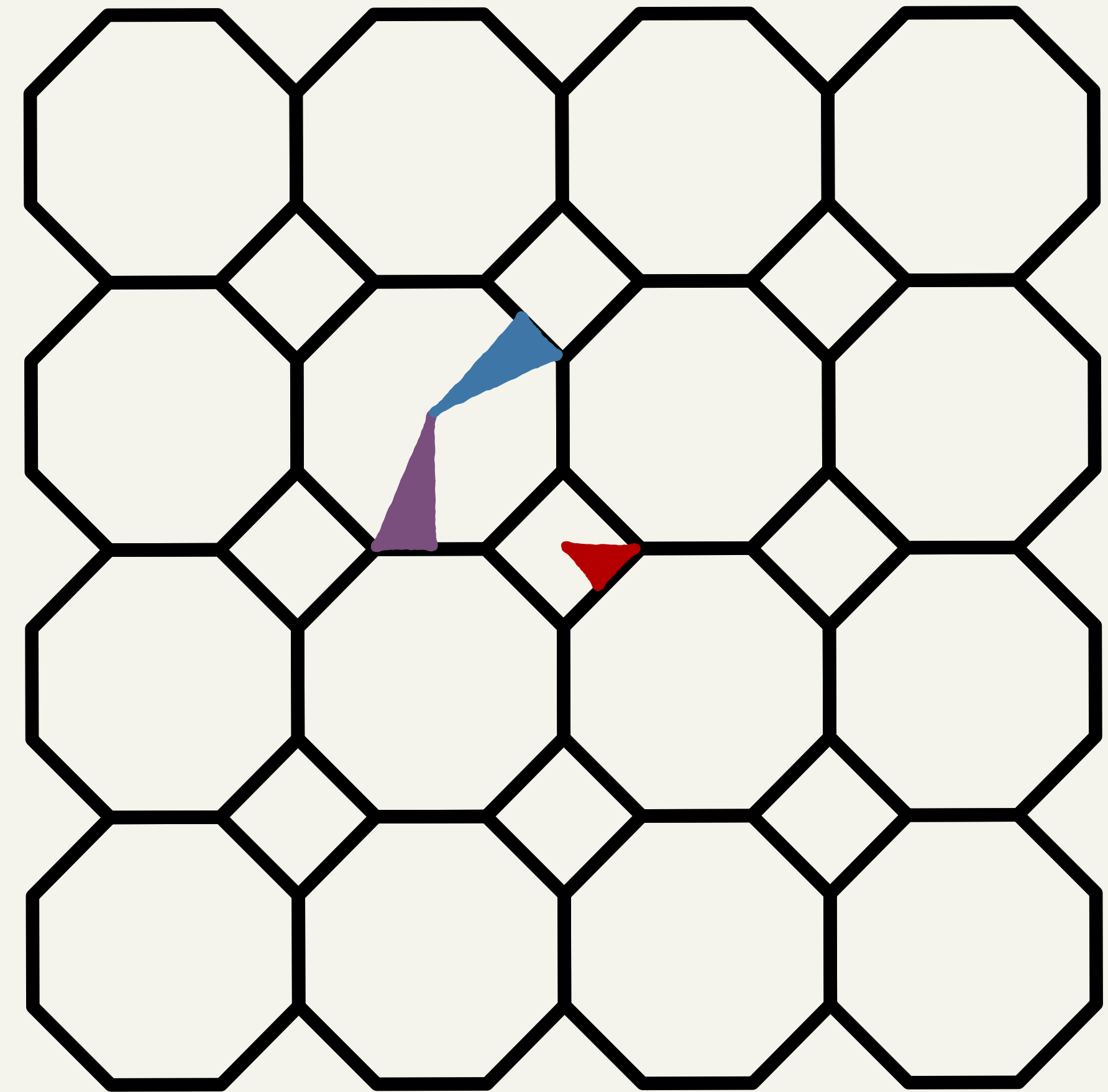
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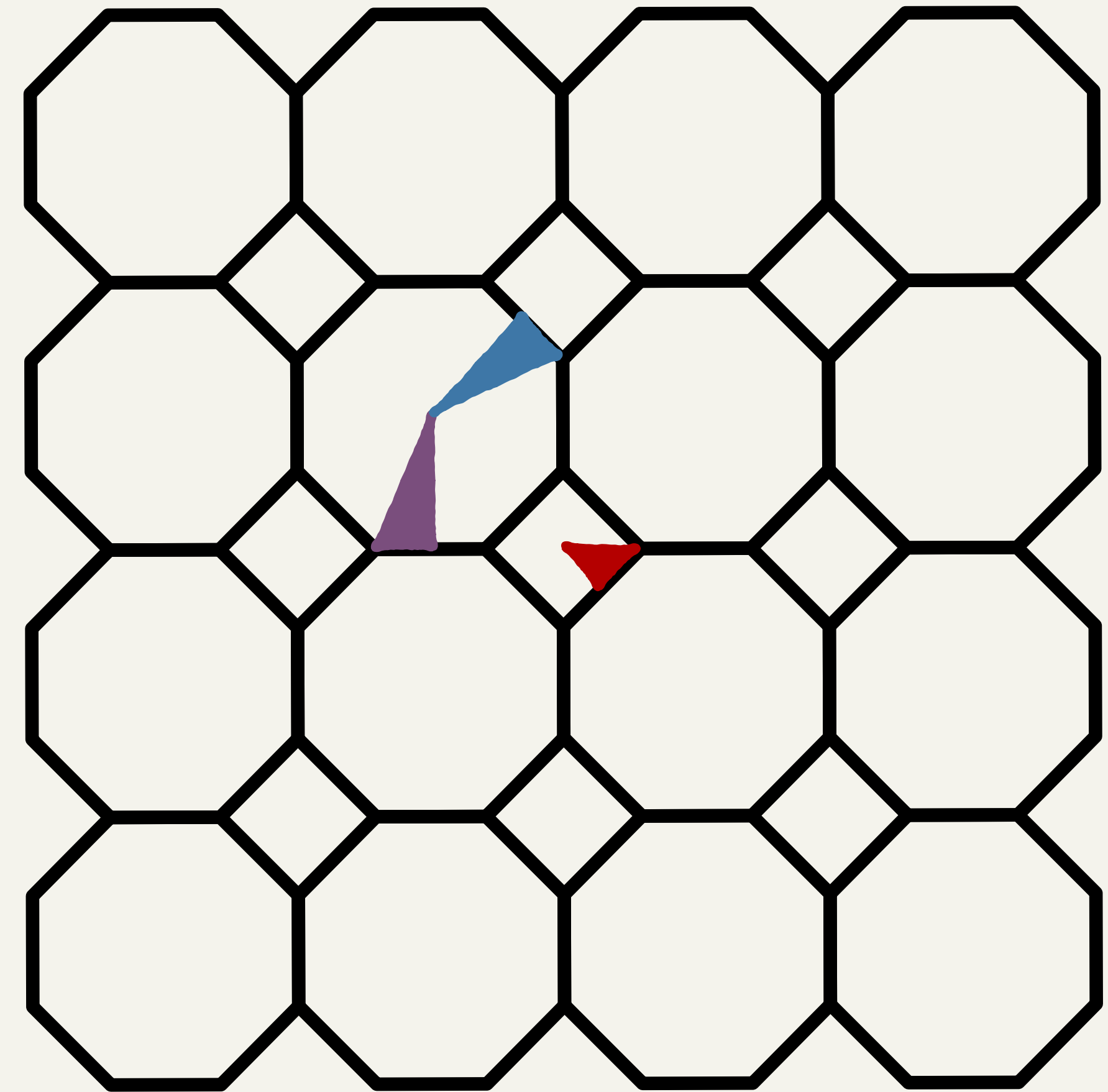


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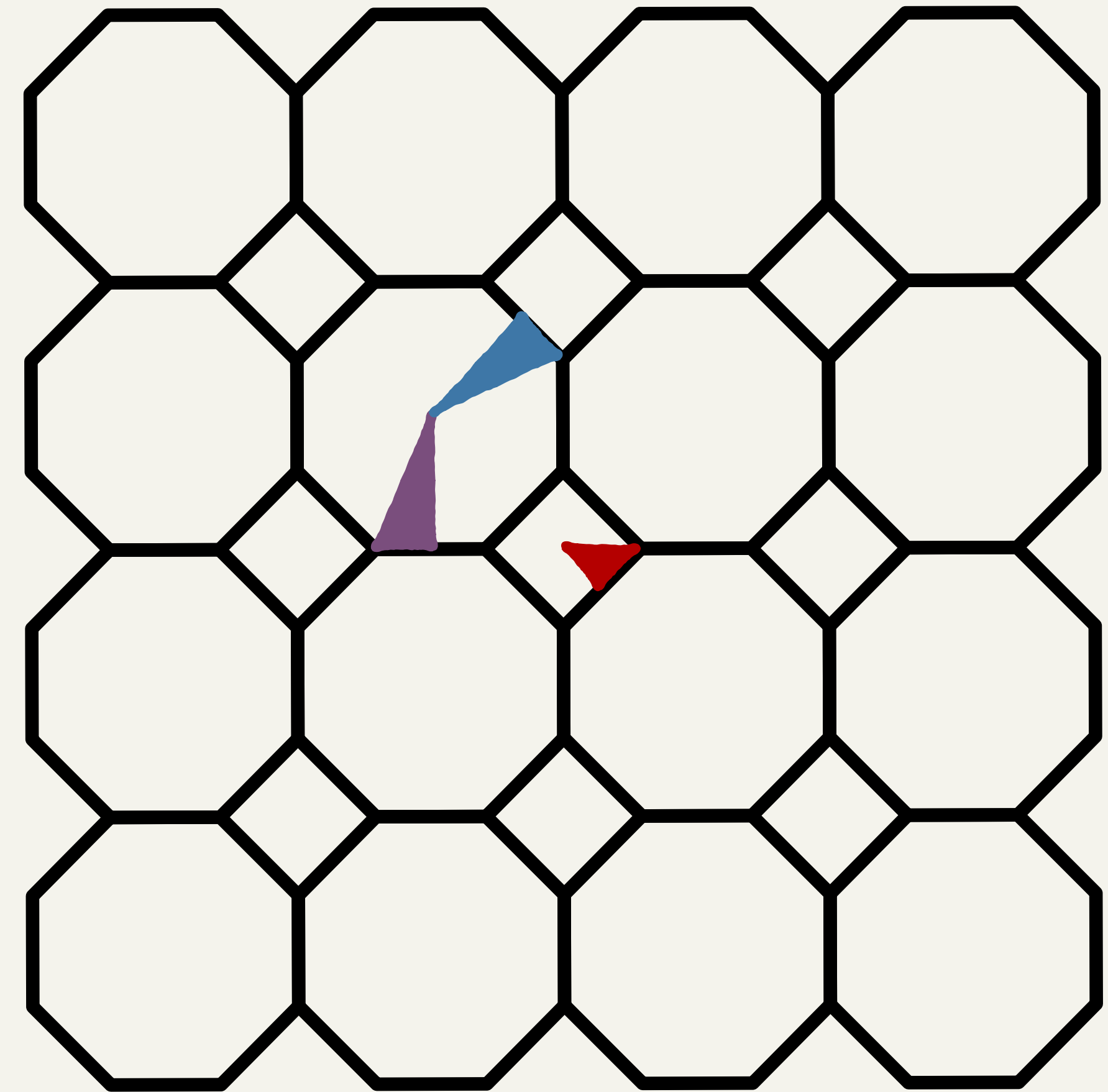
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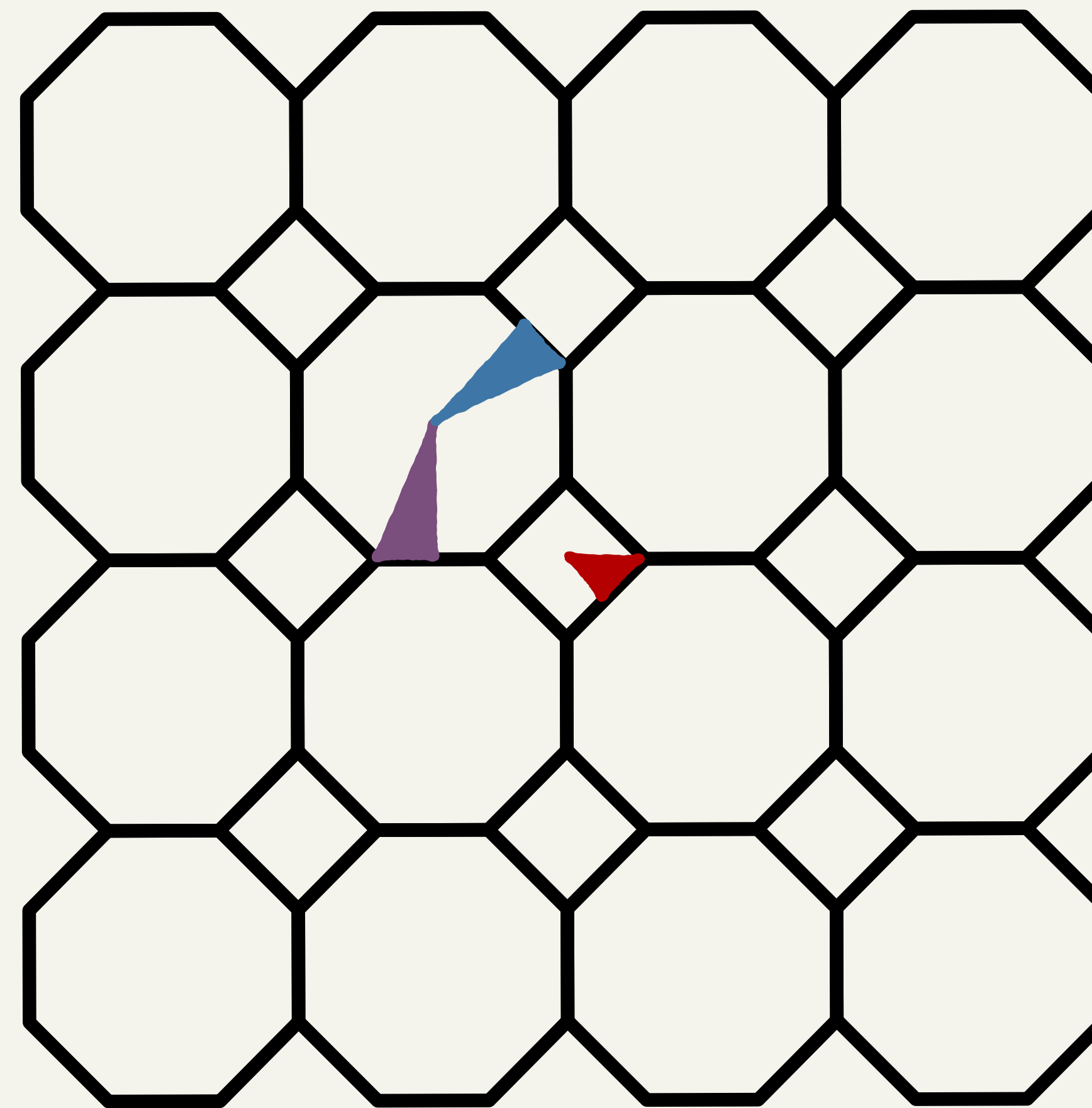
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In general, we say that a tessellation is  **$k$ -orbit** if it has  $k$  orbits on flags.



# Equivelarity vs Regularity

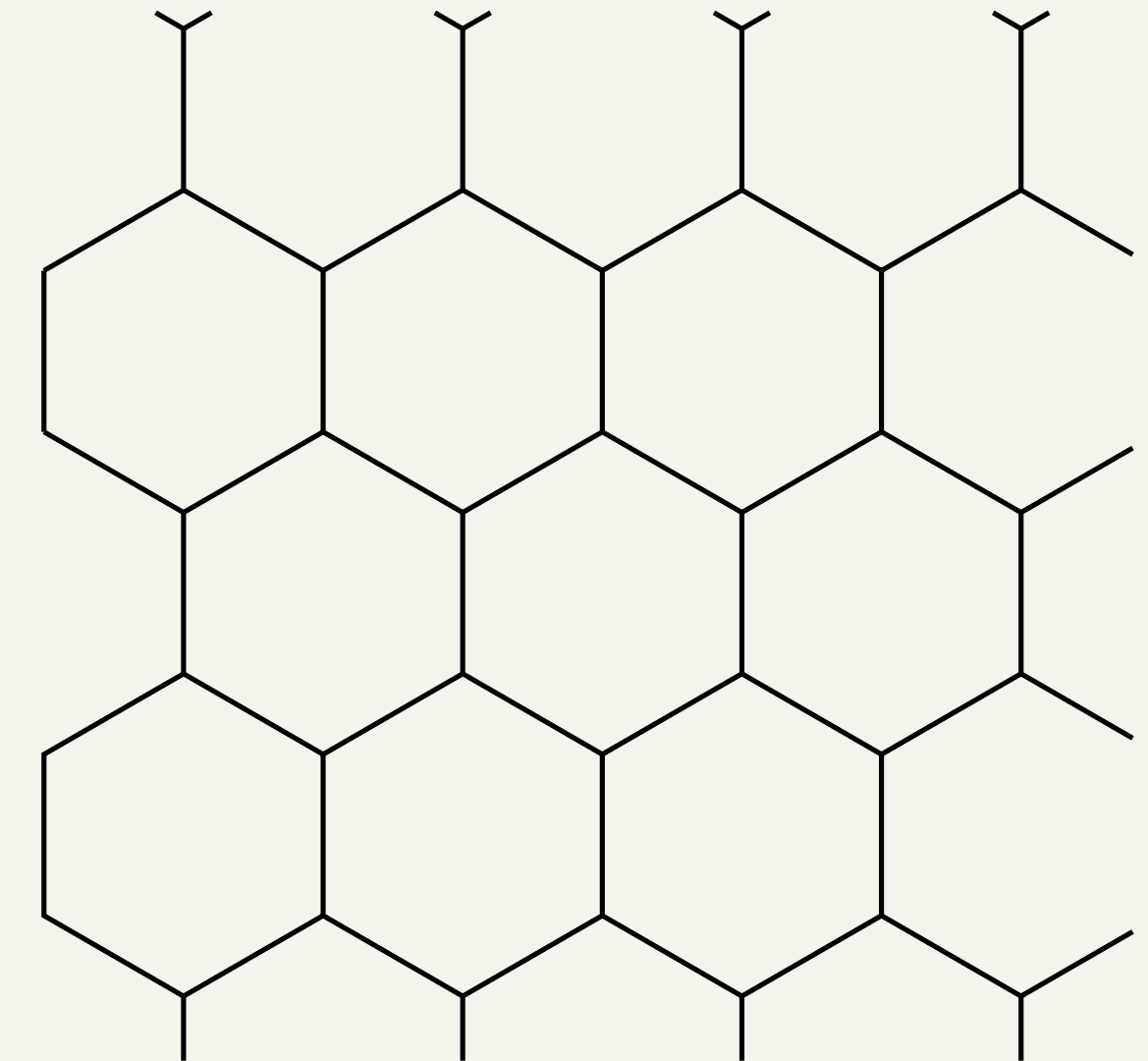
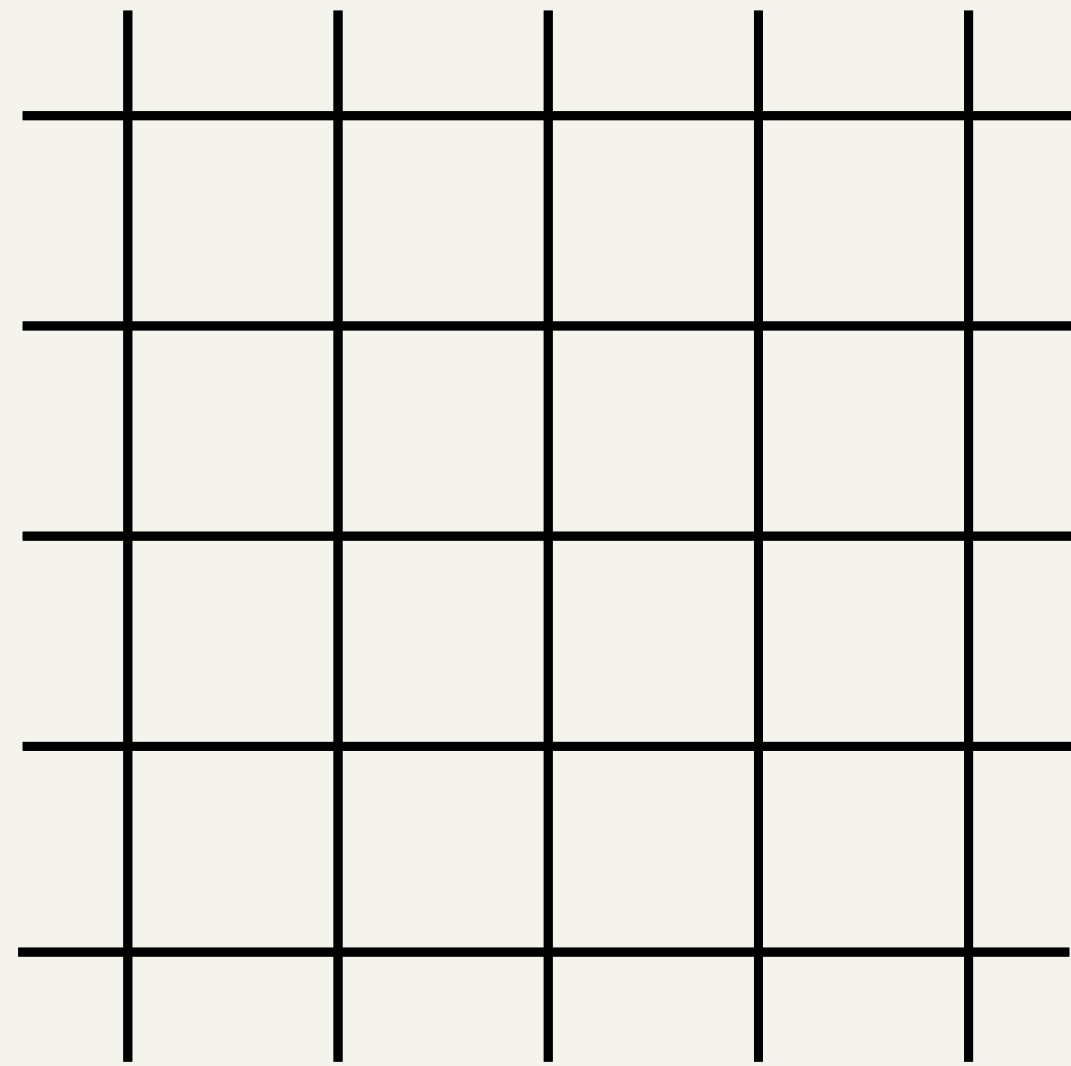
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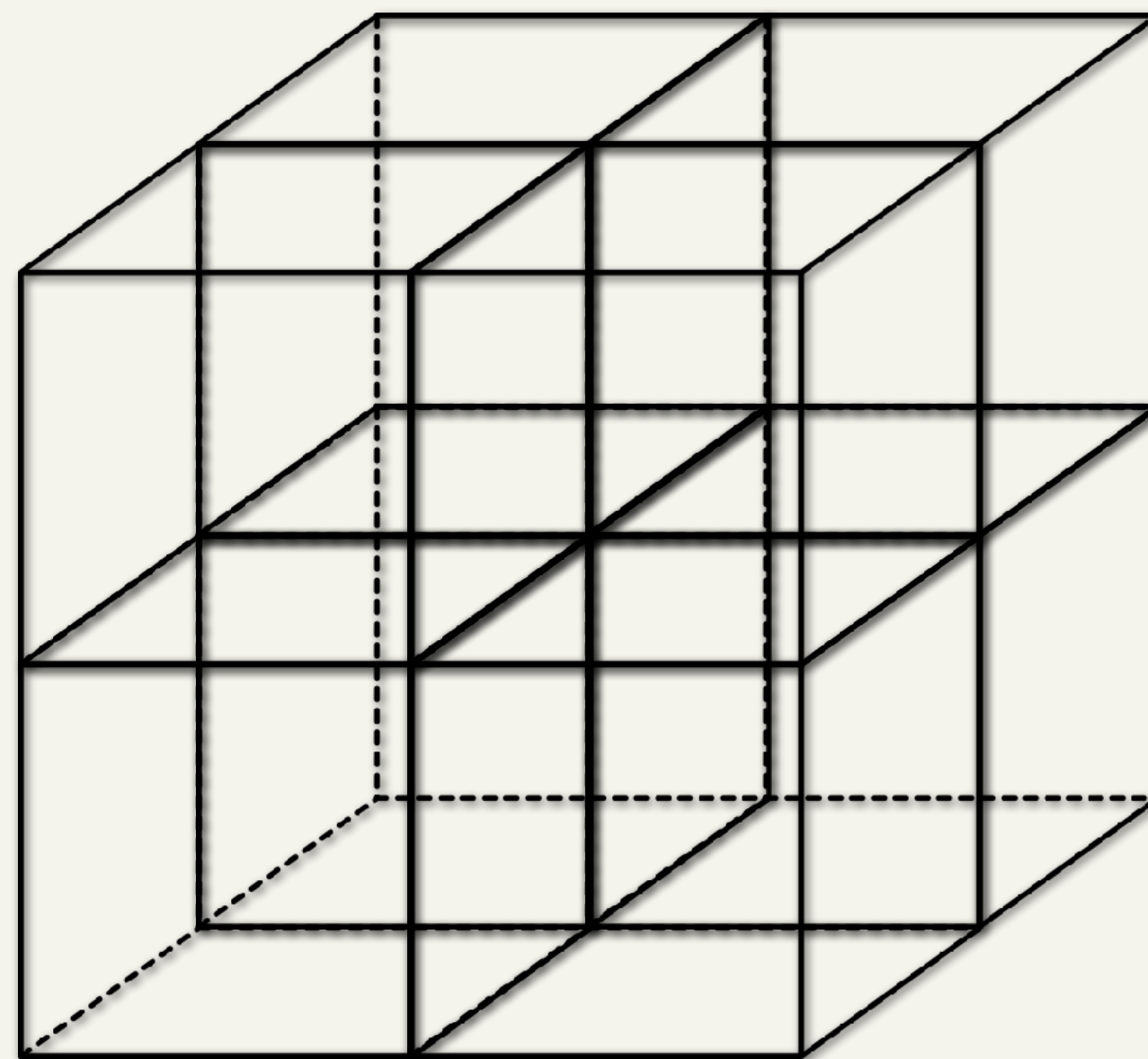


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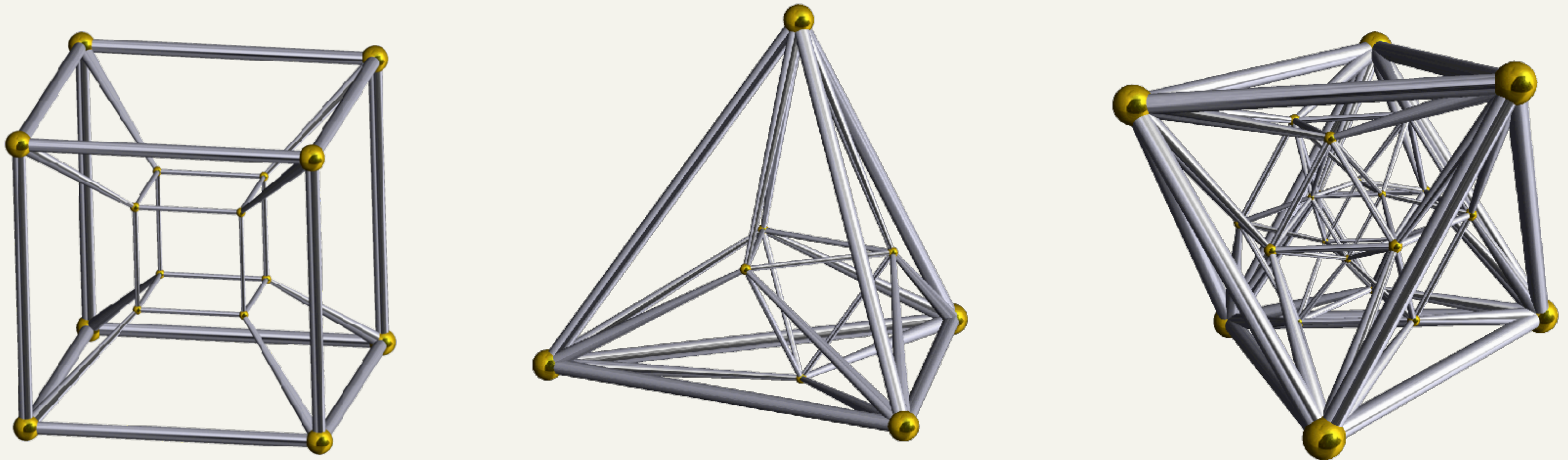


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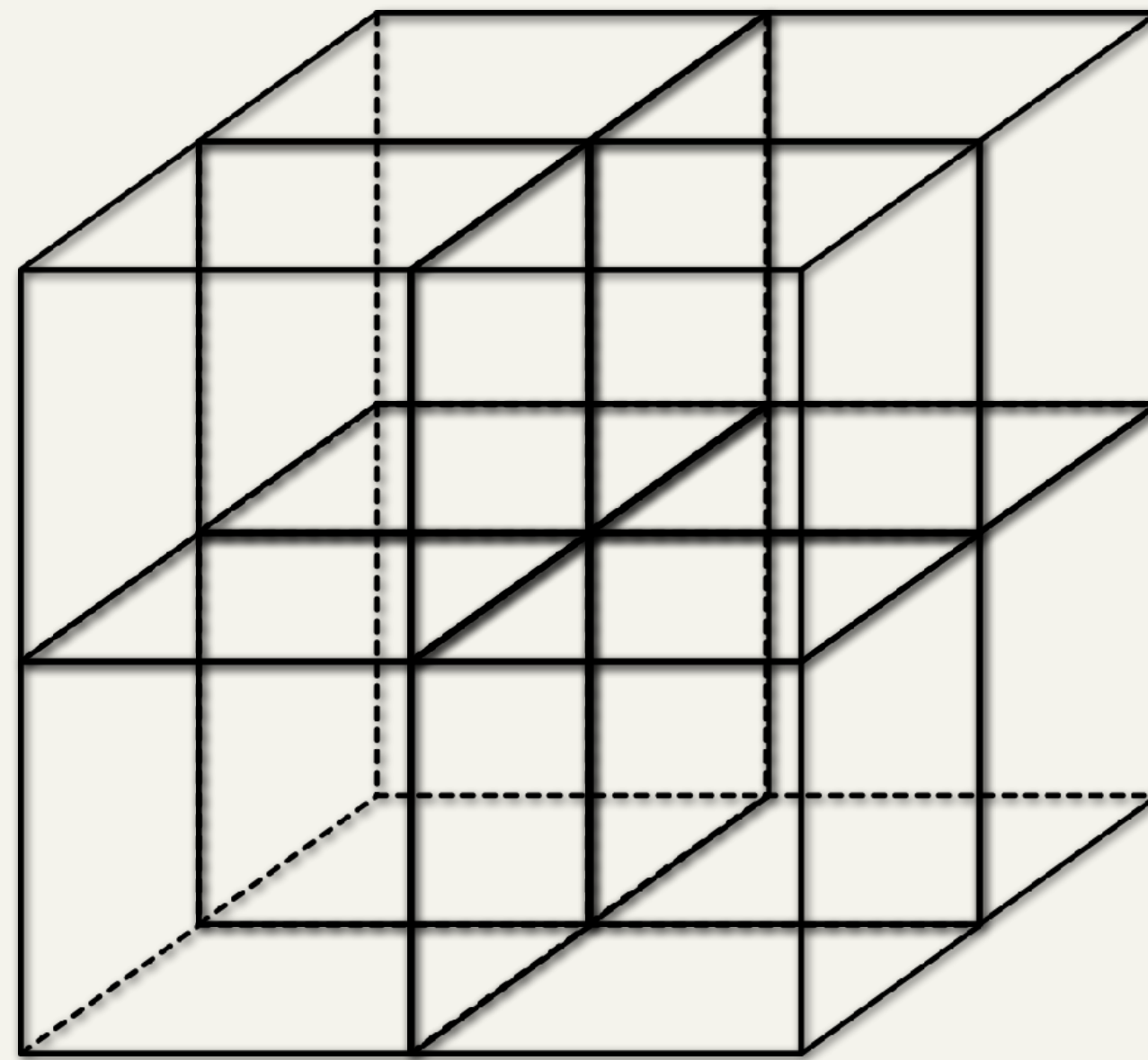


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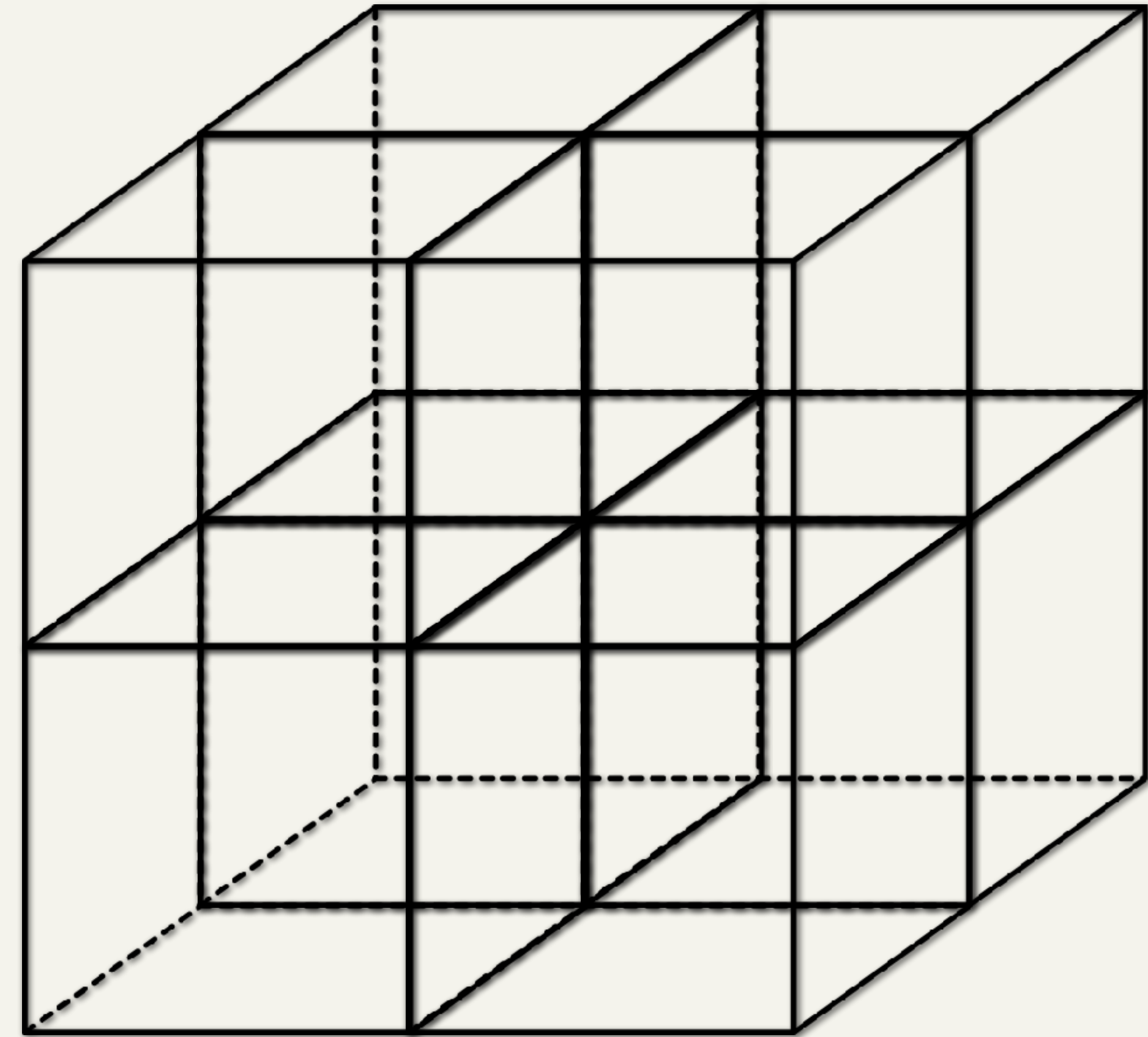
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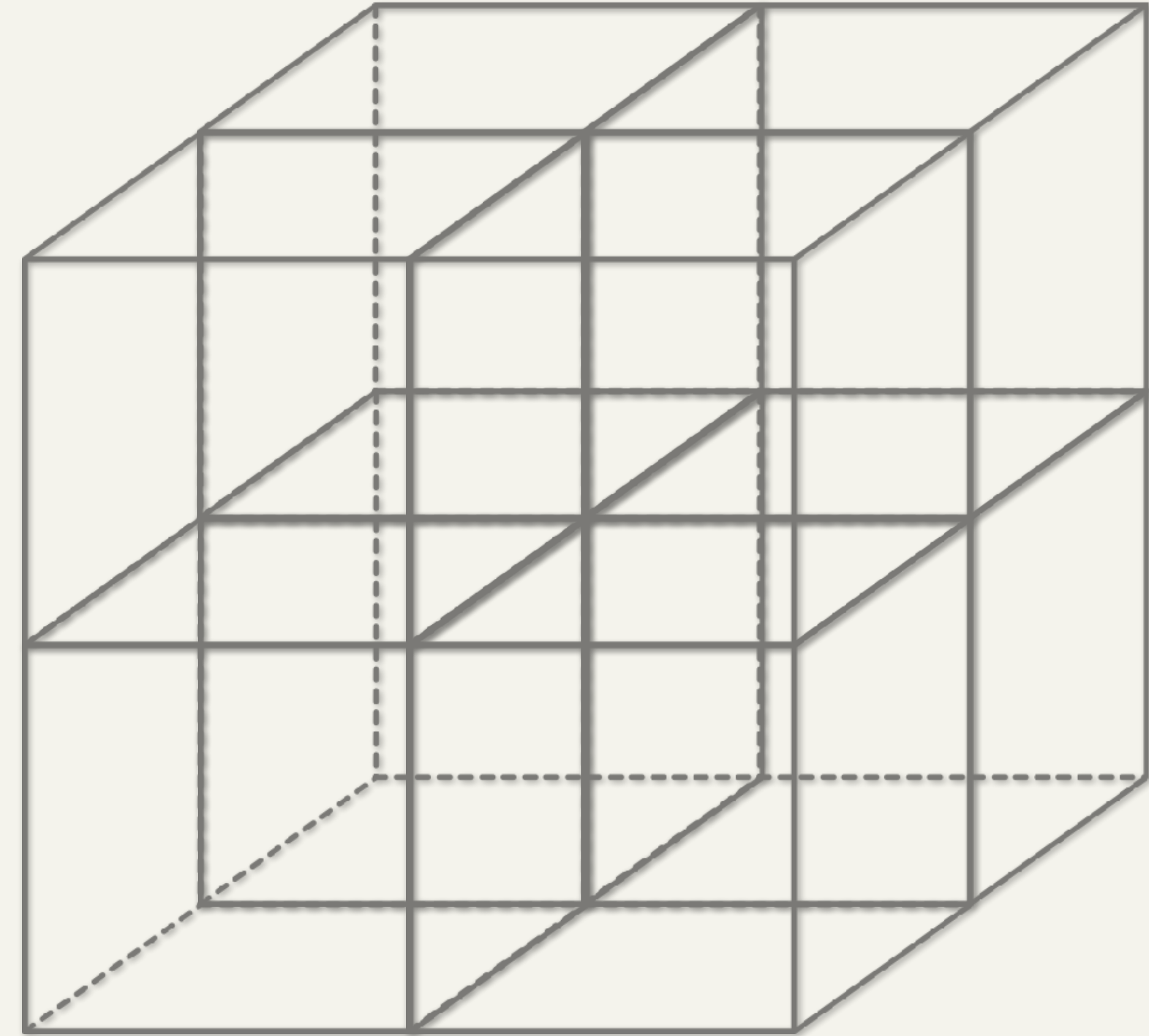
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# The cubic tessellation $\mathcal{U}$



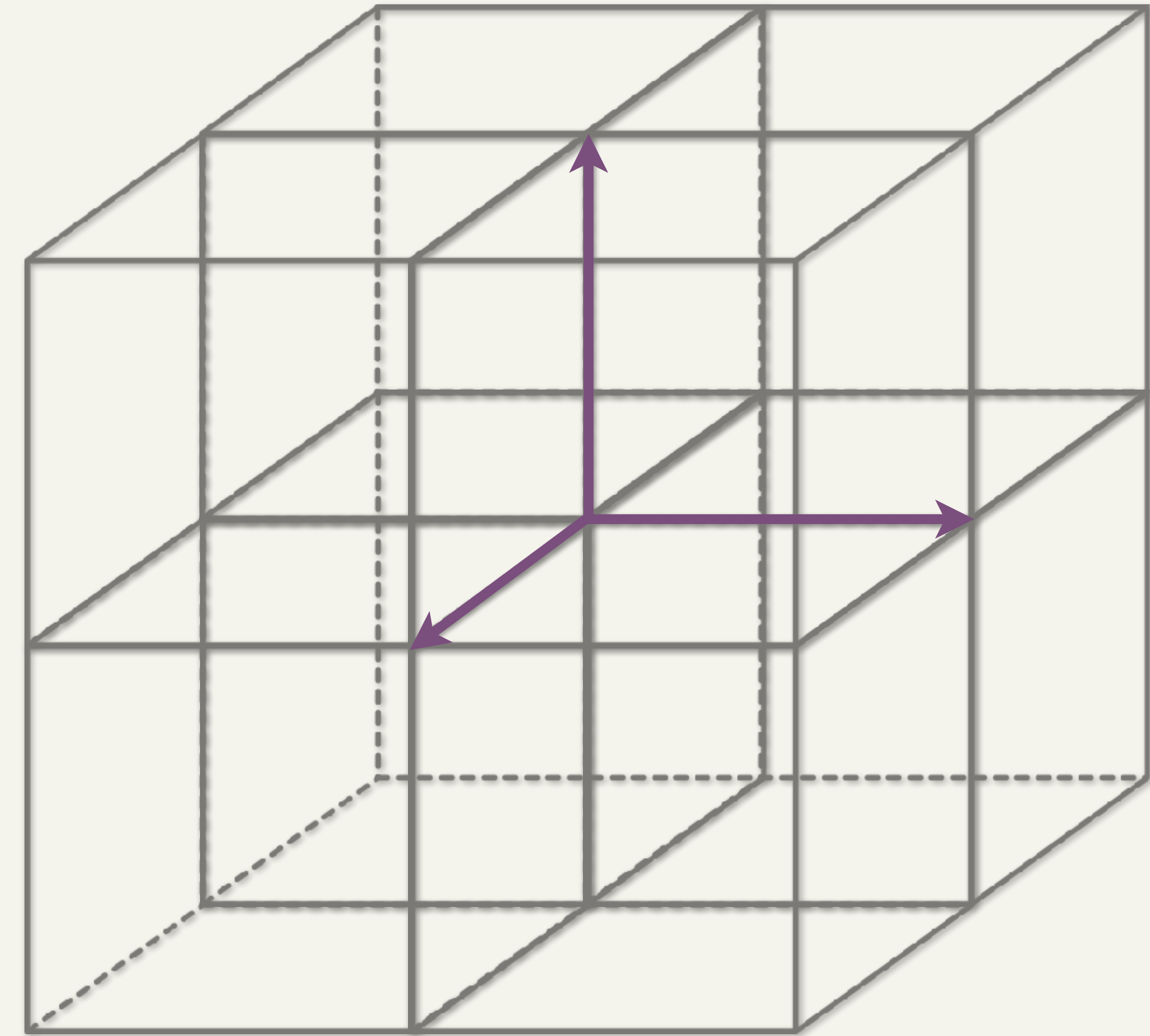
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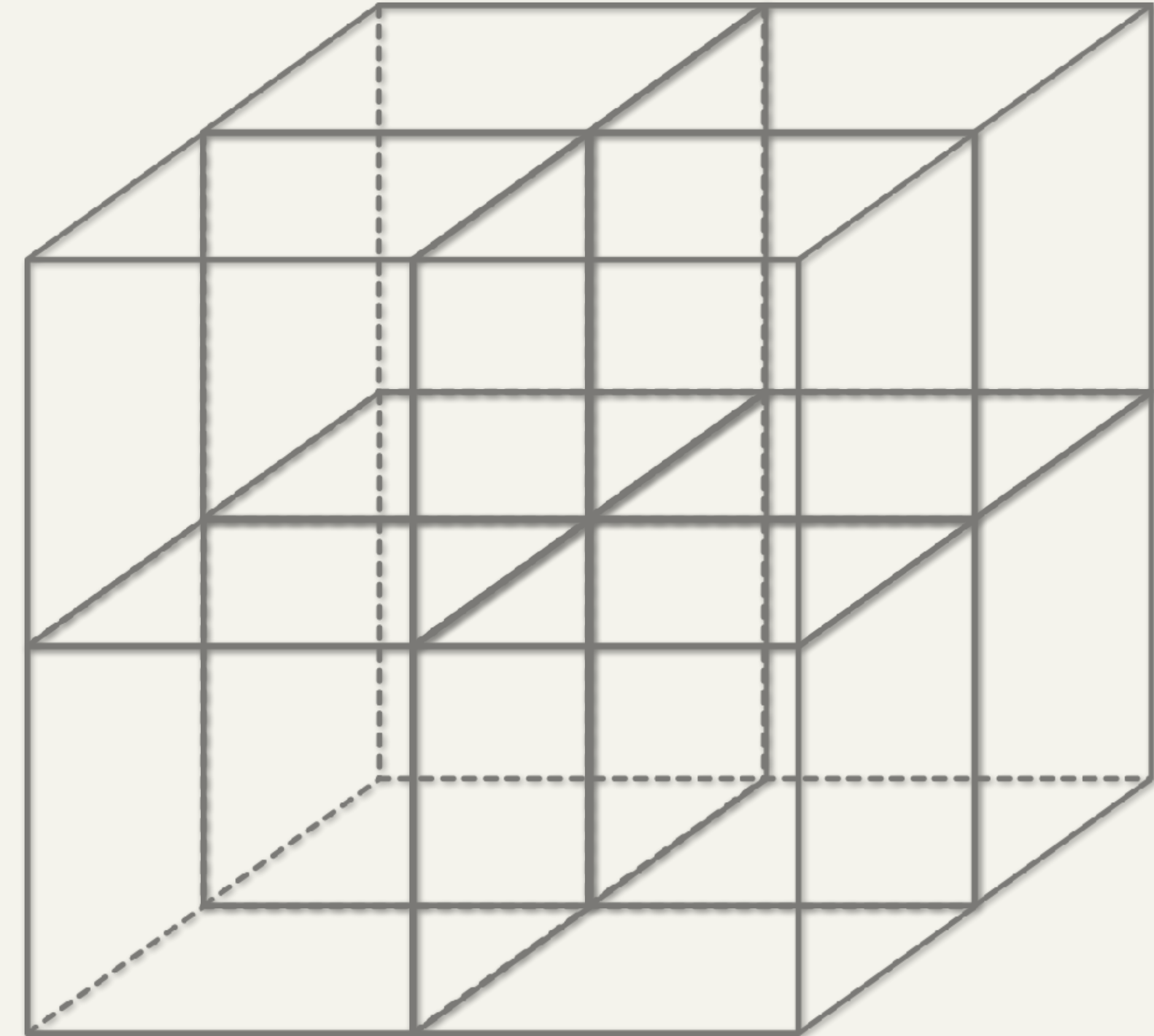
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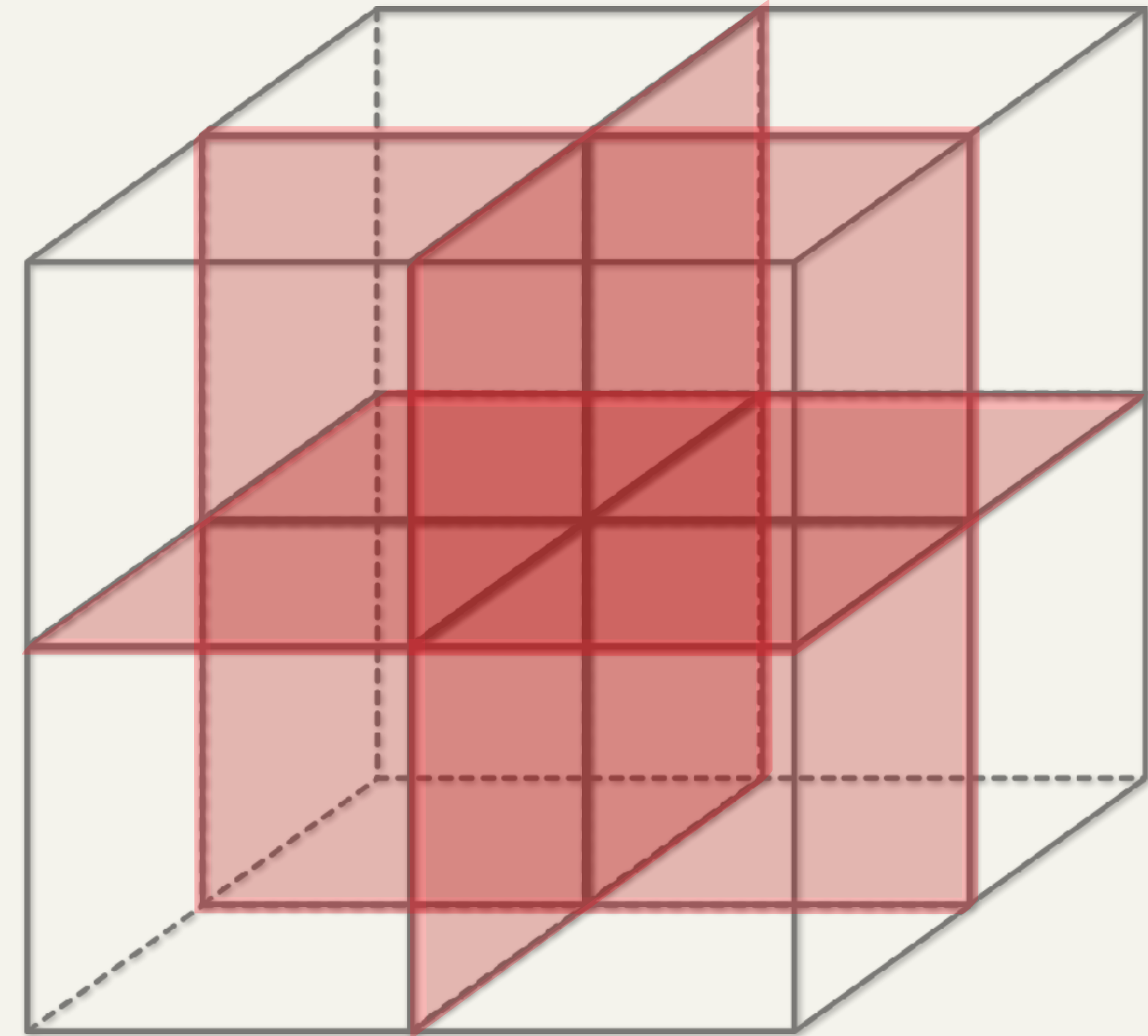
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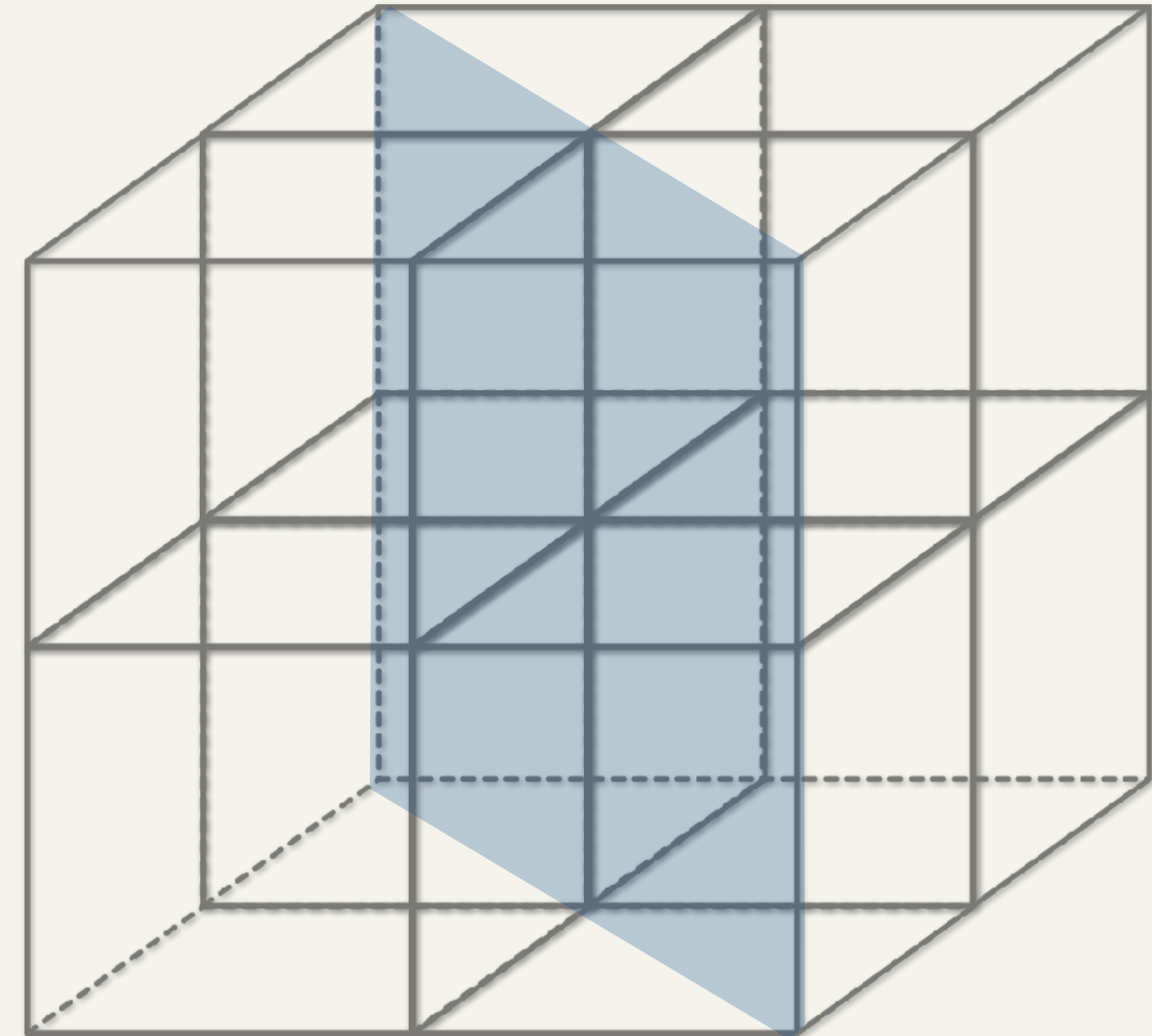
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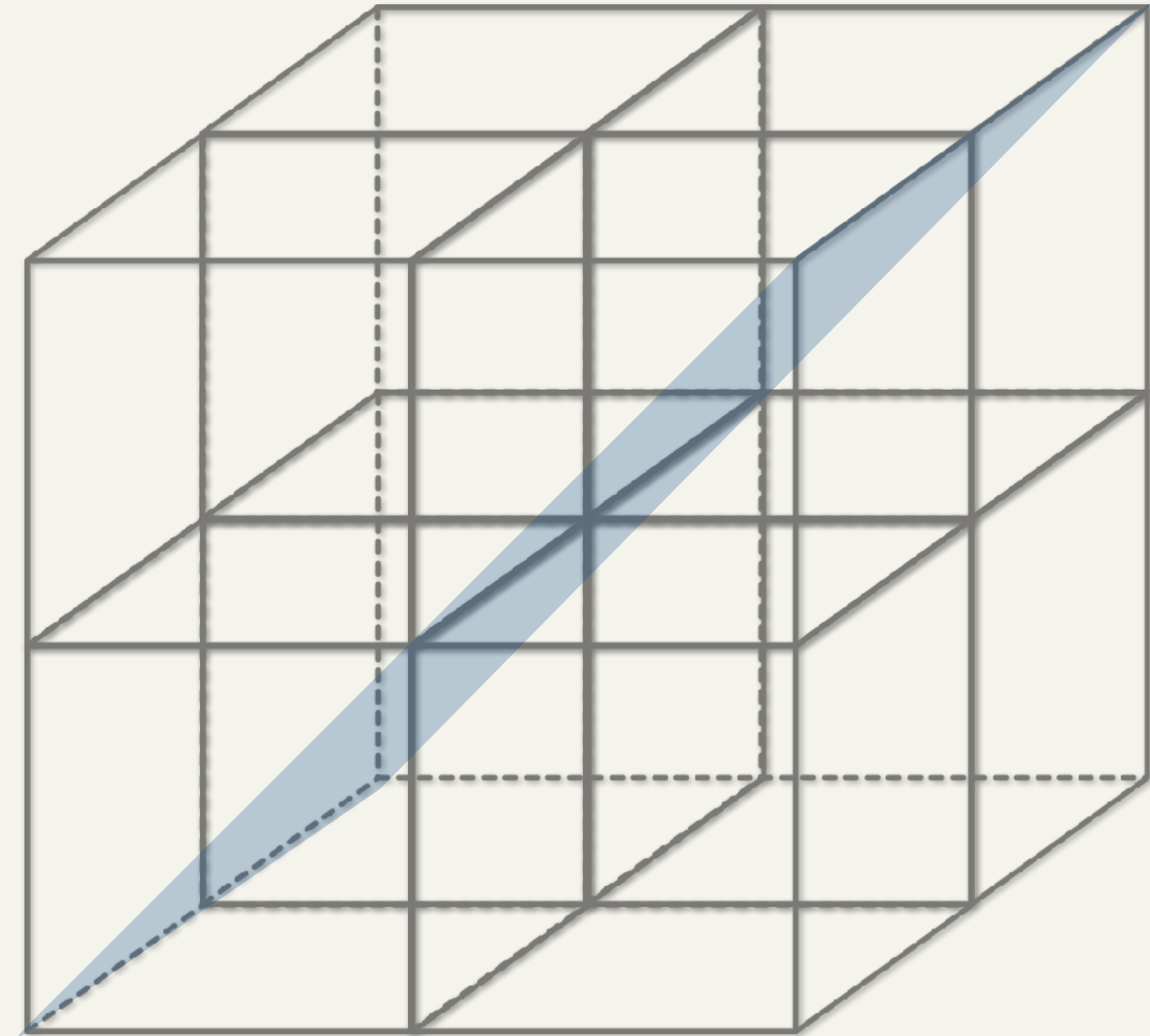
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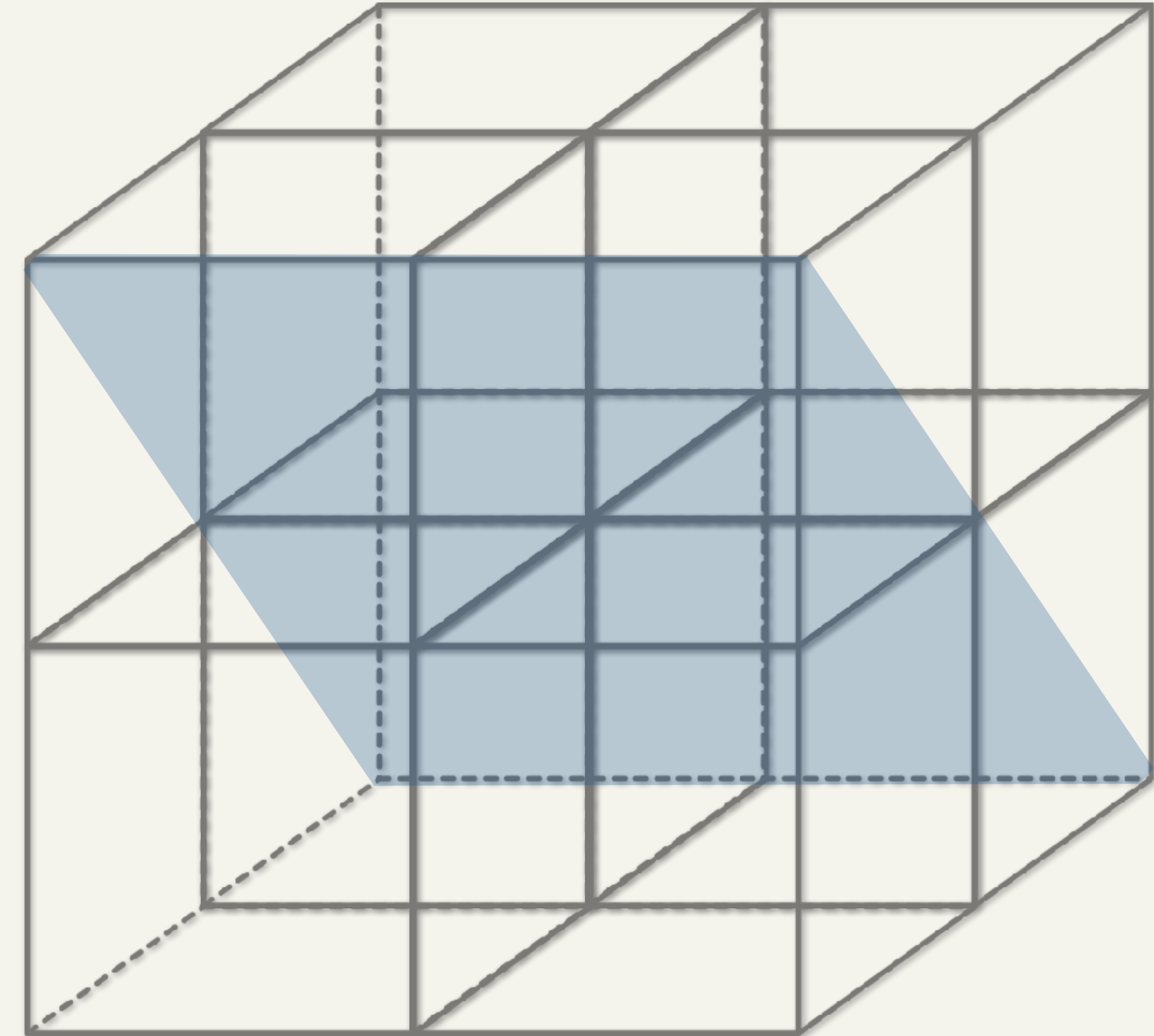
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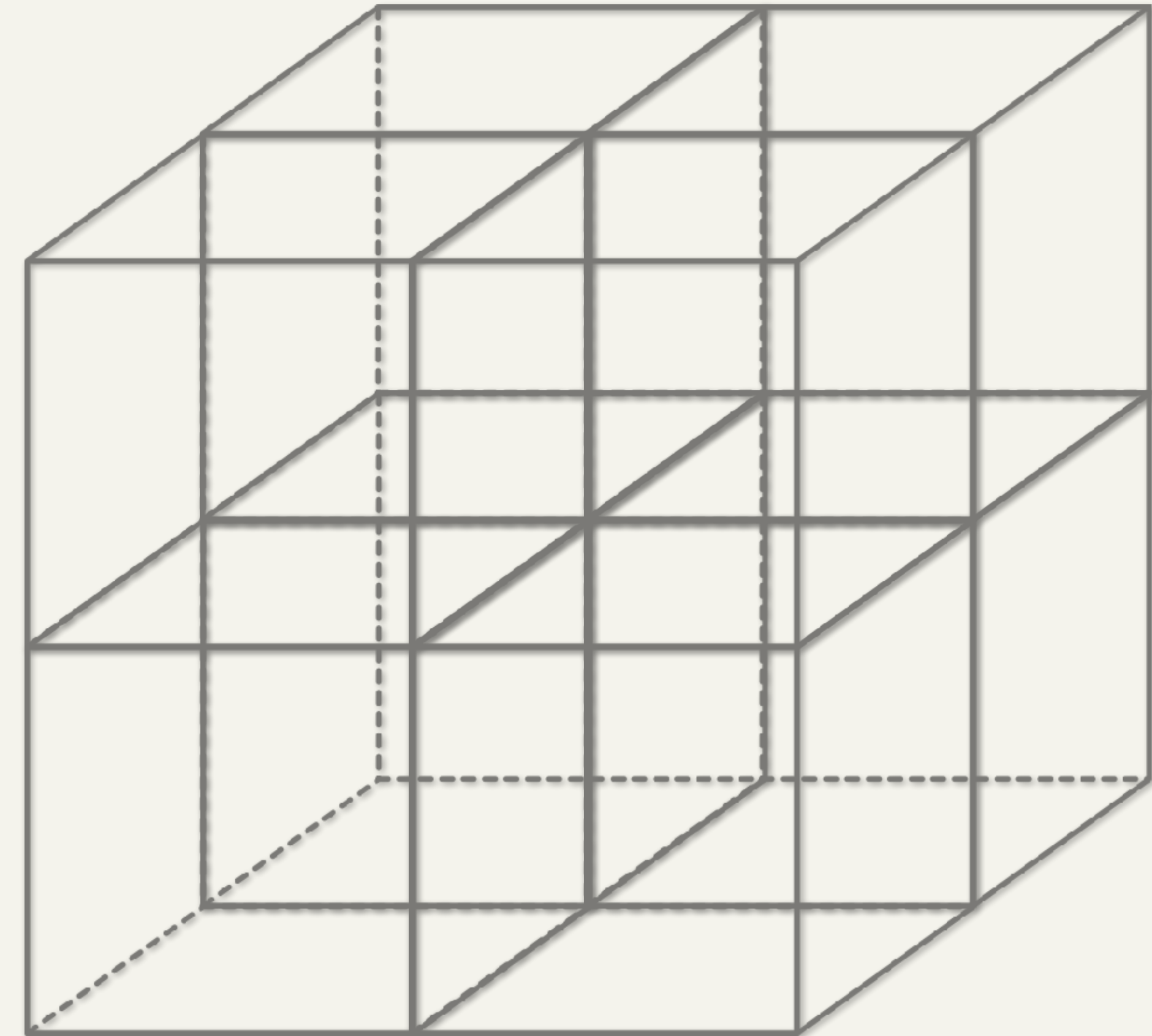
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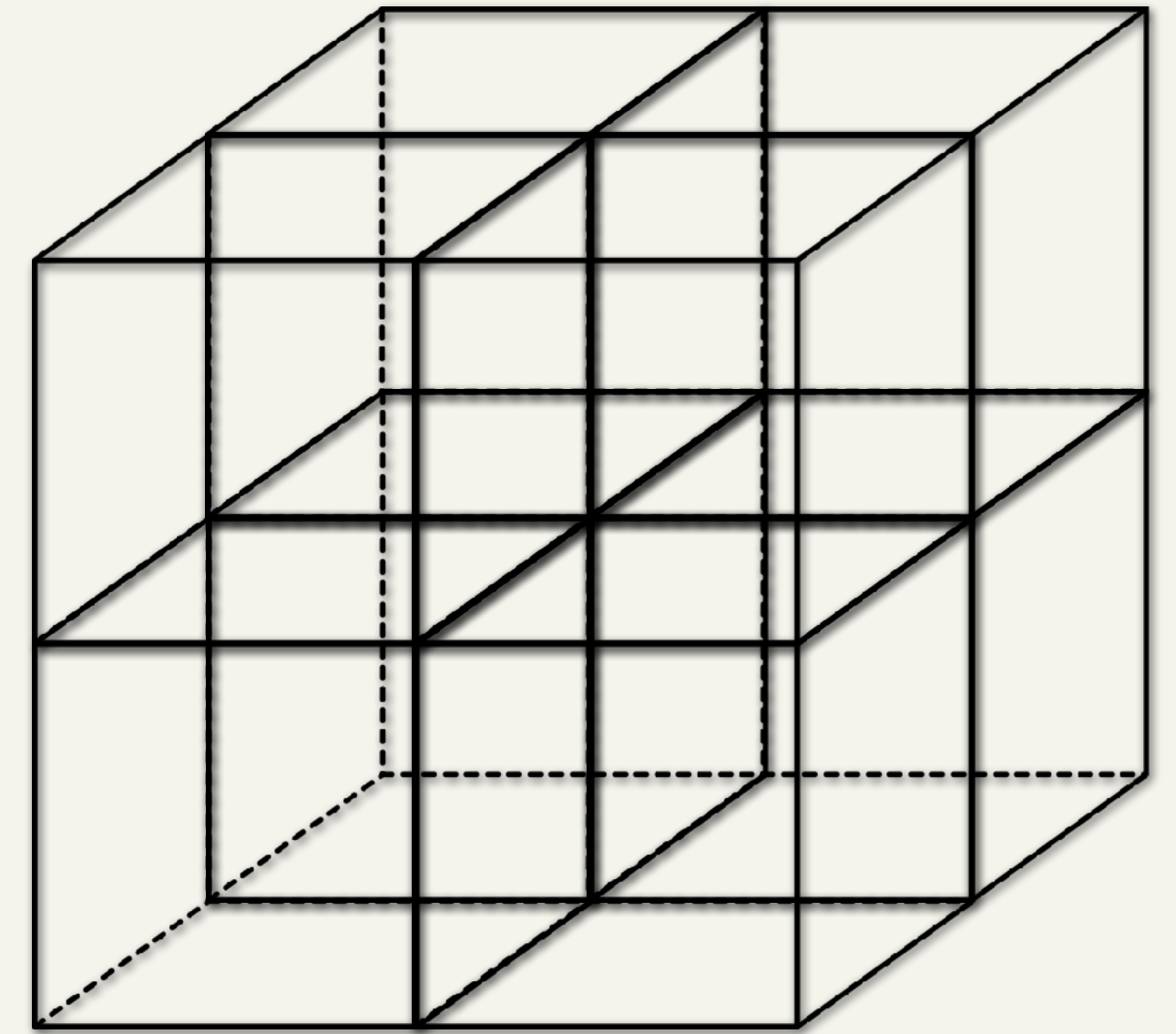
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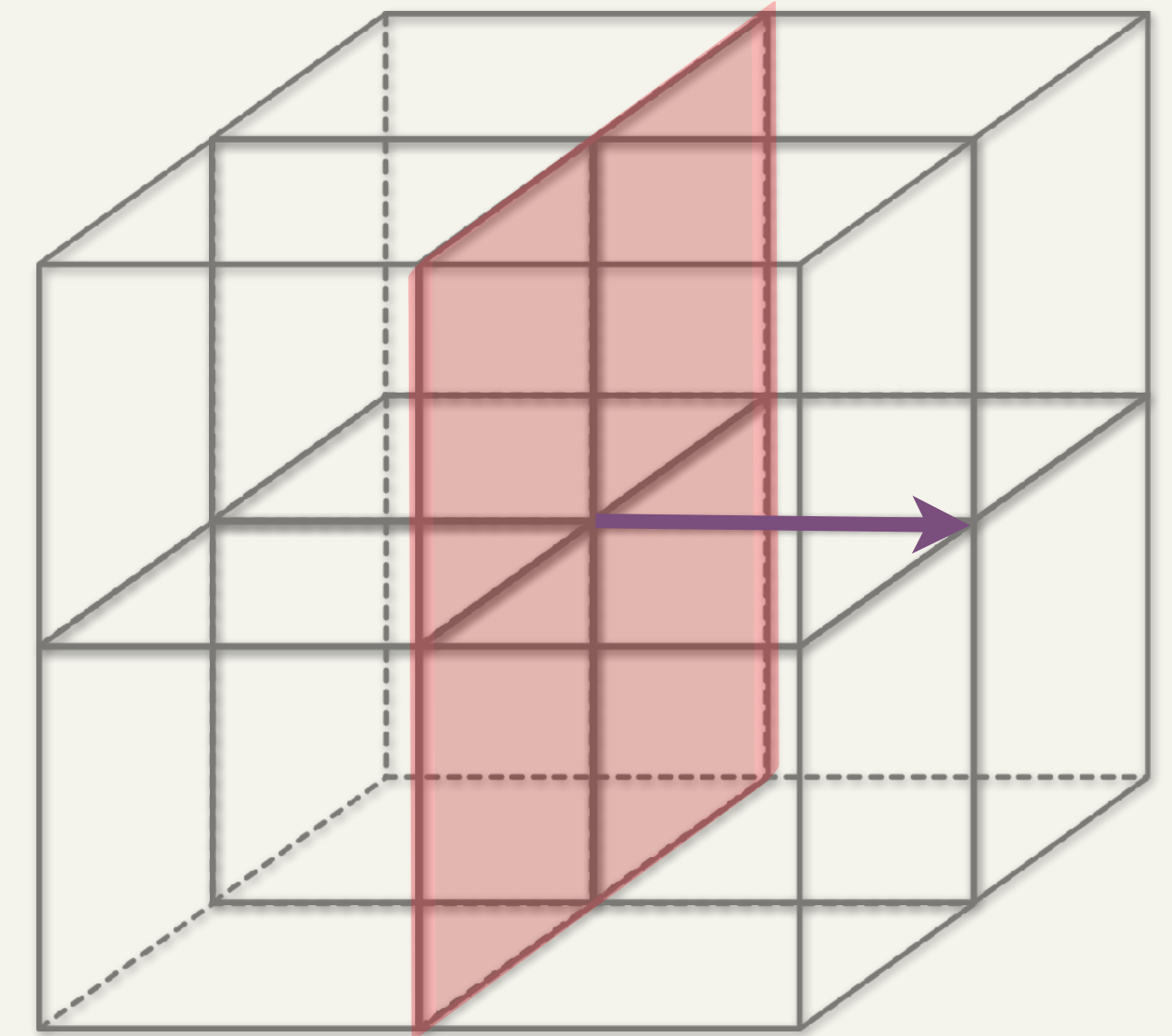
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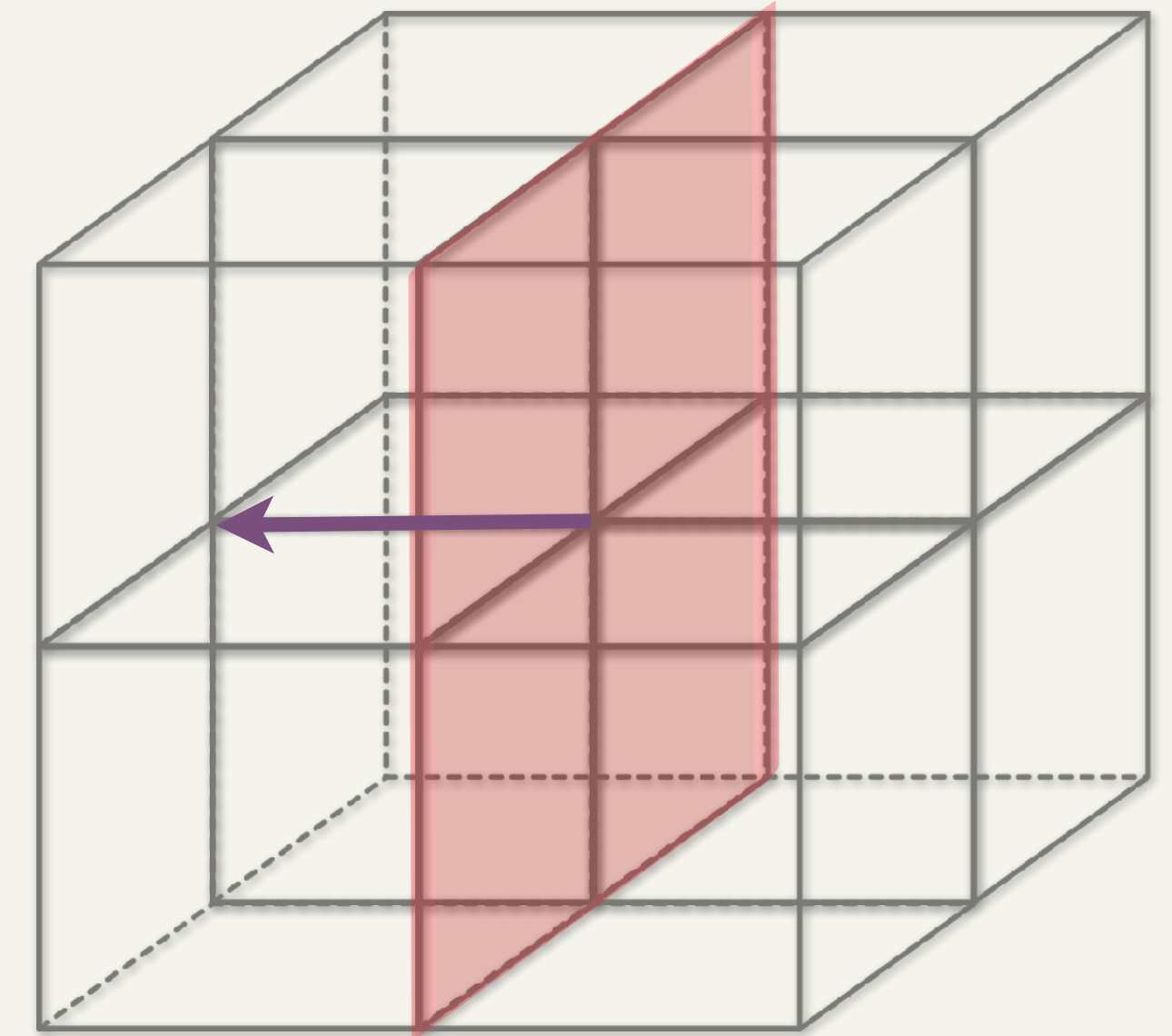
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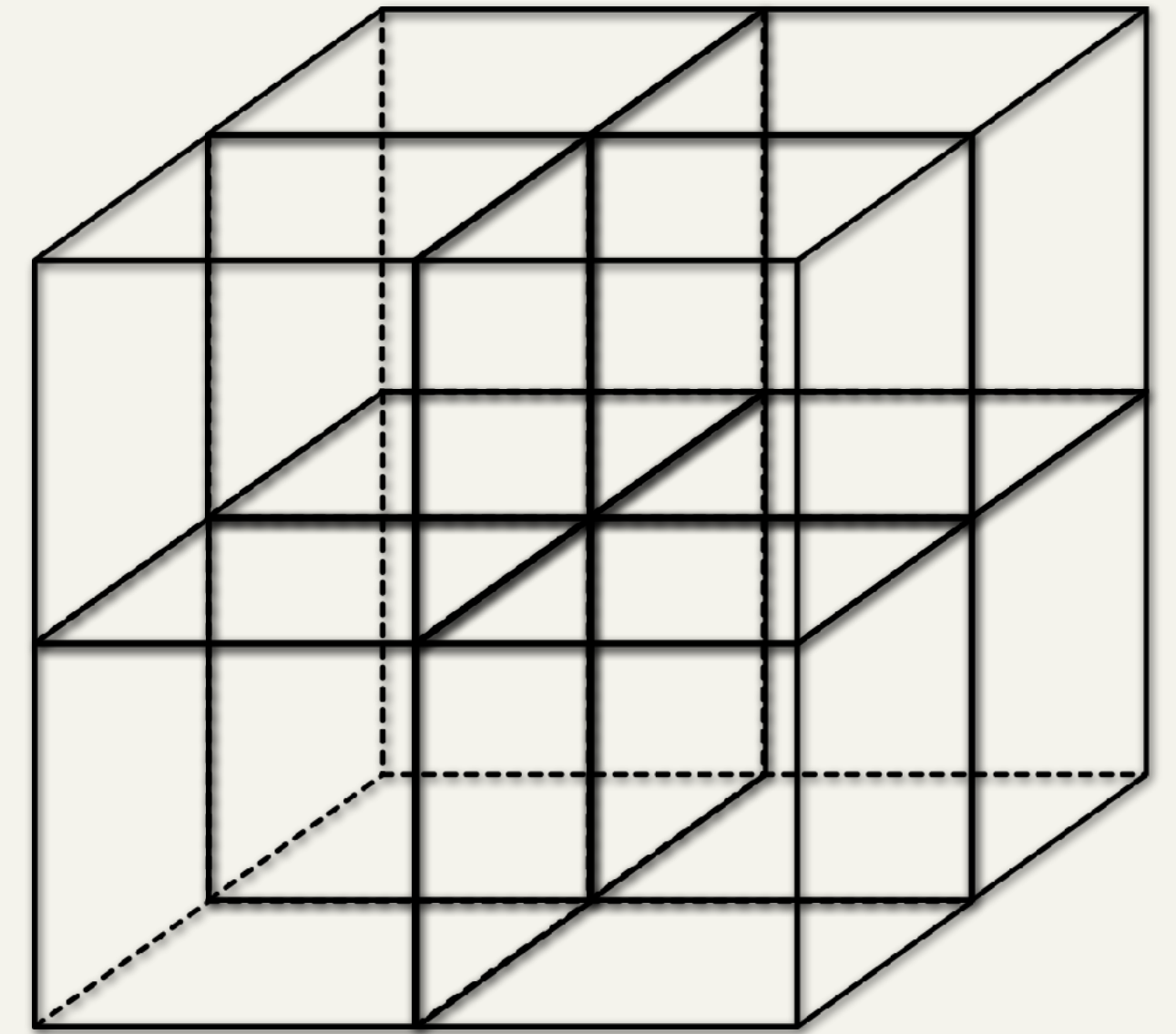
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$$u_i = \begin{cases} t_i & \text{if } x_i = 1 \\ -t_i & \text{if } x_i = -1 \end{cases}$$

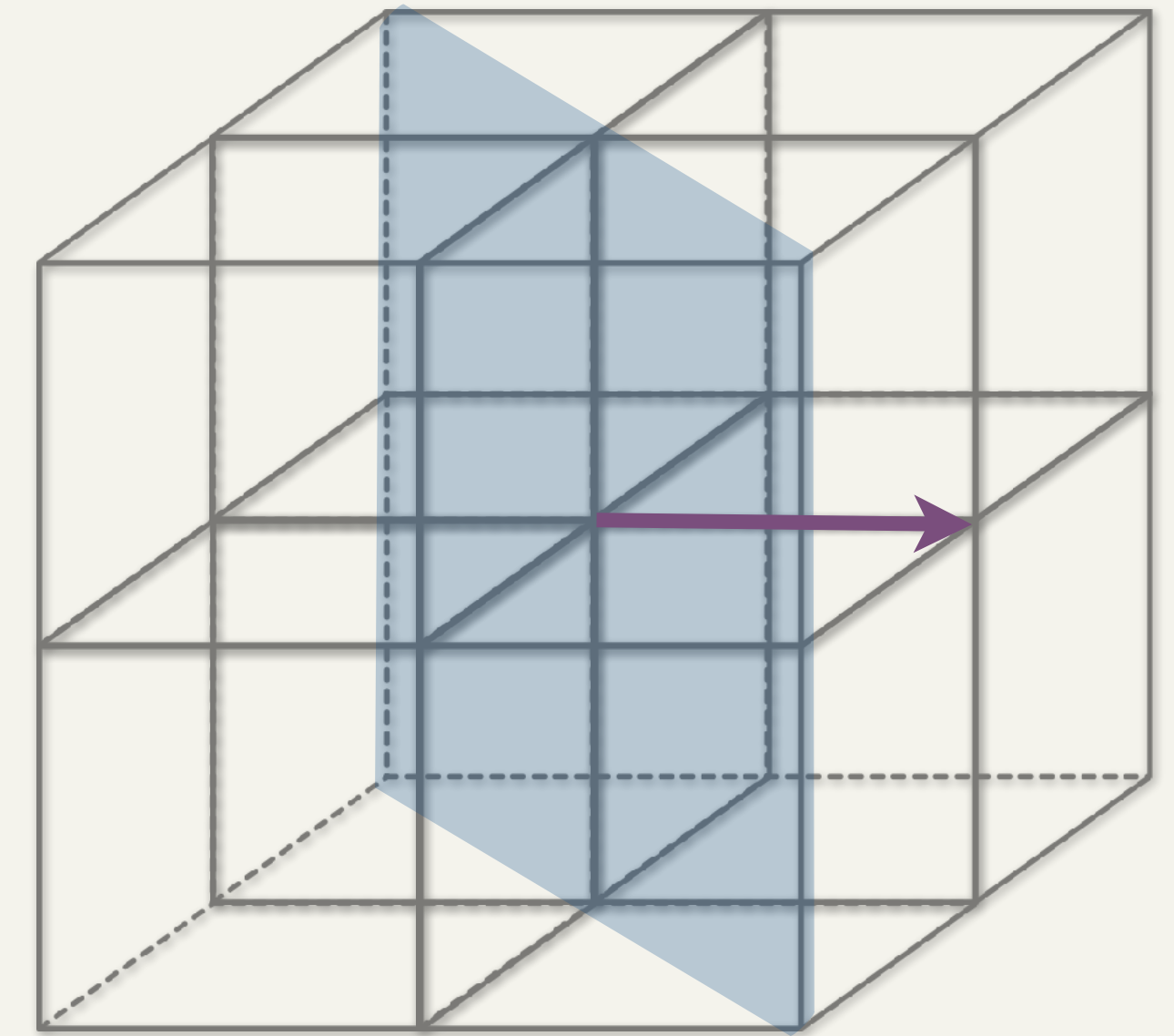
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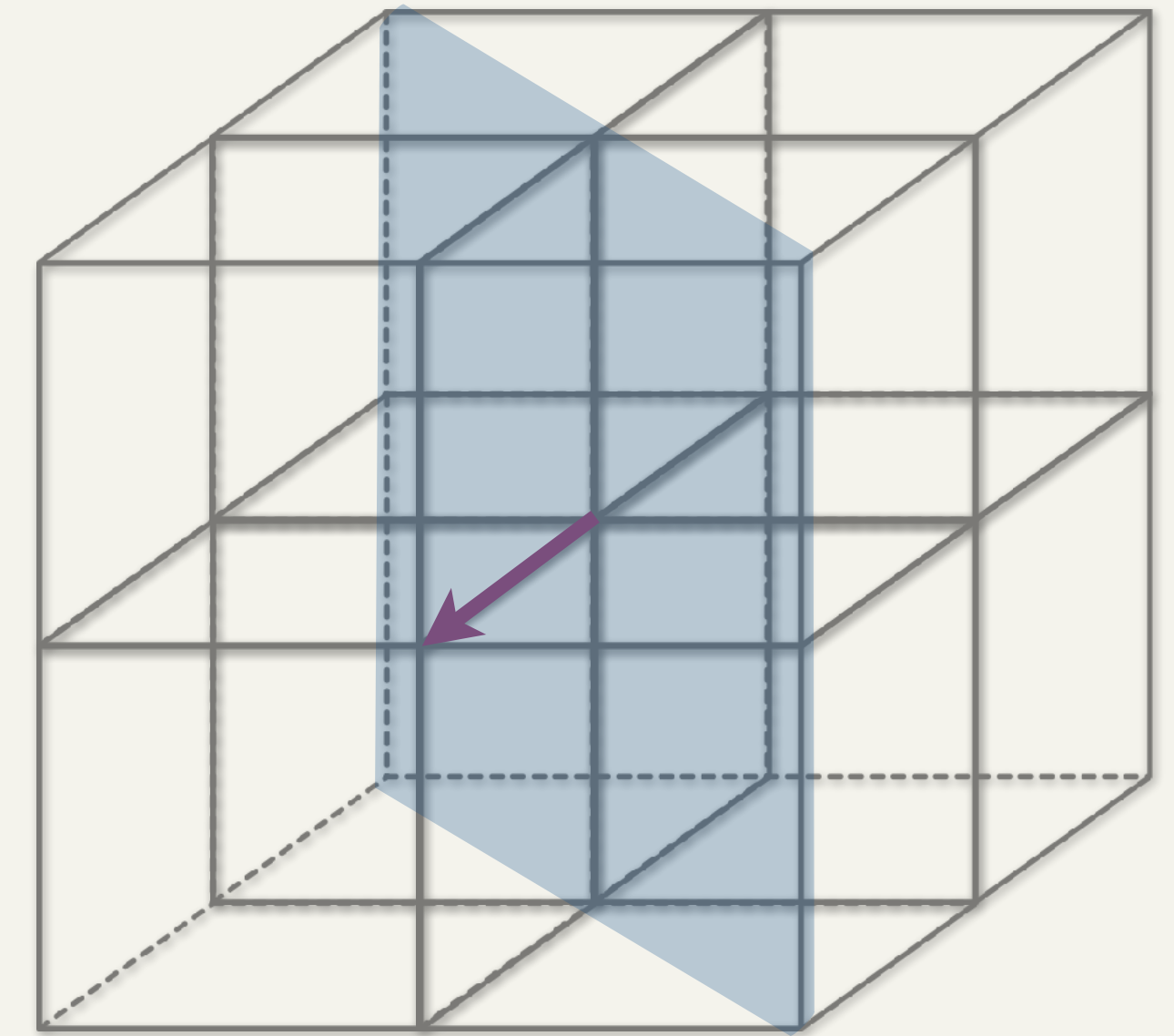


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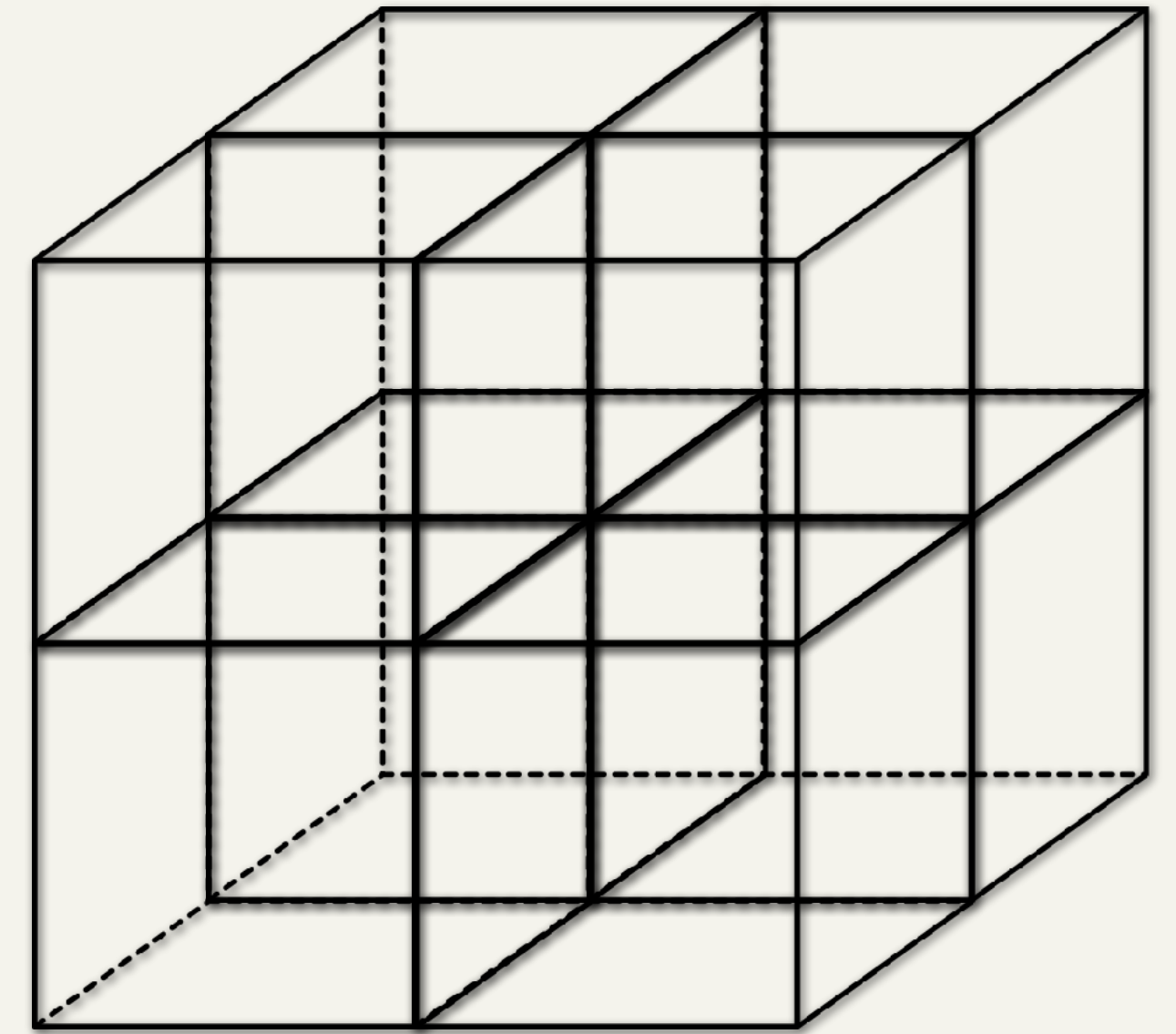


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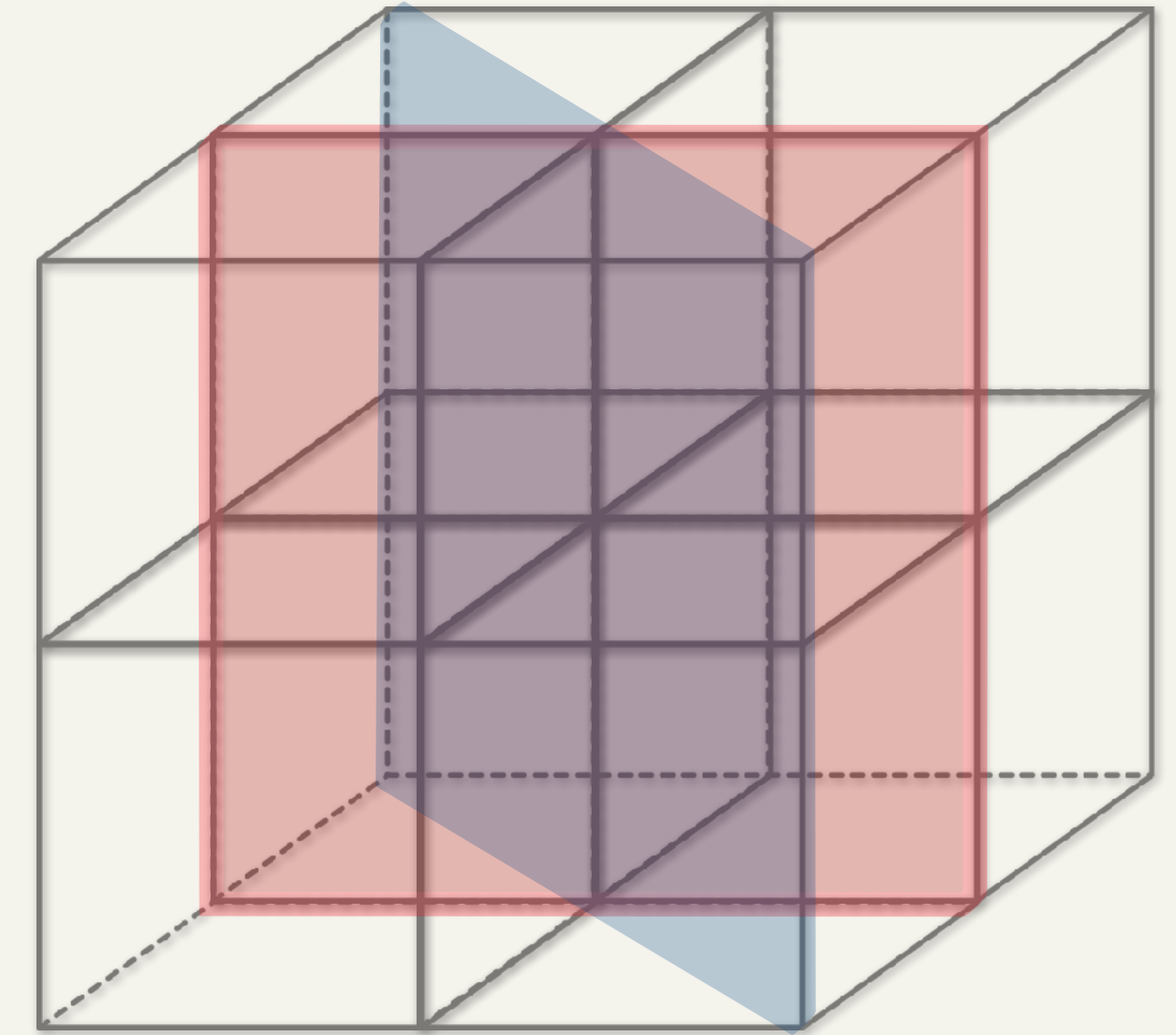
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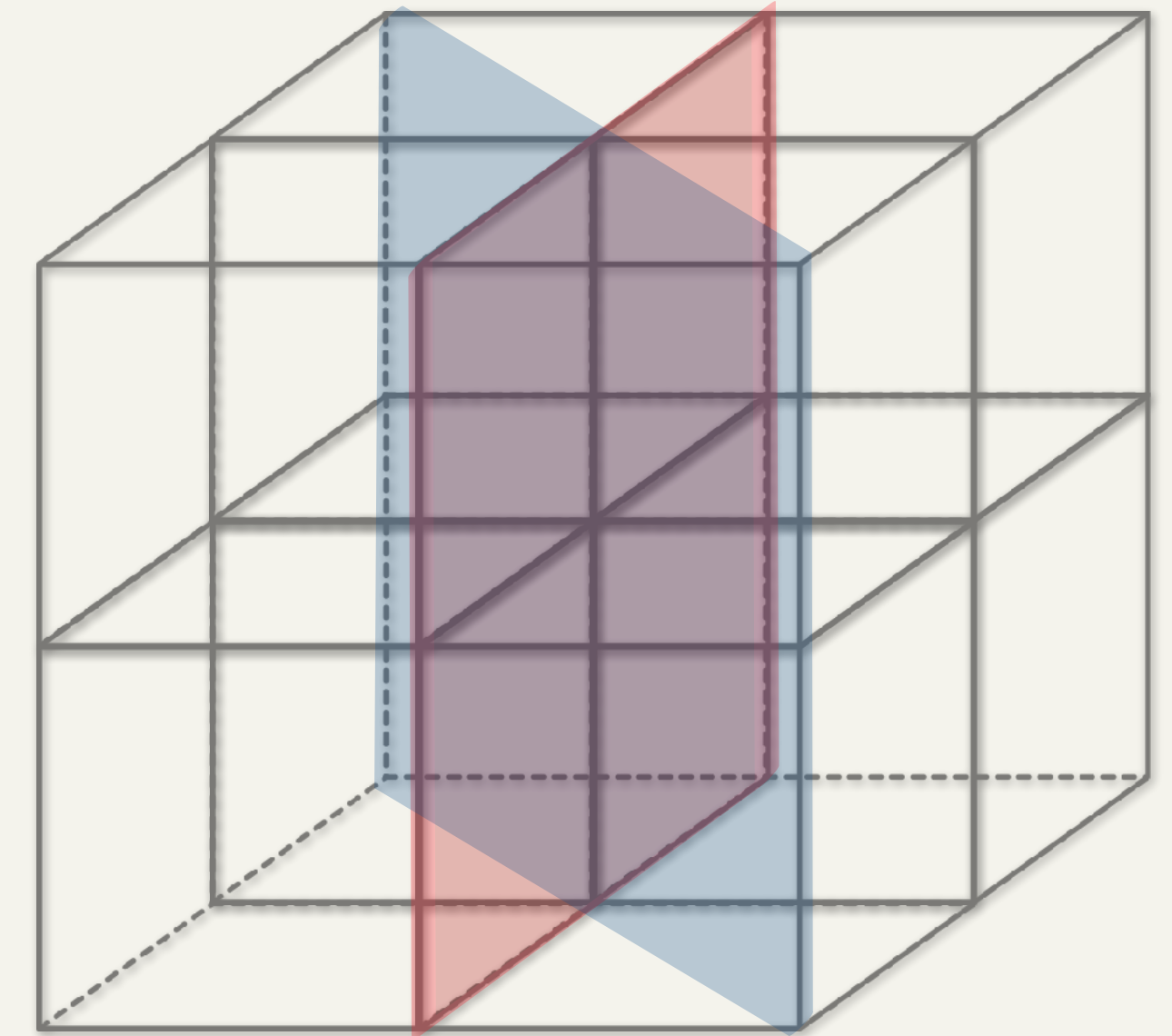
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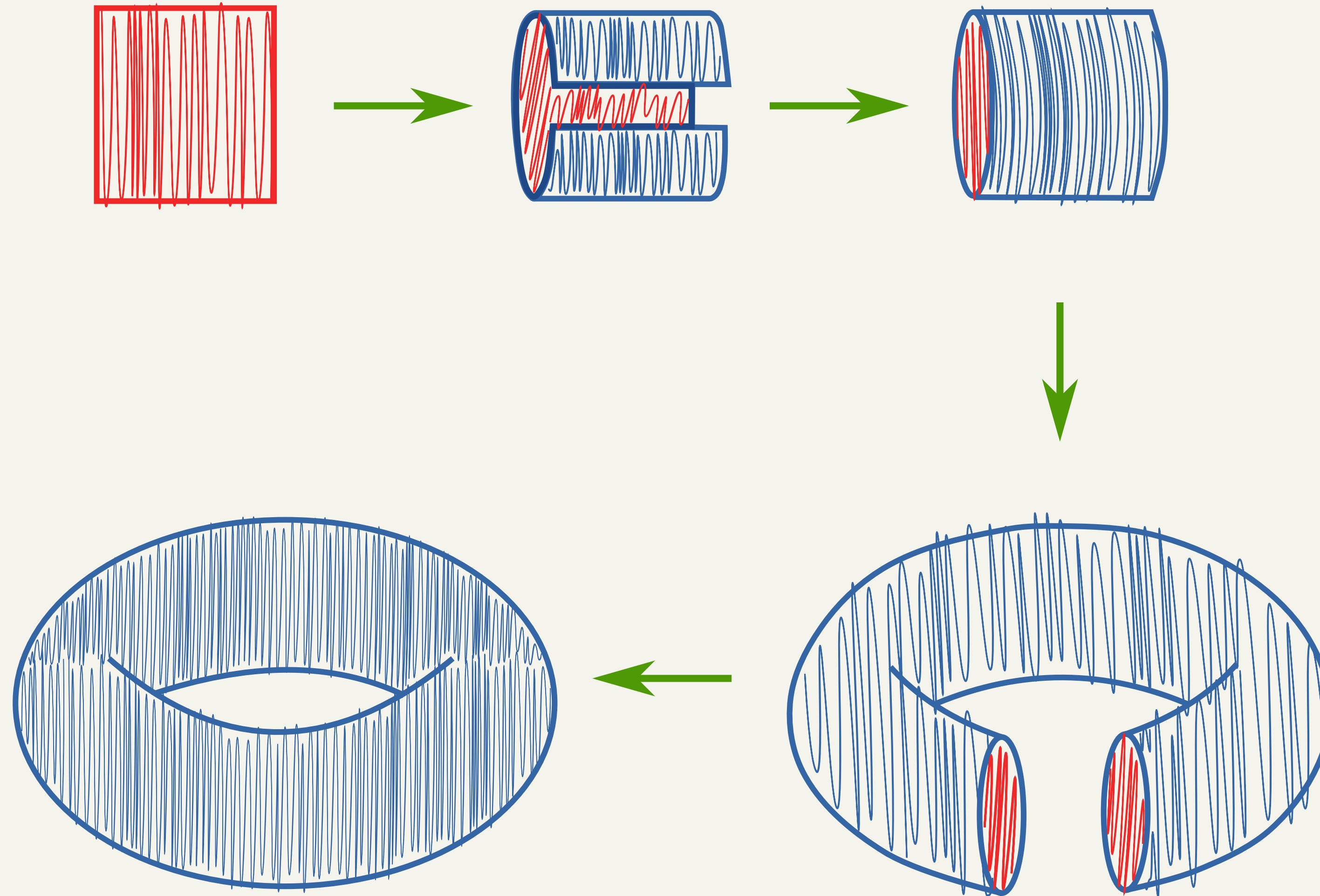


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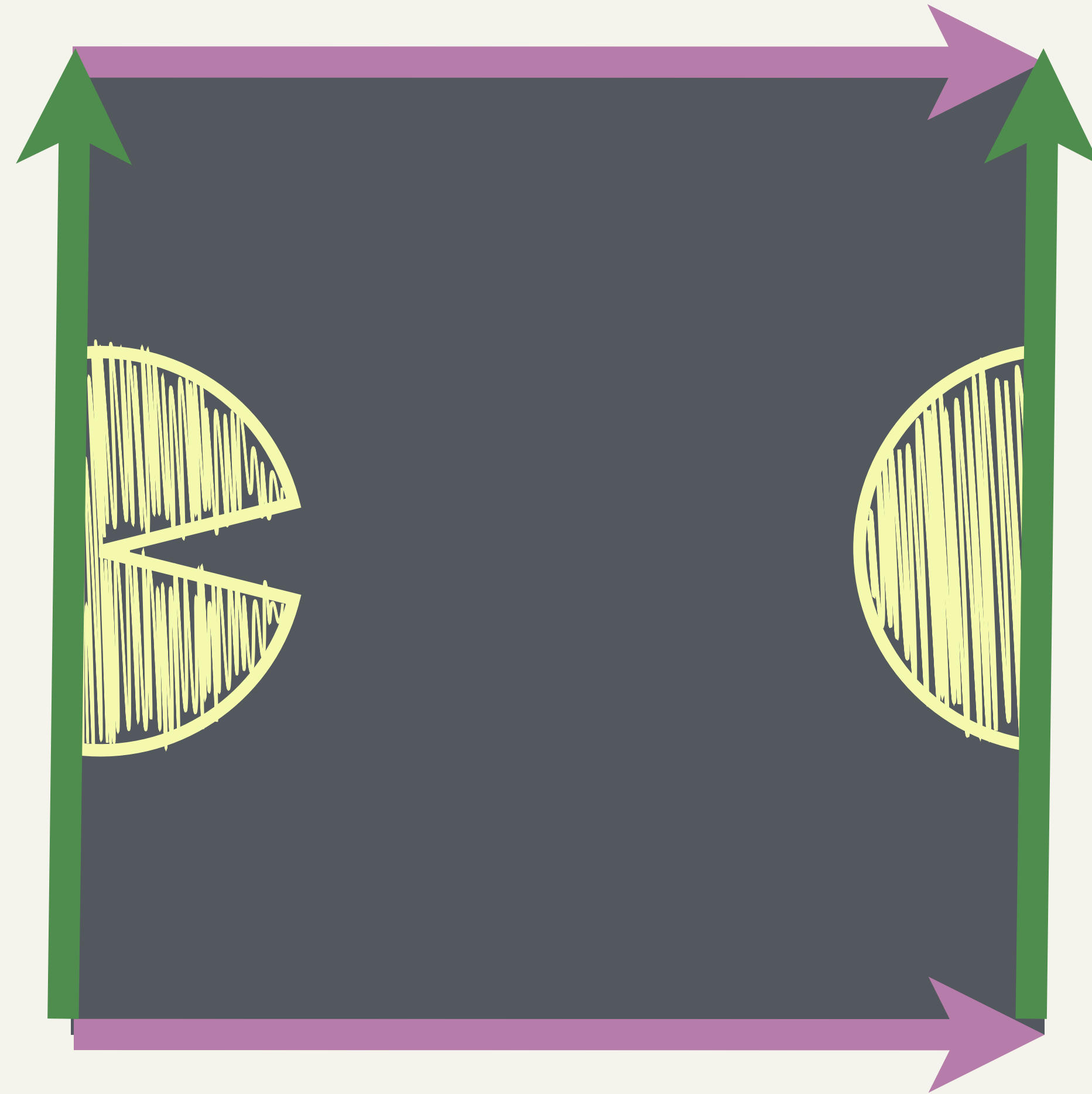
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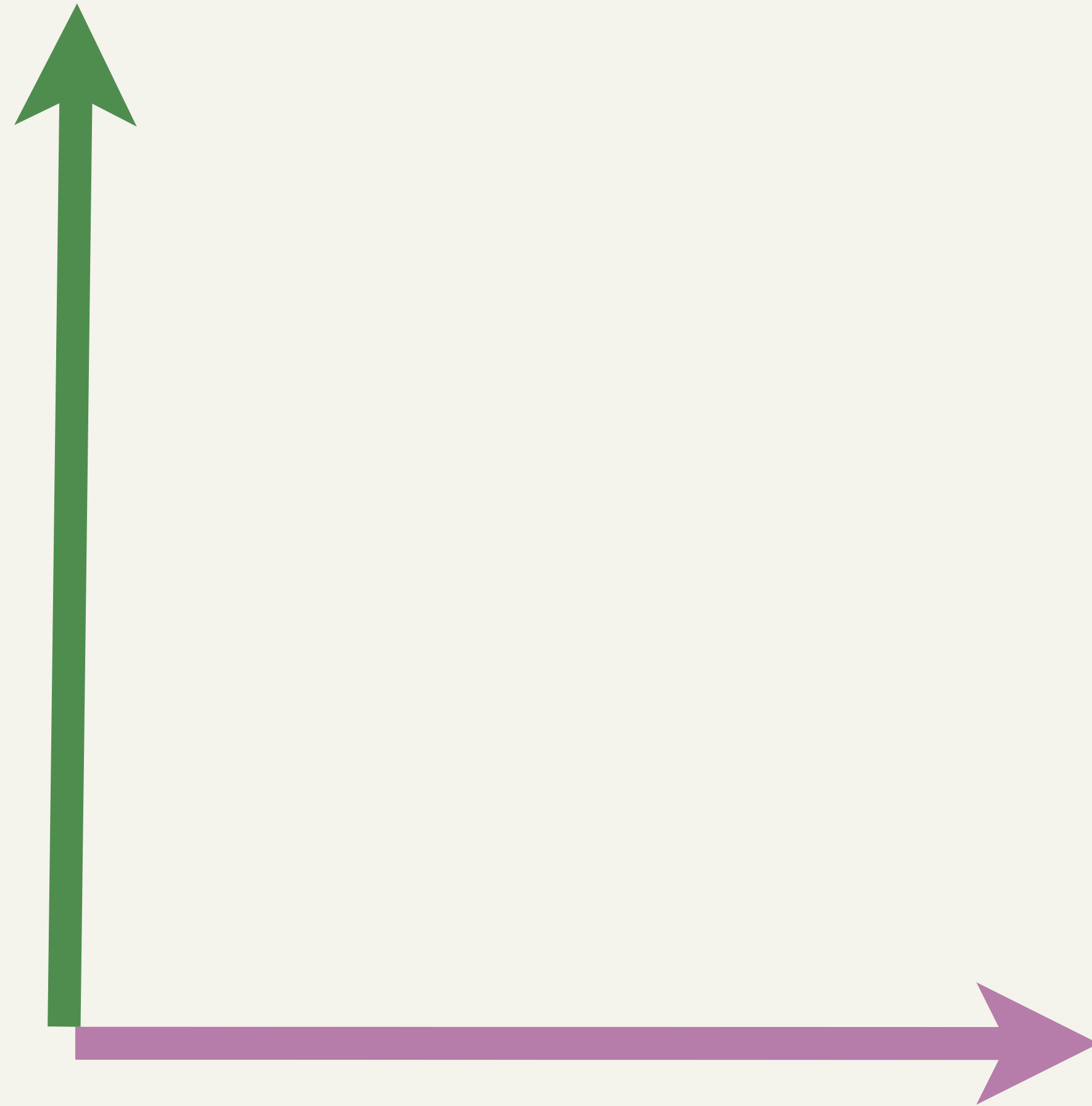


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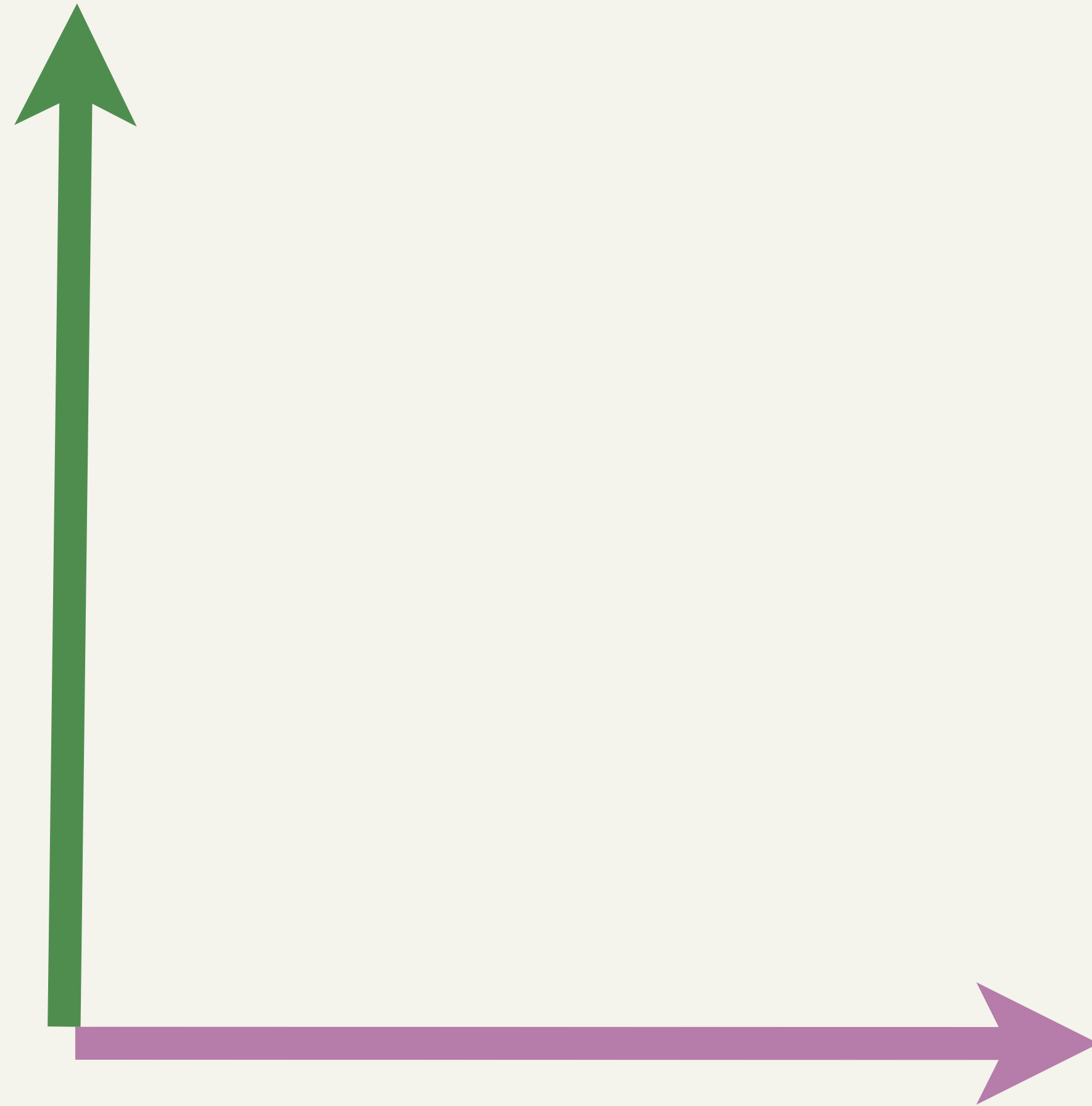
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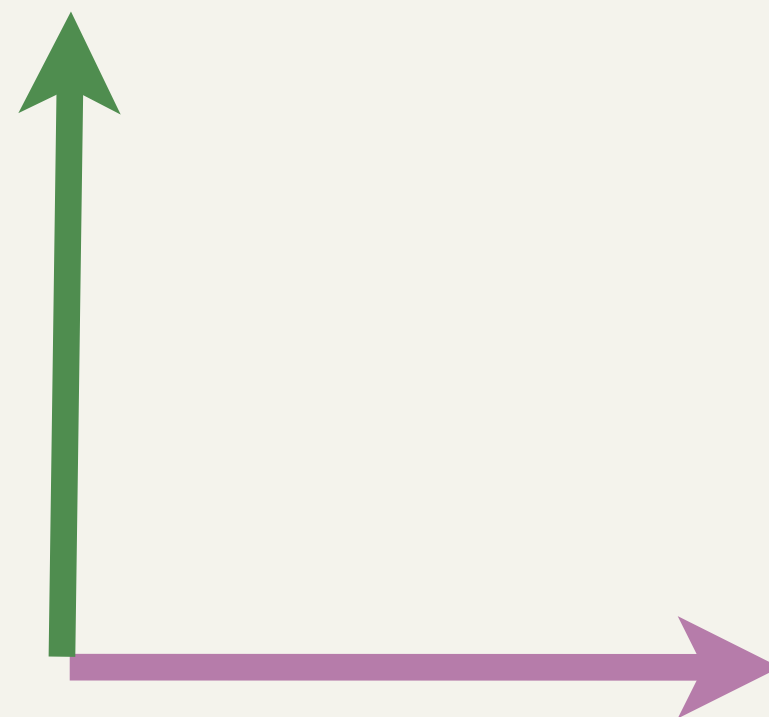
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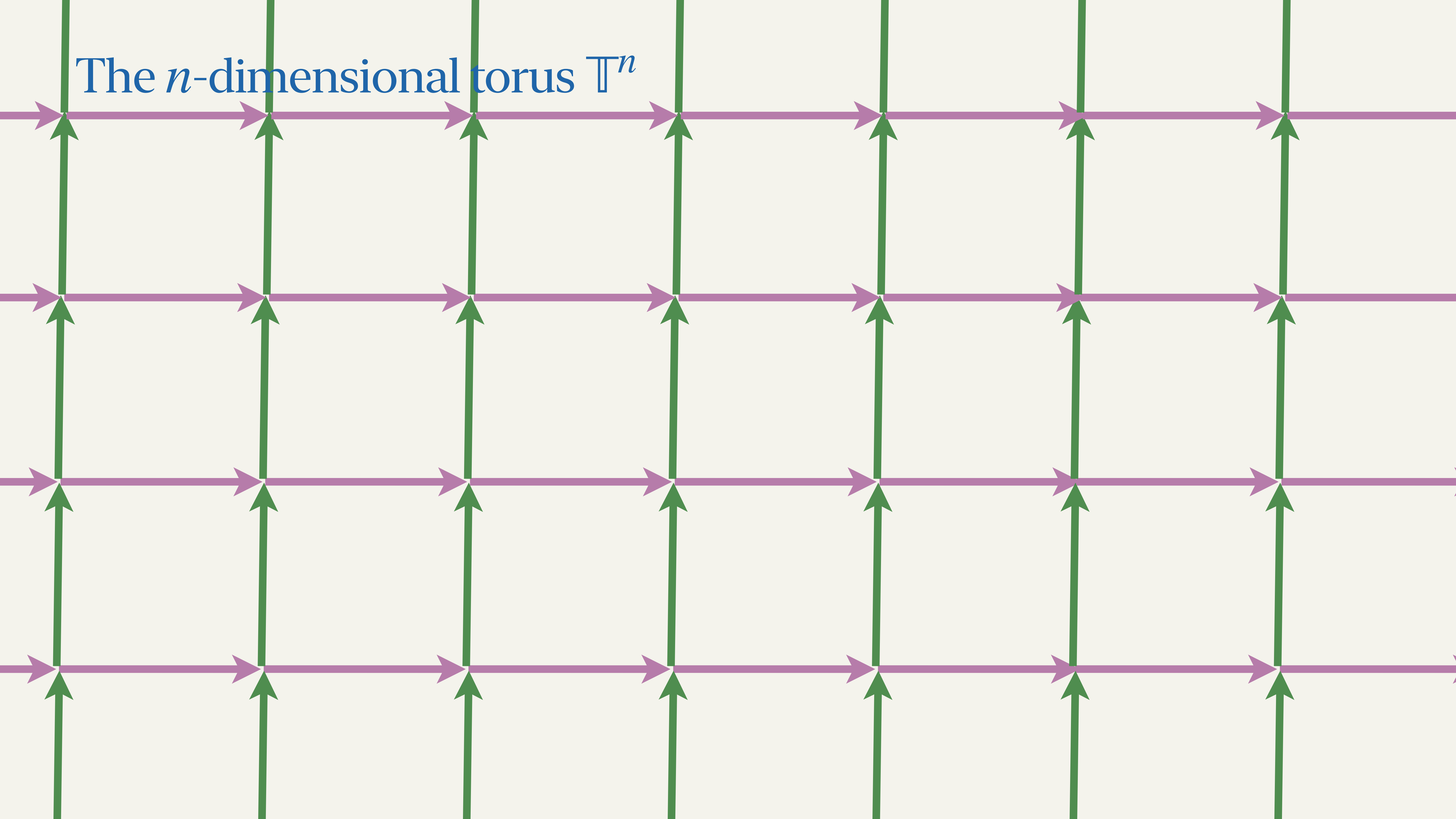
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$$\mathbb{T}^2 = \mathbb{E}^2 / \sim$$
A diagram illustrating the quotient space  $\mathbb{E}^2 / \sim$ . It shows a 2D coordinate system with a green vertical arrow pointing upwards and a purple horizontal arrow pointing to the right, representing the fundamental domain of the torus.

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$$\mathbb{T}^n = \mathbb{E}^n / \Lambda$$

A **lattice group**

(a group generated by  $n$  linearly independent translations)

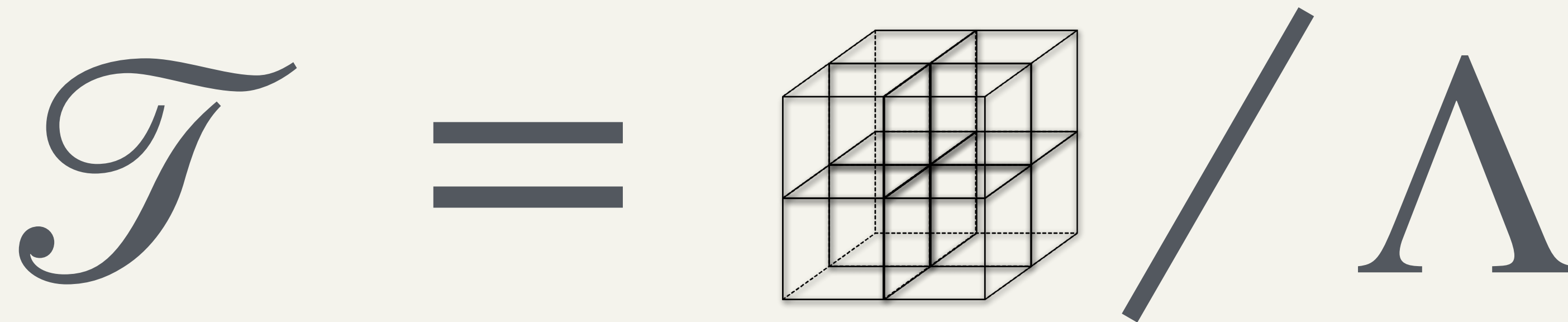
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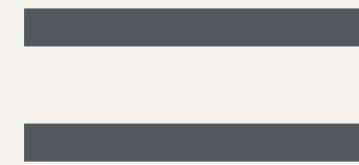
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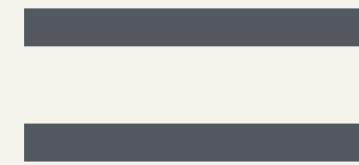
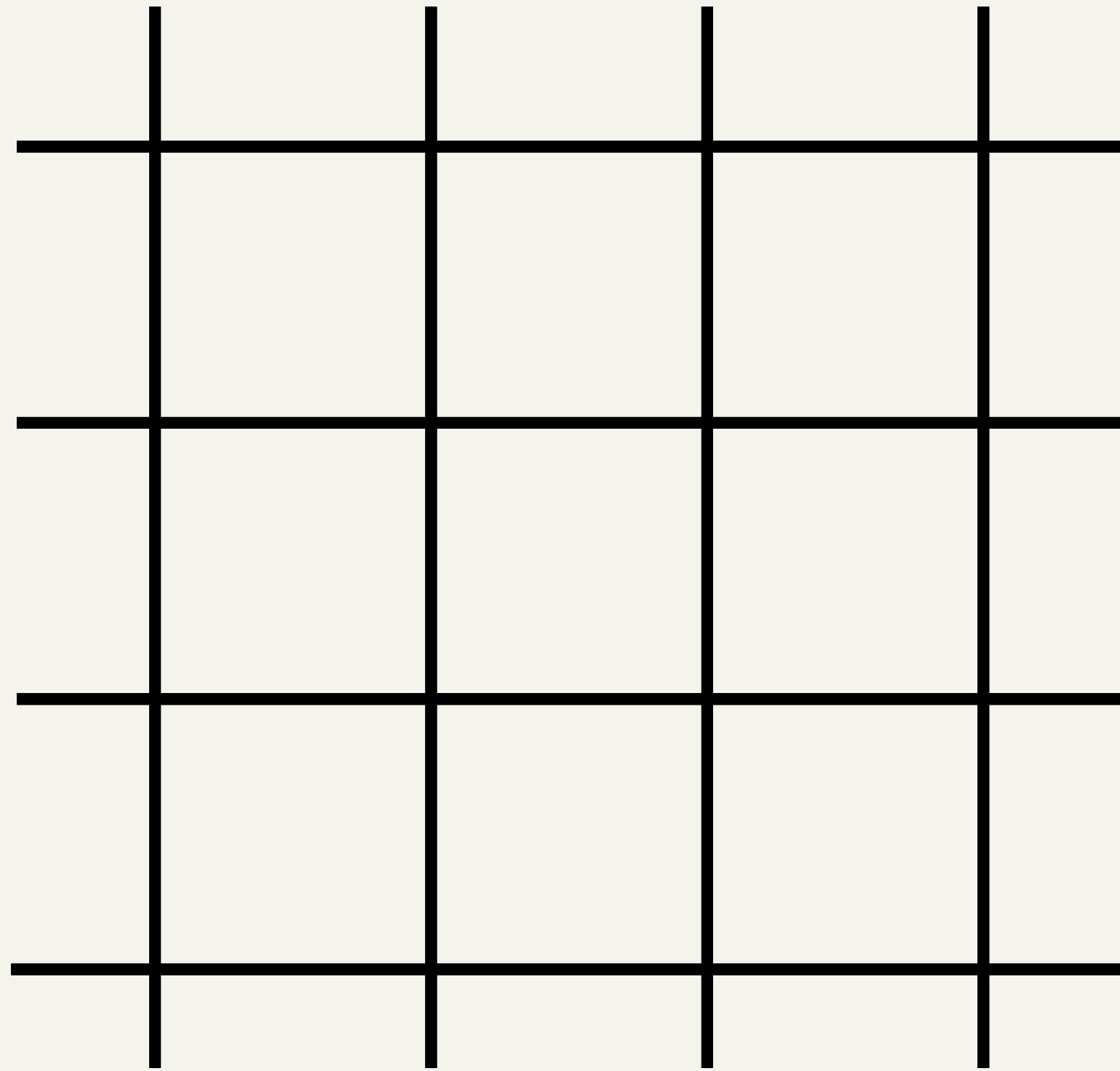
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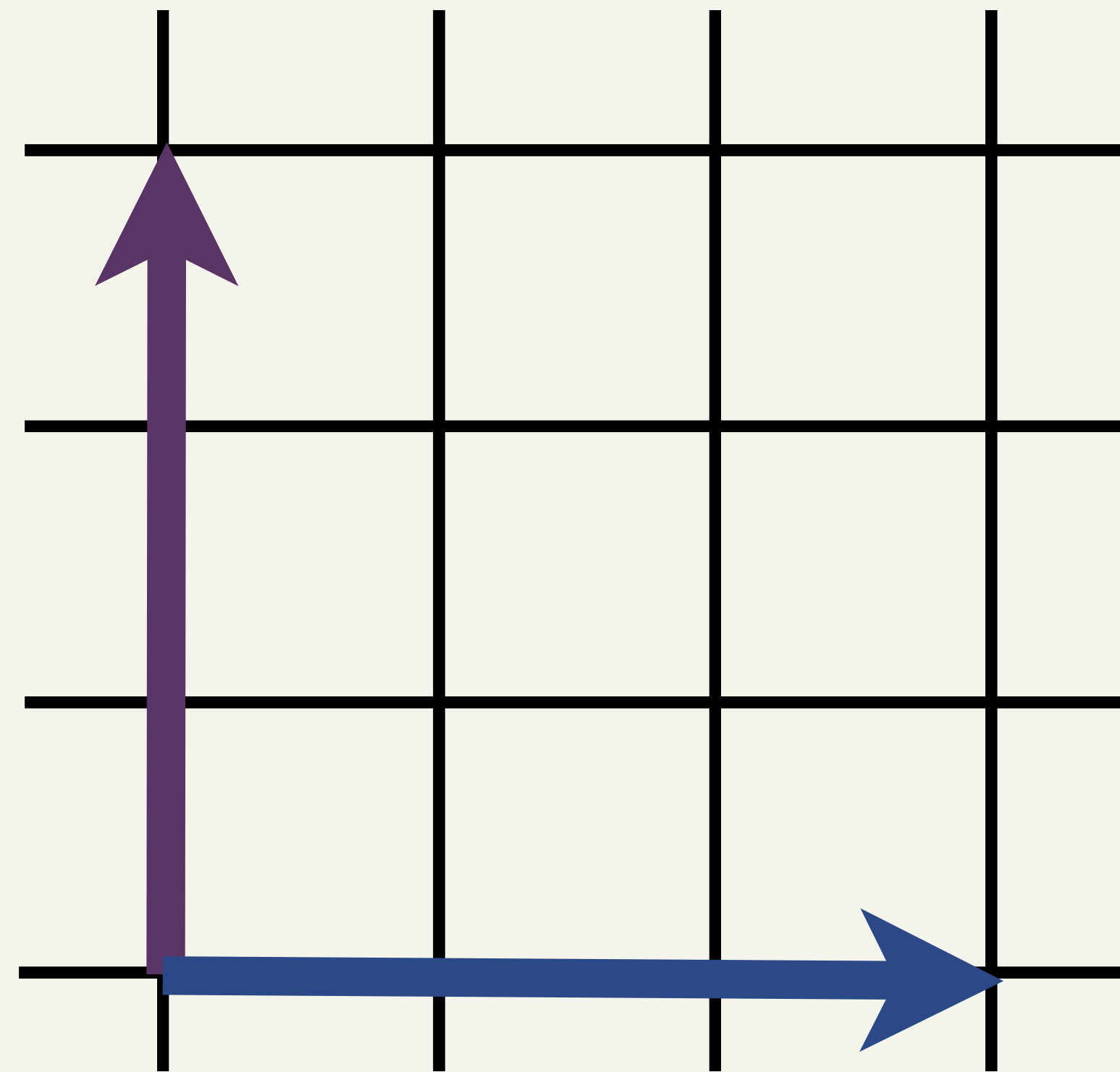
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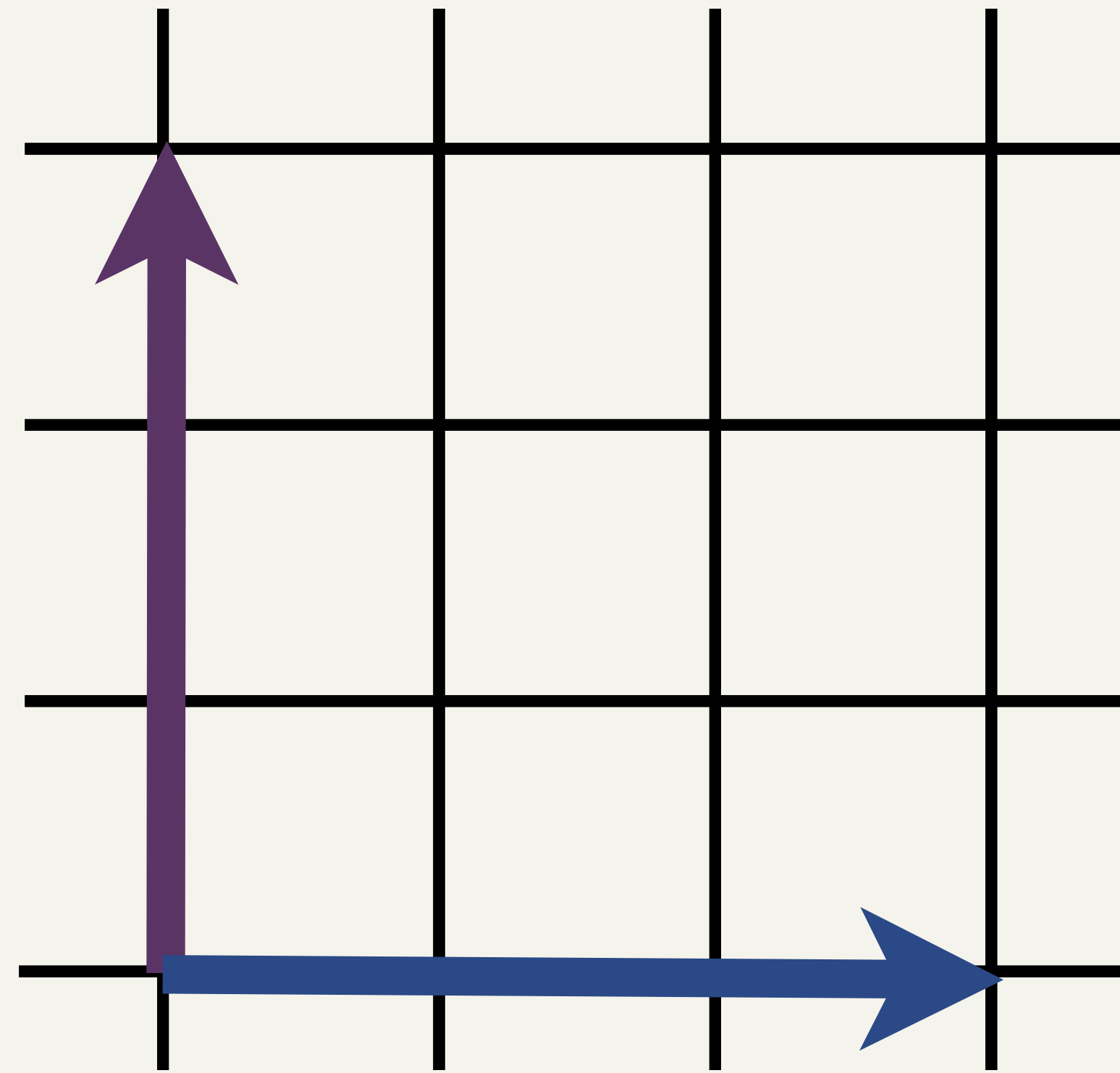


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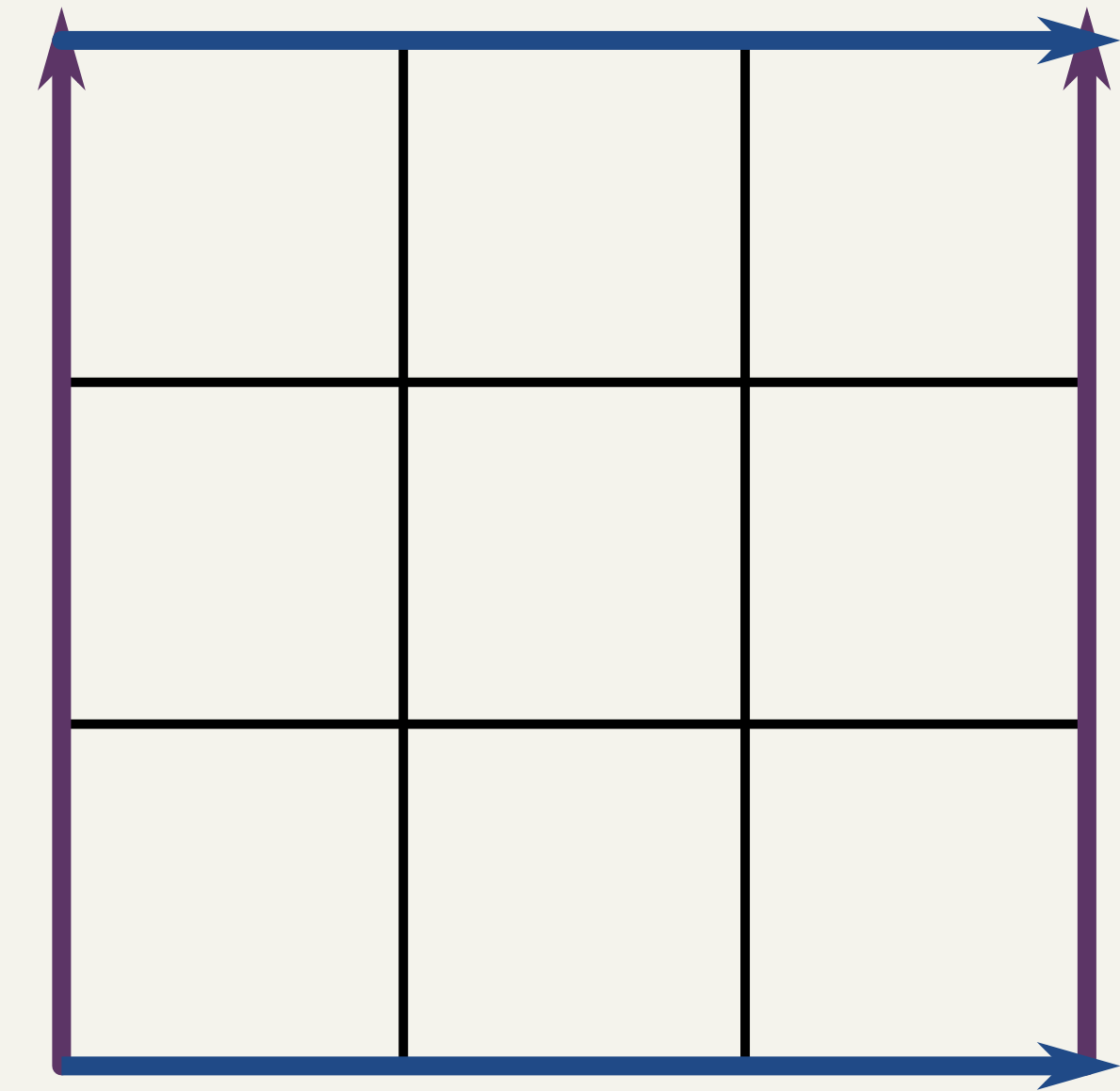


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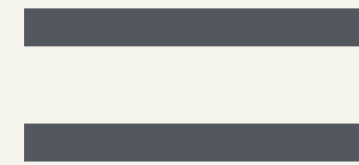
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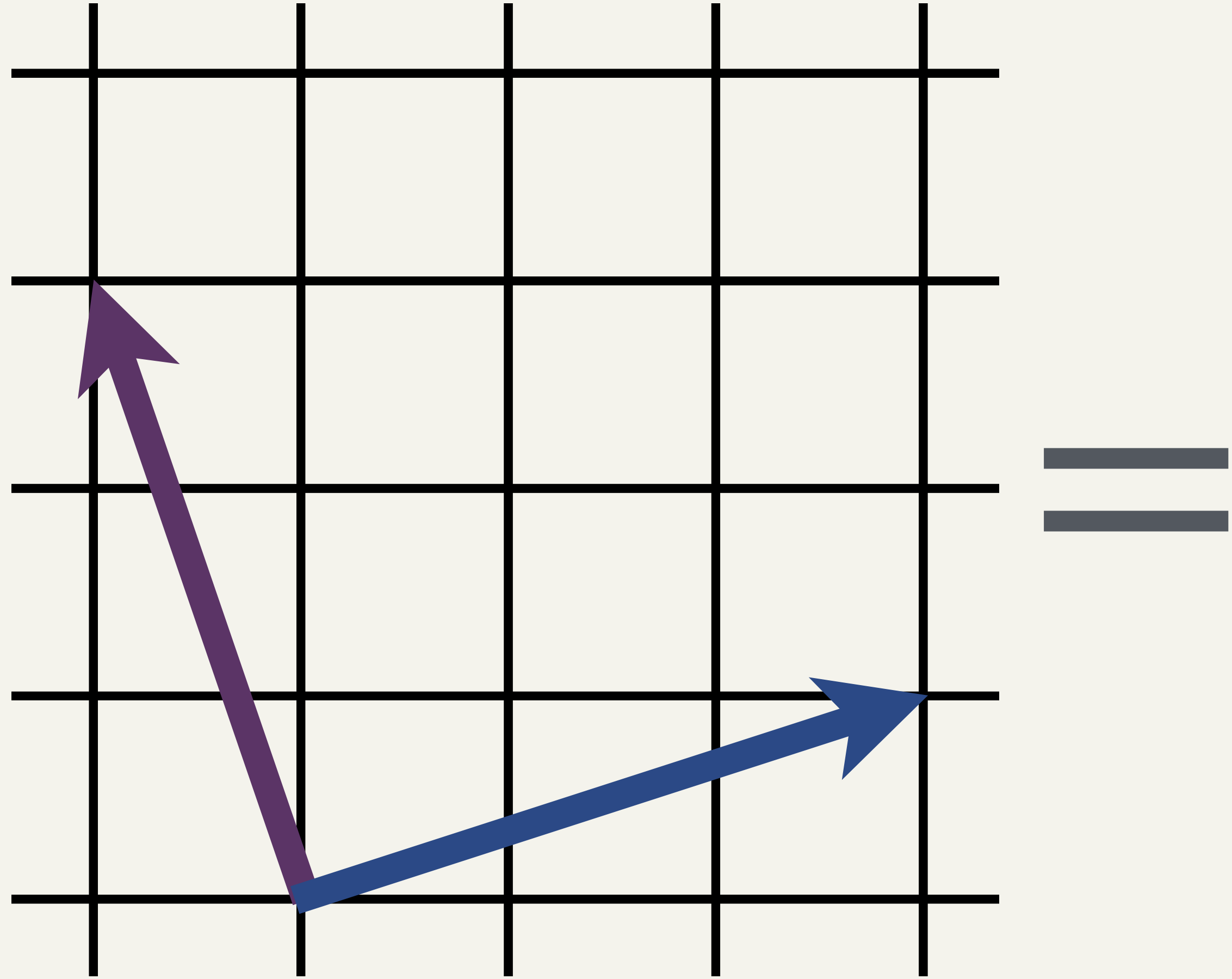
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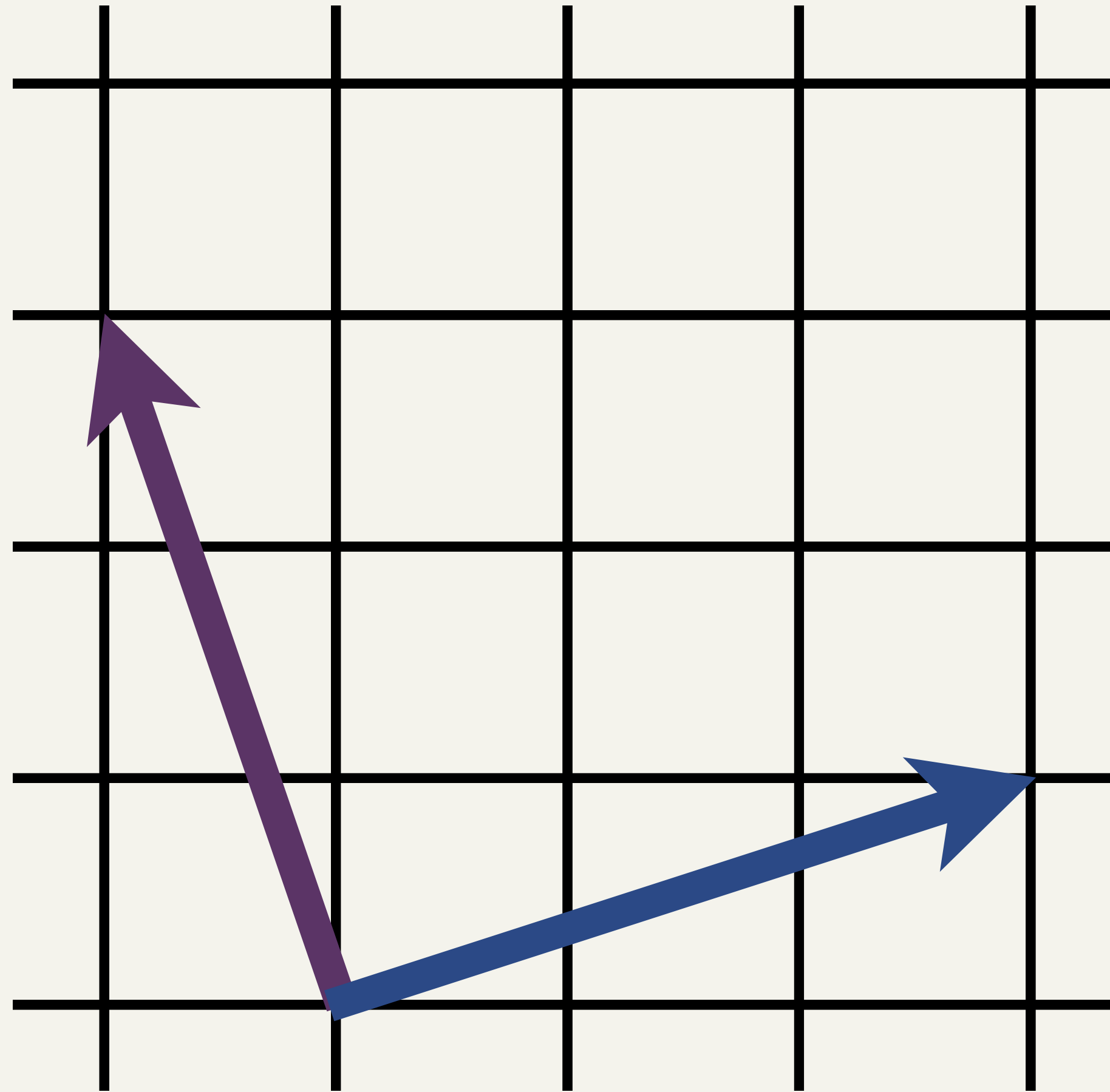
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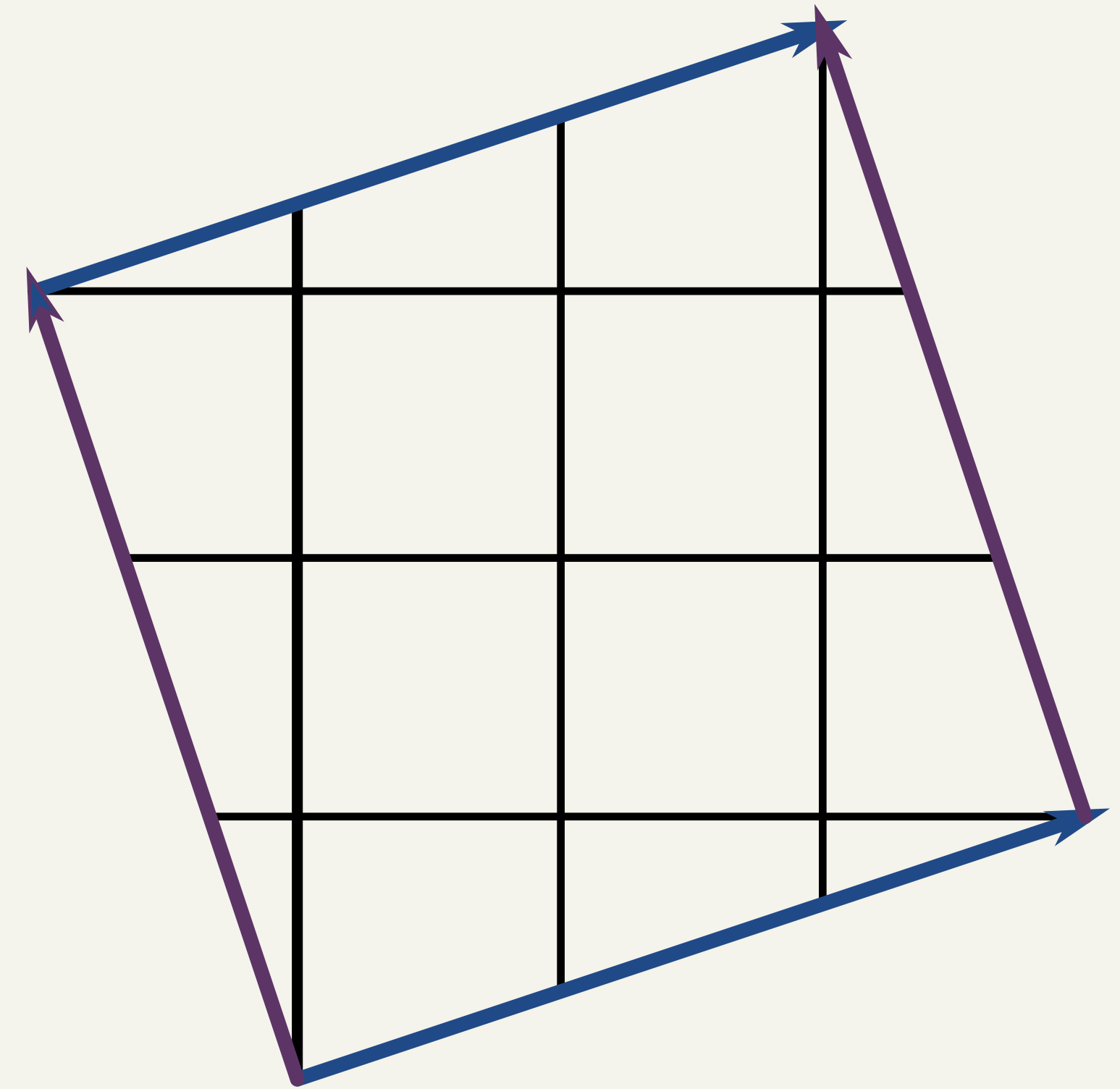
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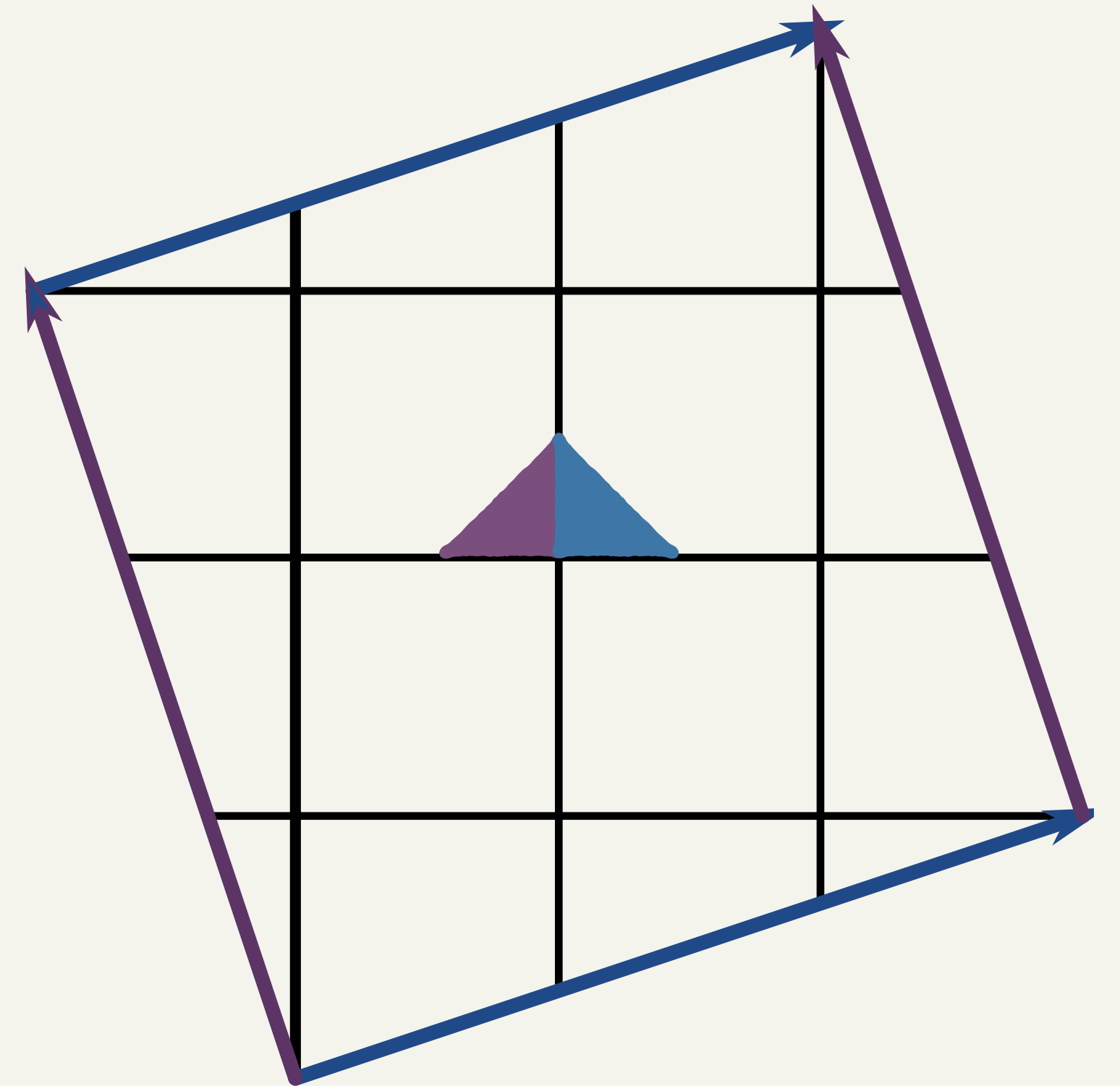
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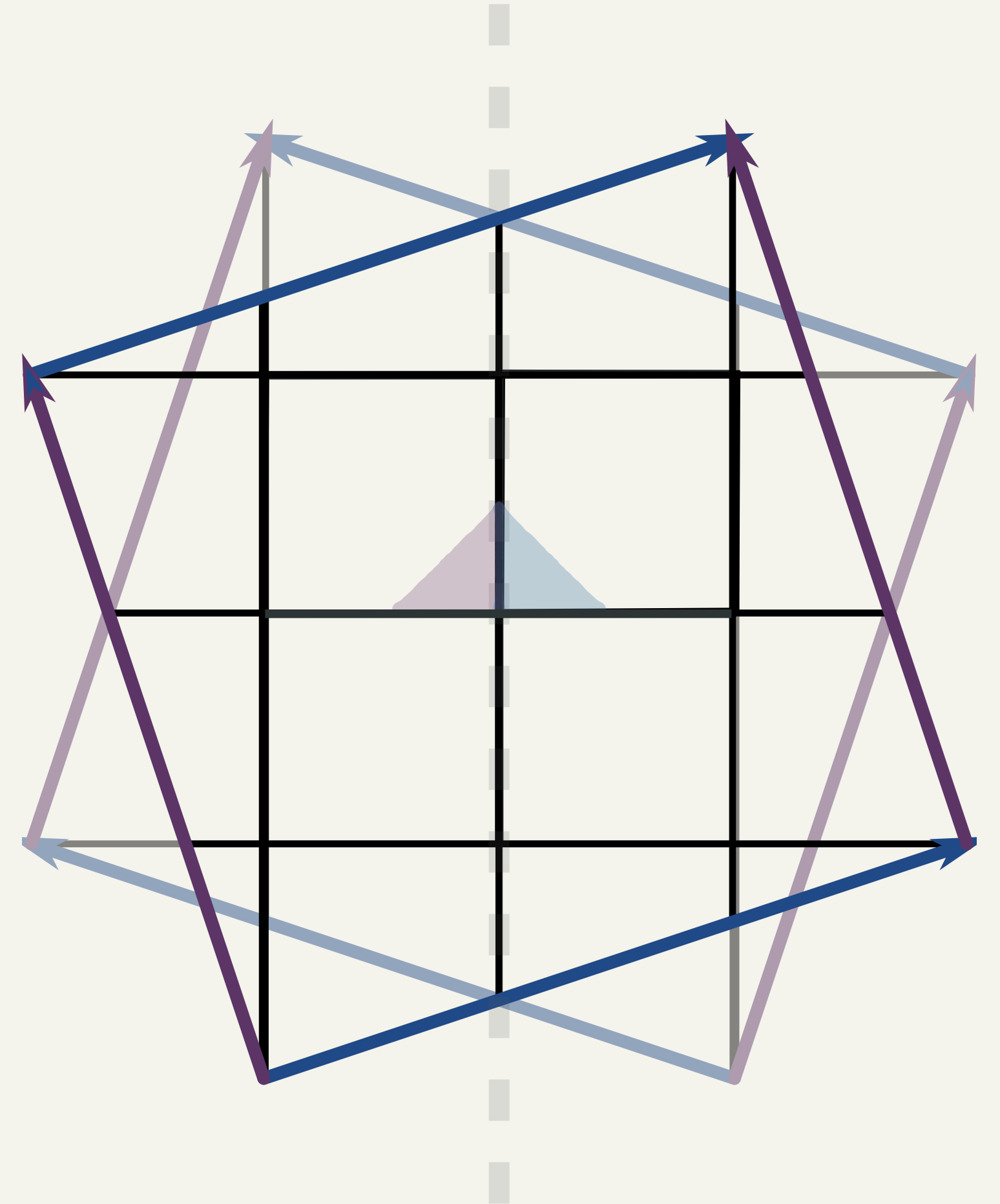


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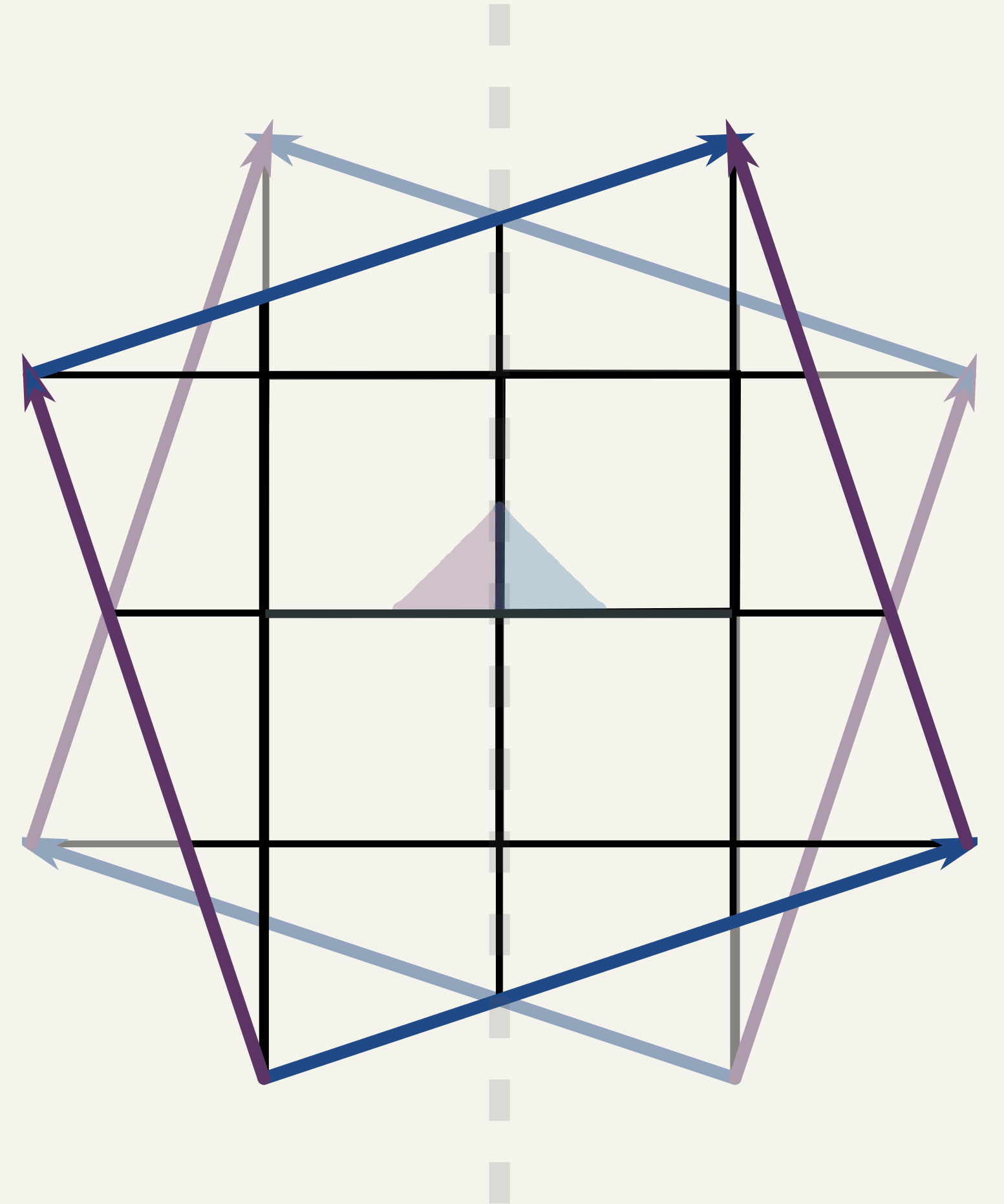


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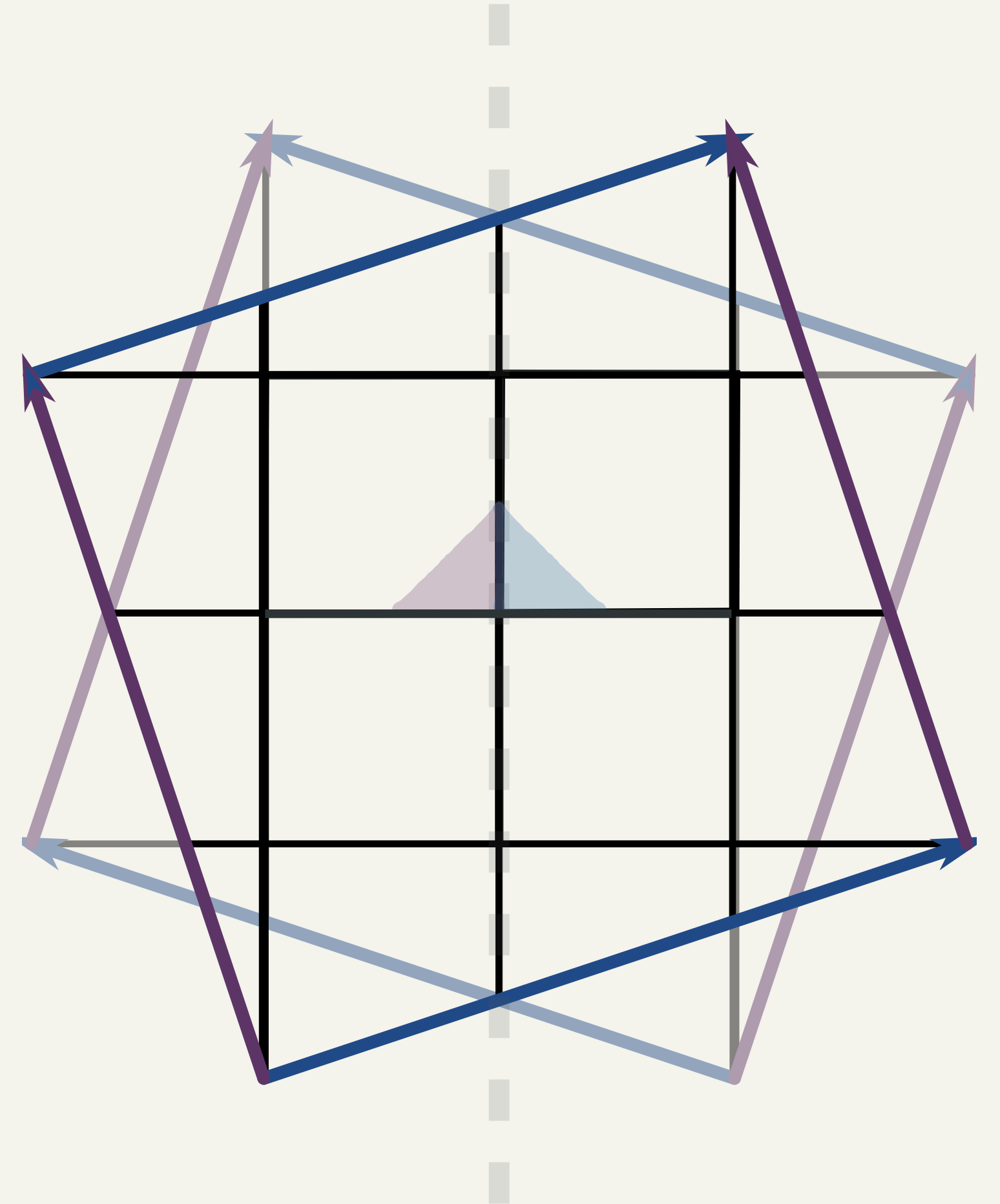


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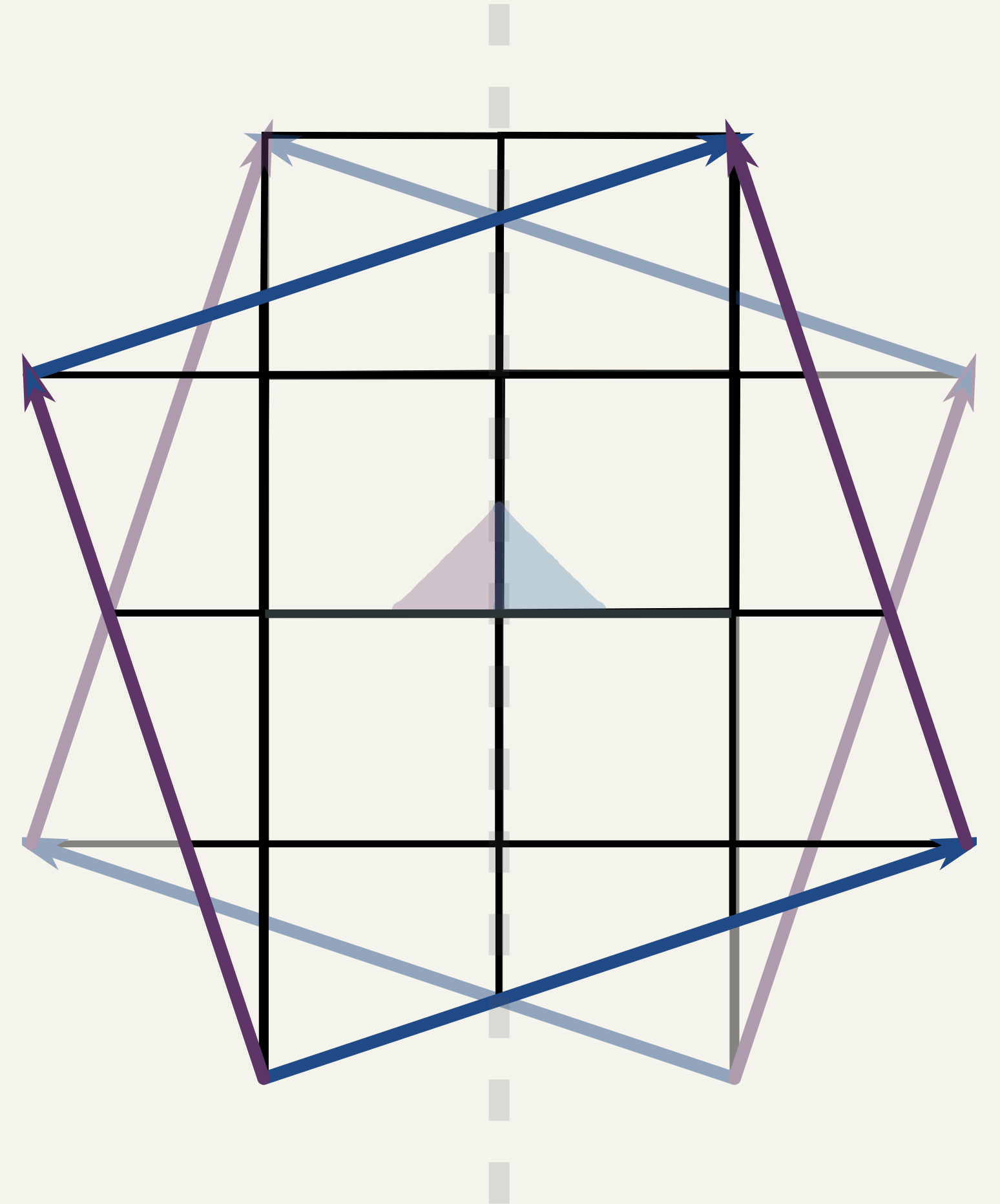


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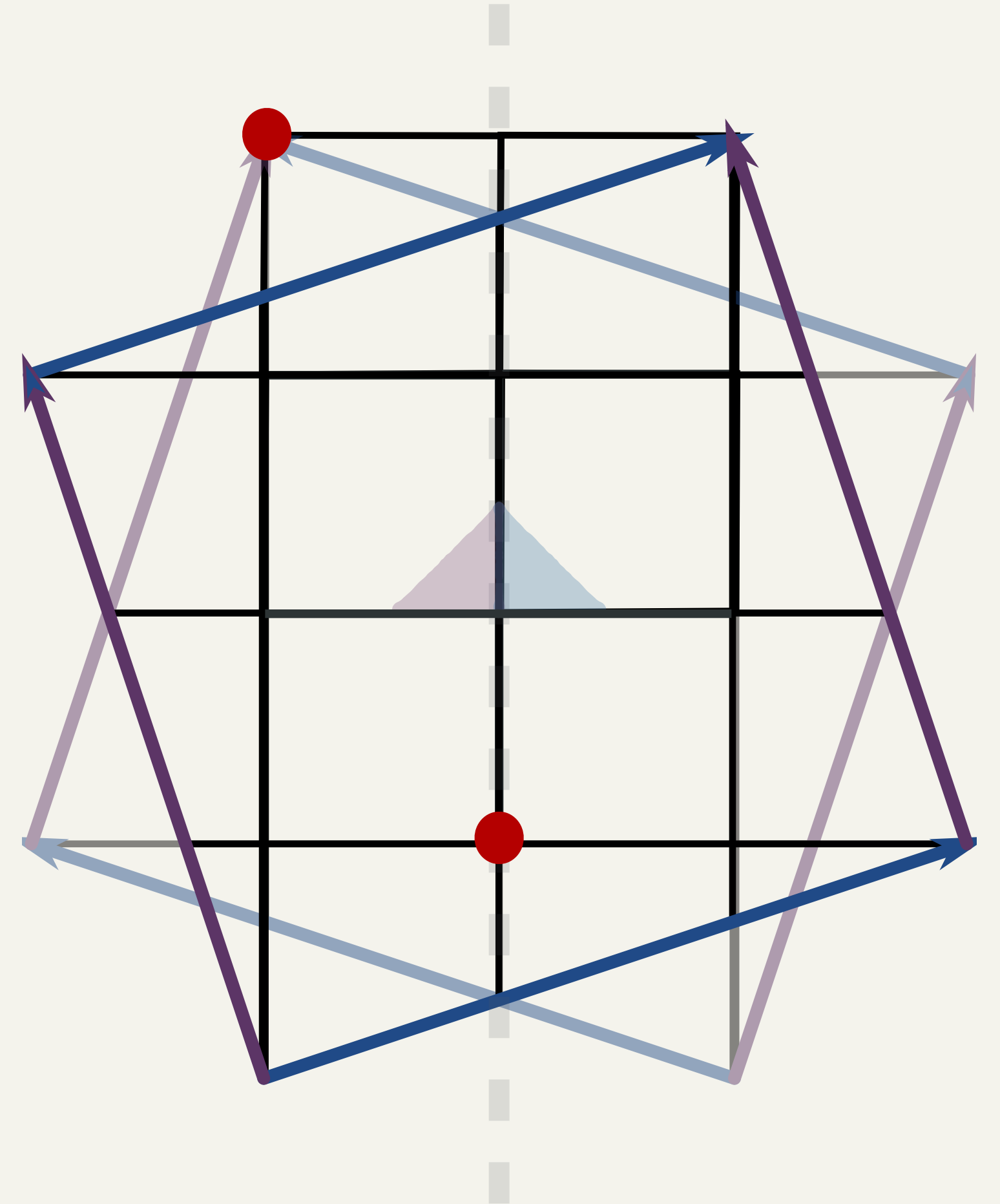


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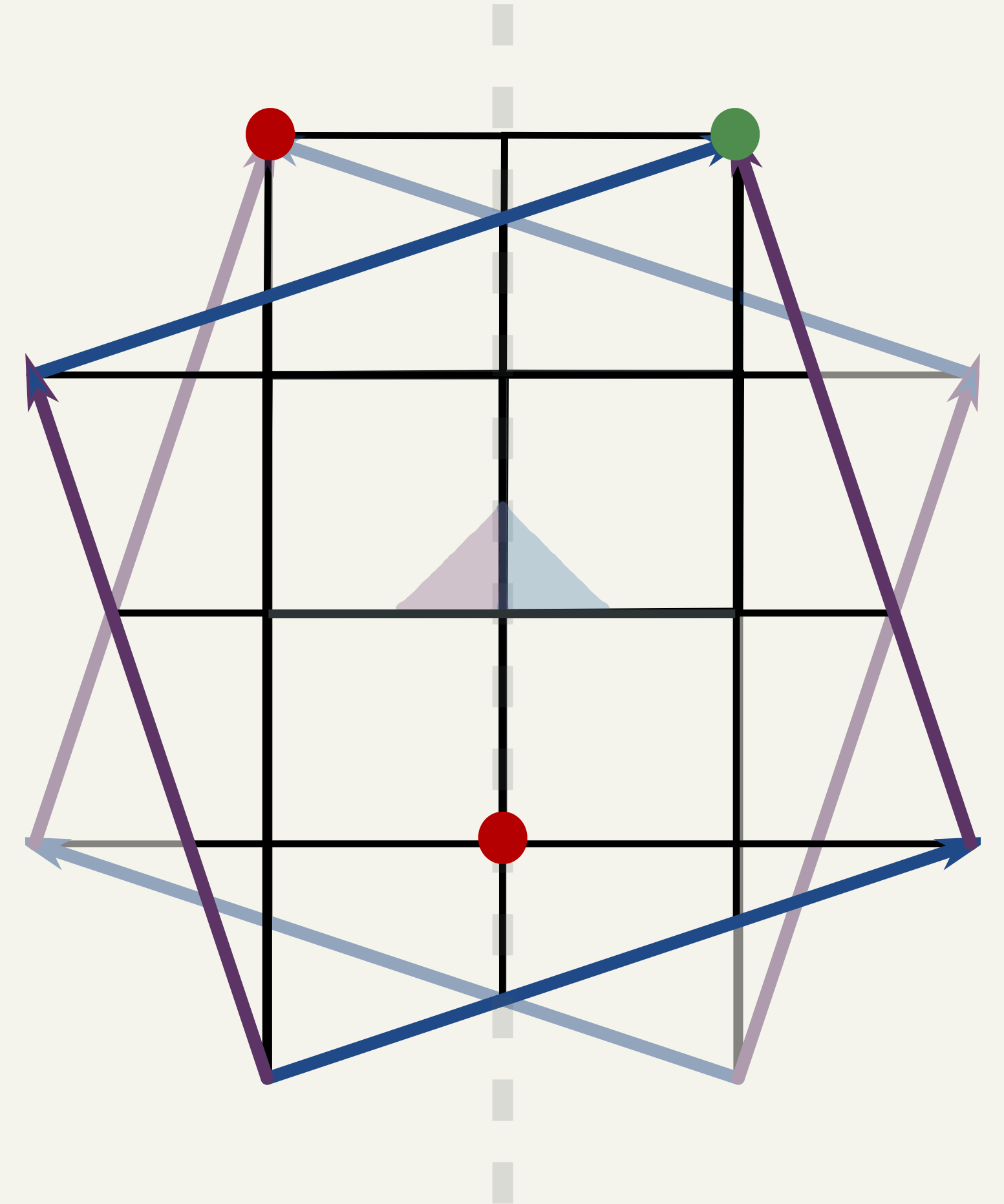


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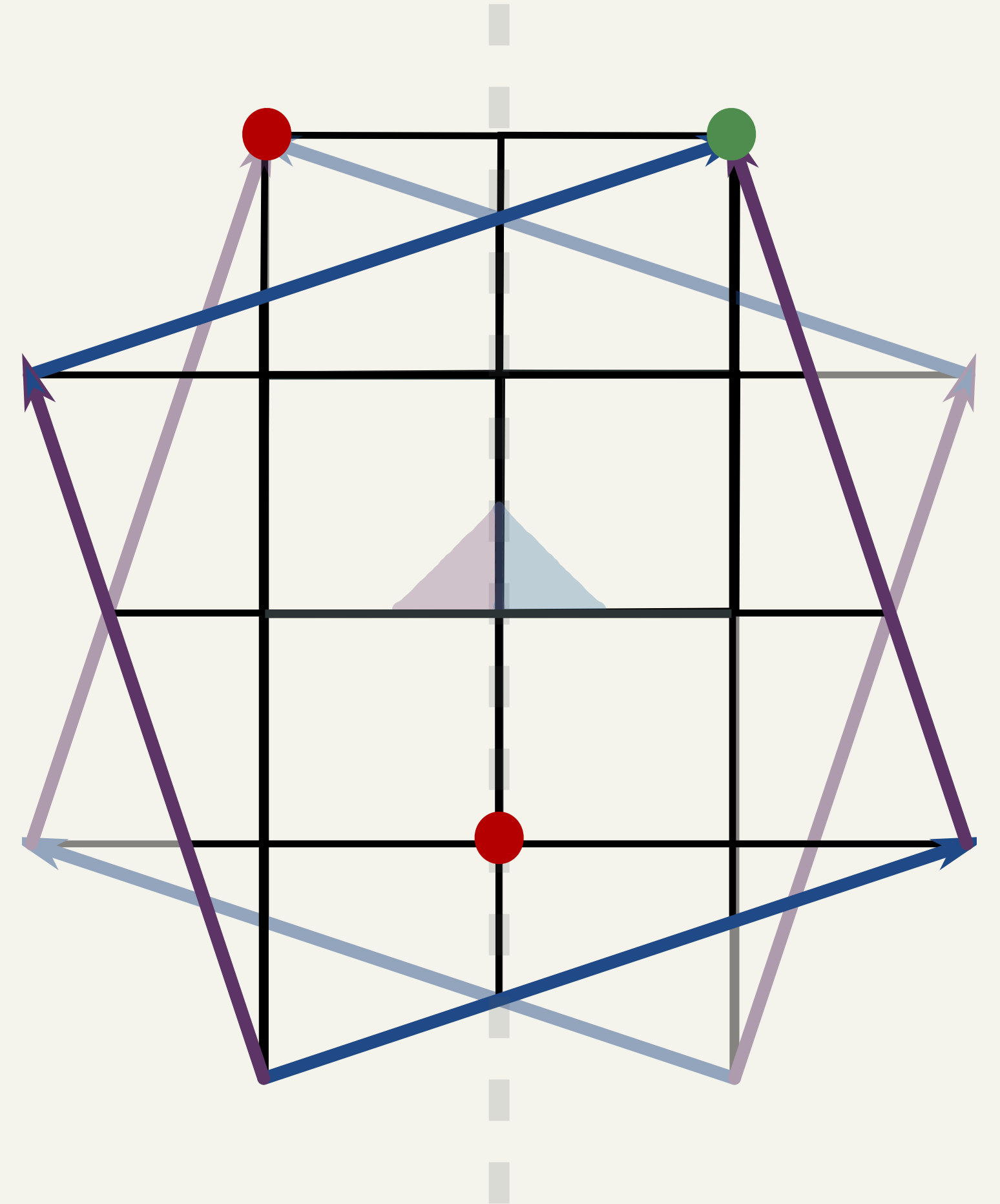


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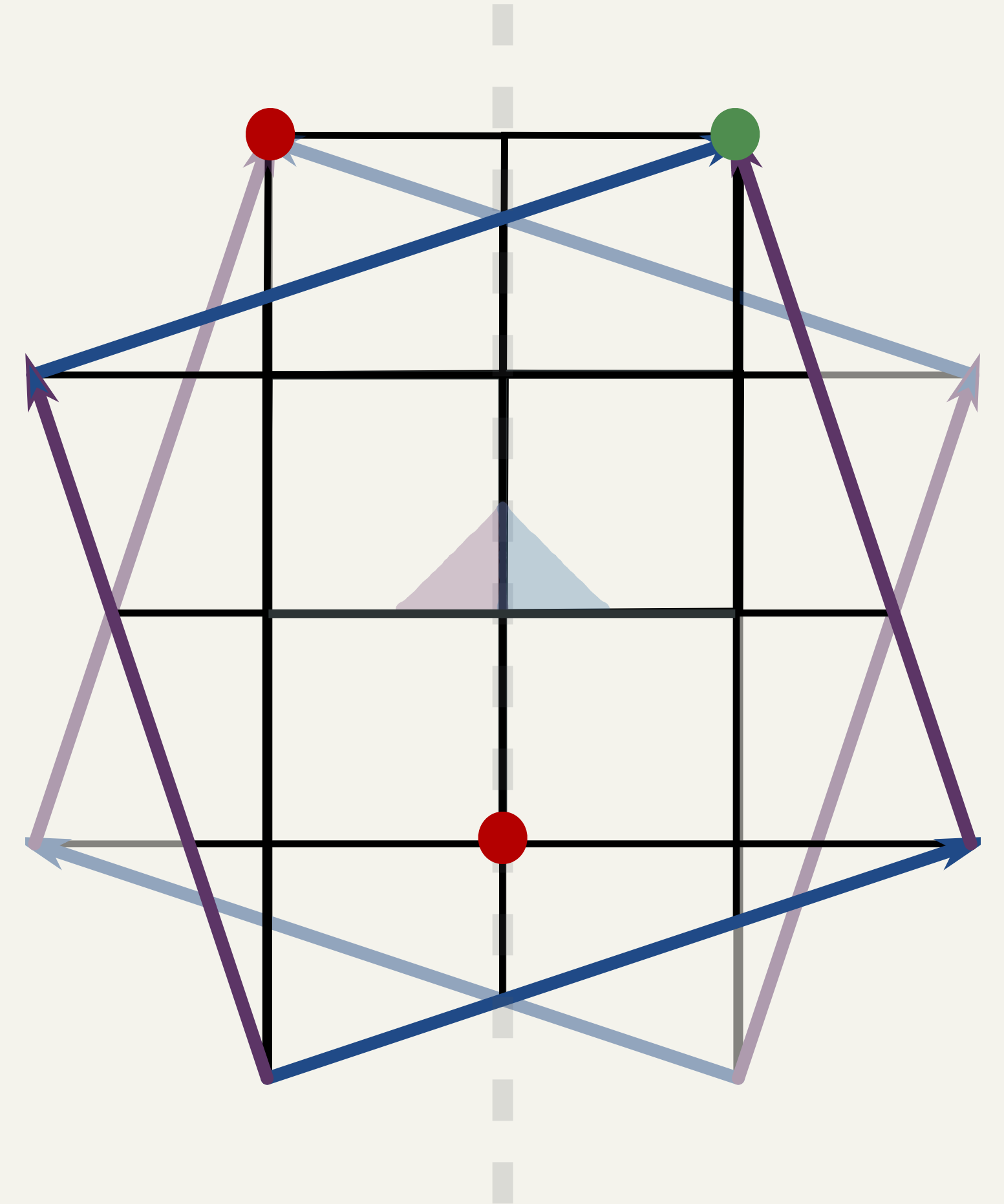


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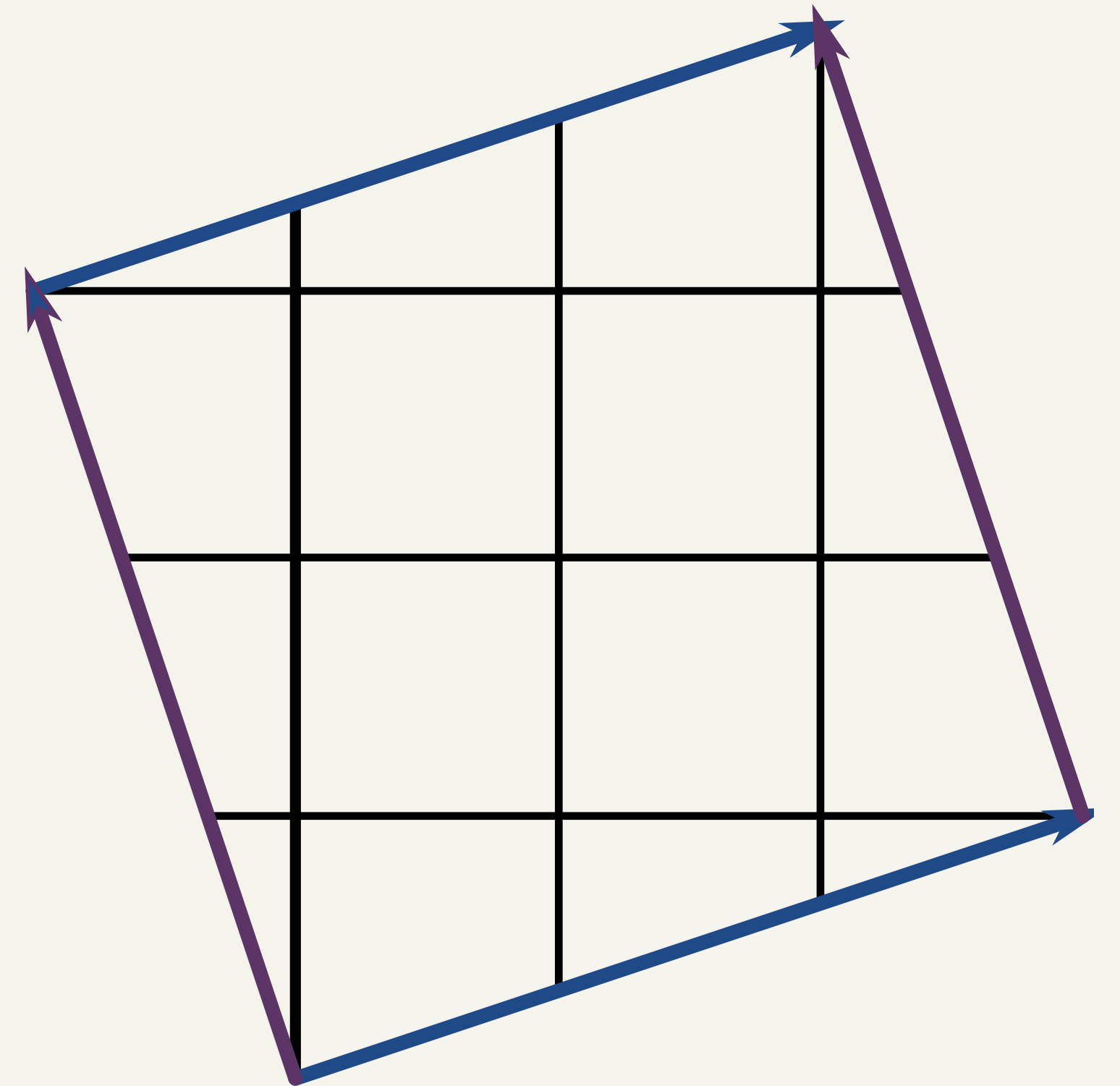


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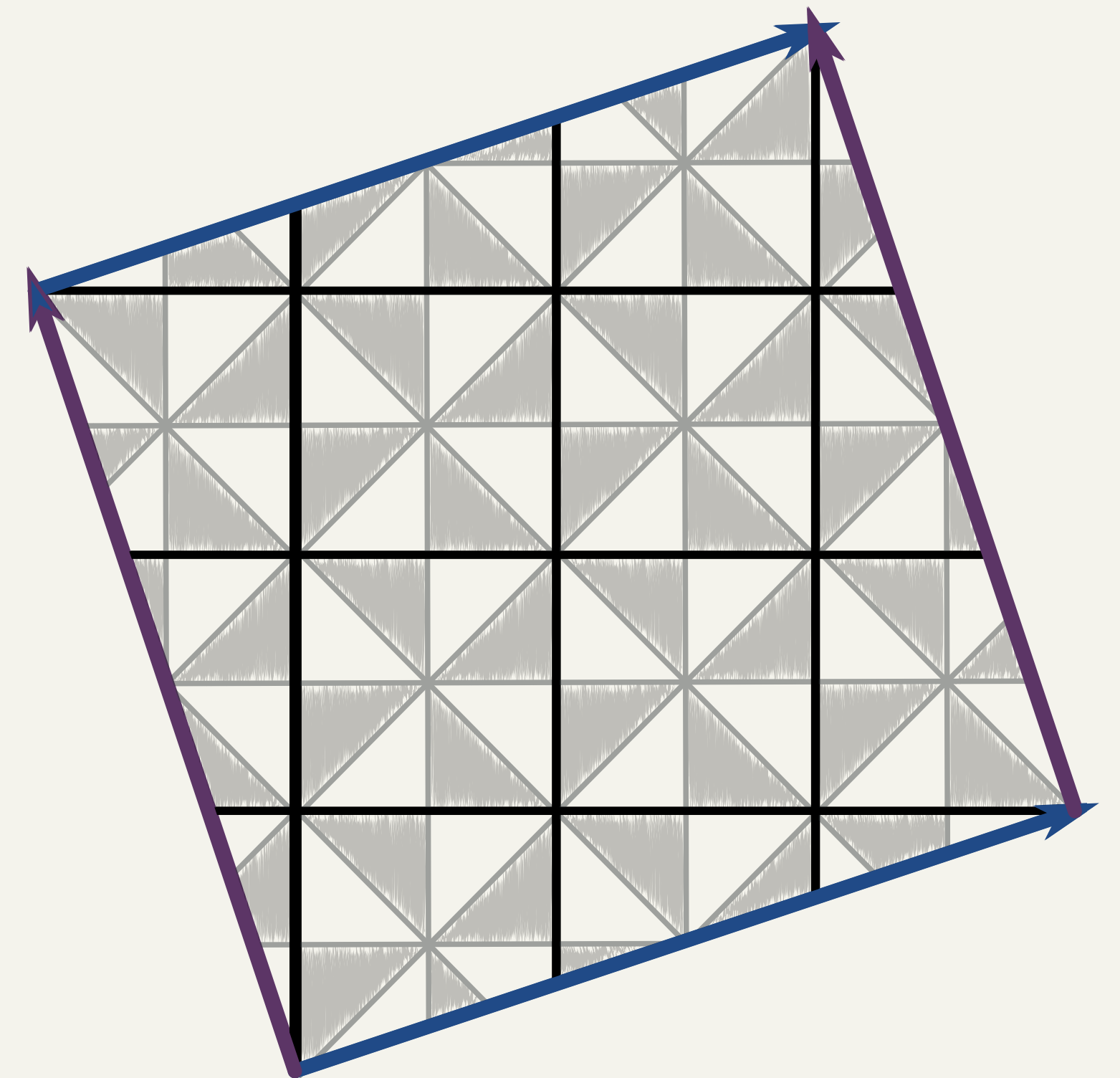


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Branko Grünbaum  
1929 - 2018

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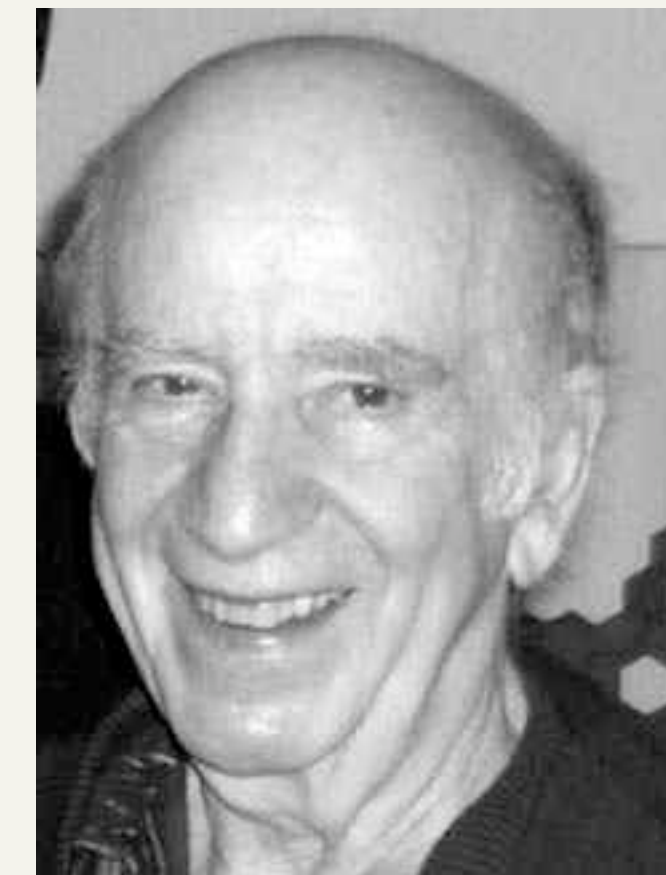
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H.S.M. Coxeter  
1907 - 2003



W. O. J. Moser  
1927 - 2009

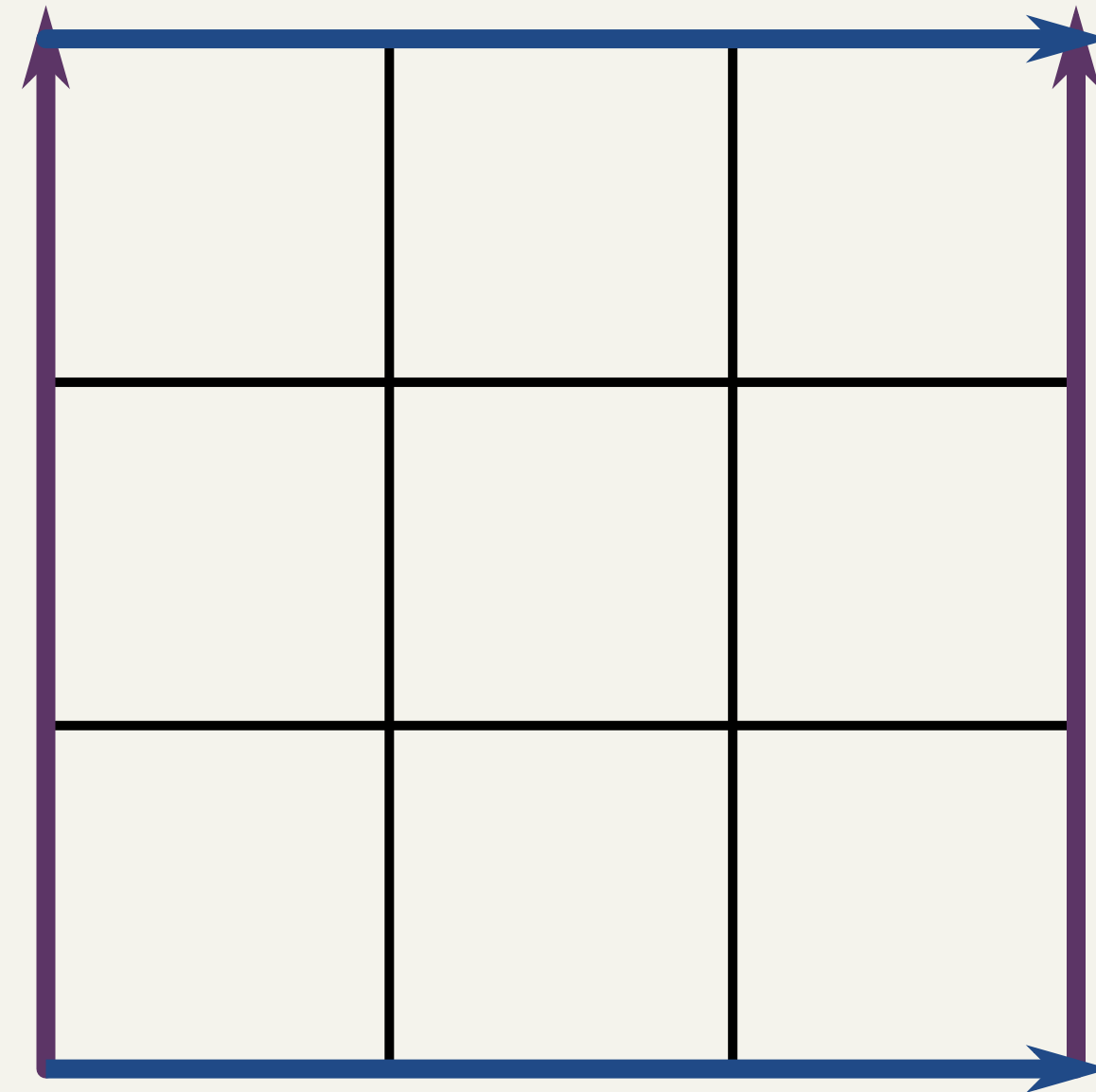
# Symmetries of $n$ -dimensional **toroids**

What are the possibilities for the symmetry type of an equivelar tessellation of the 2-torus?

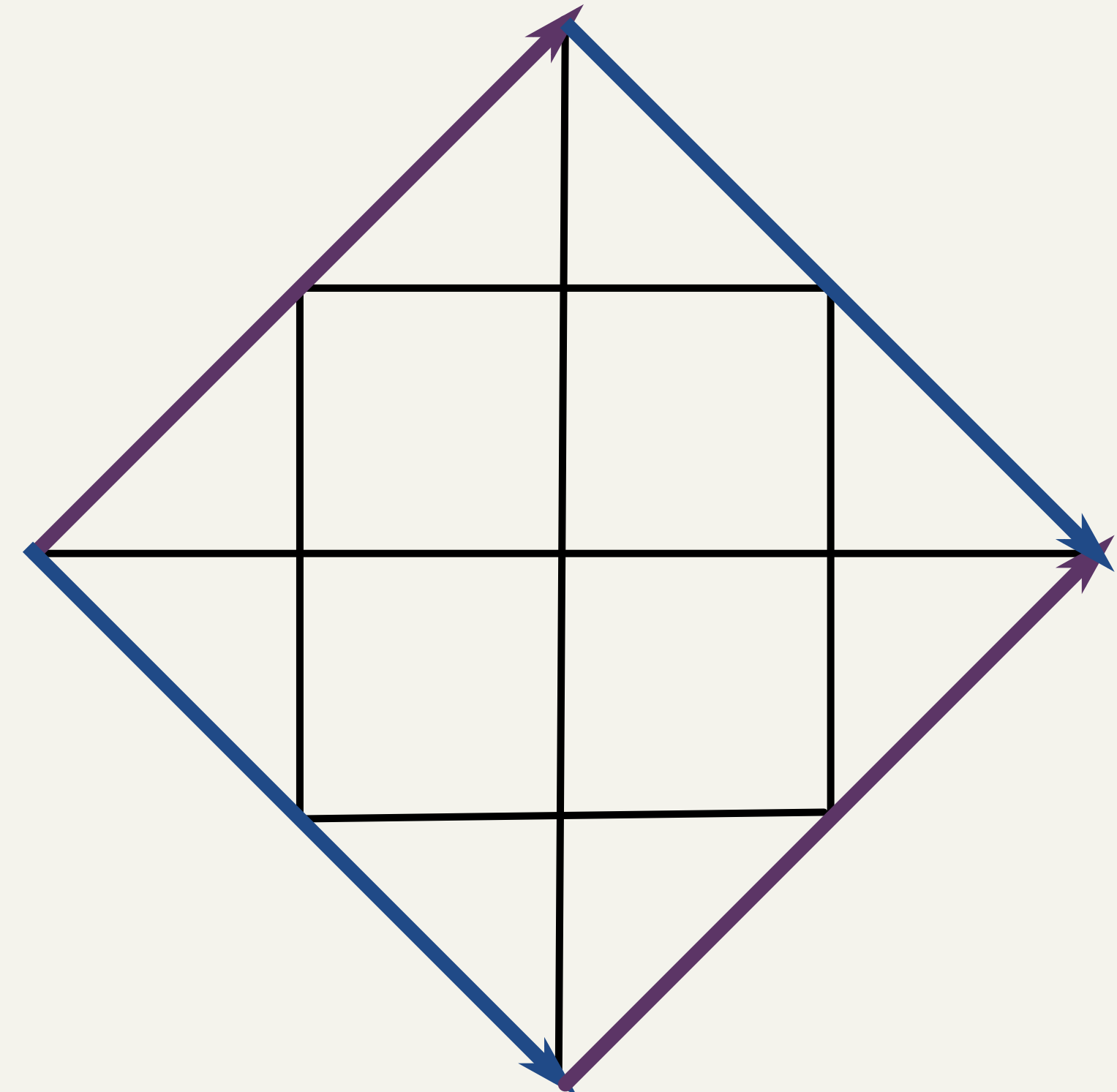


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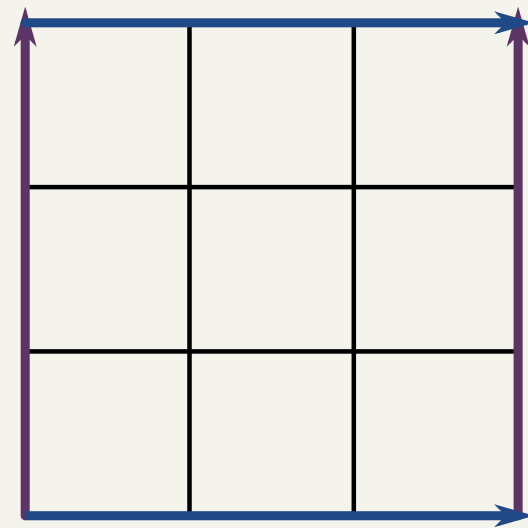
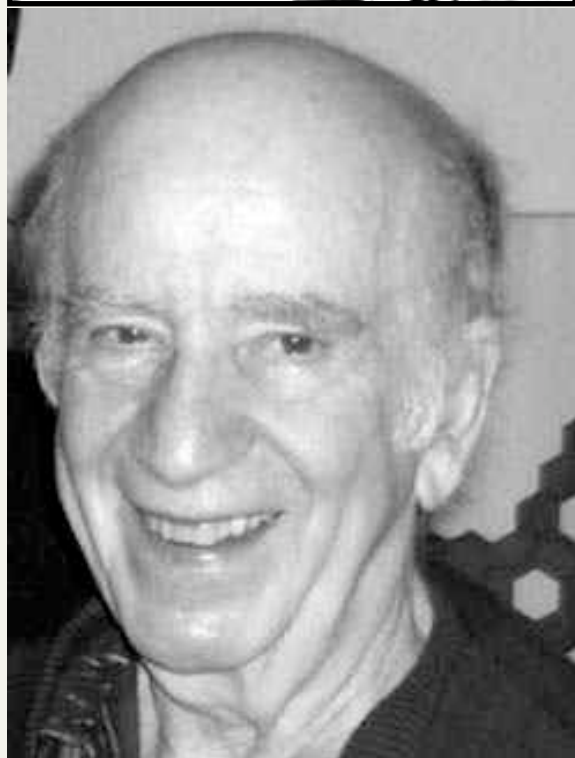
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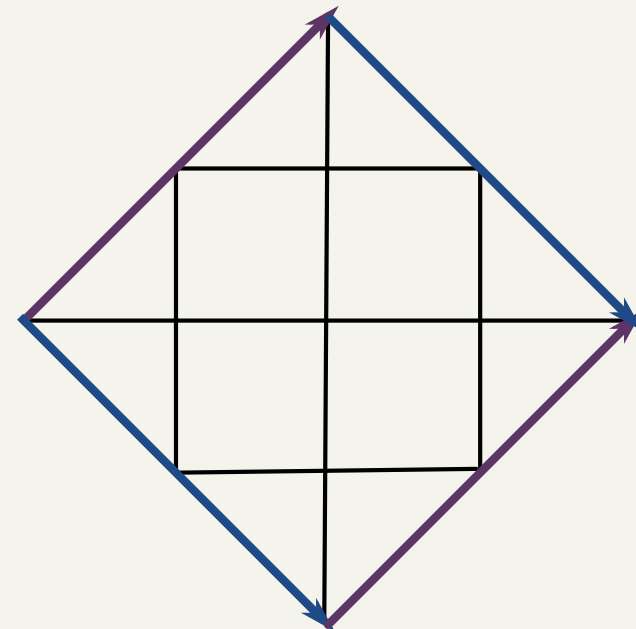
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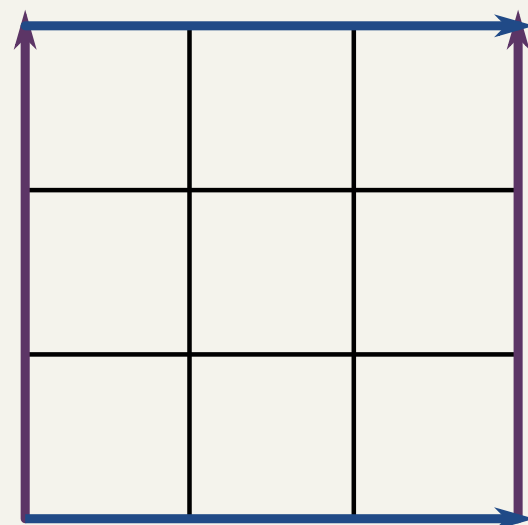
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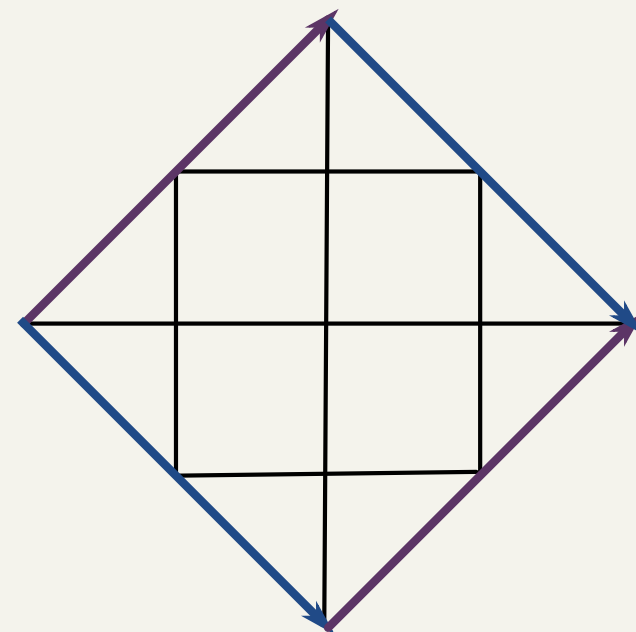
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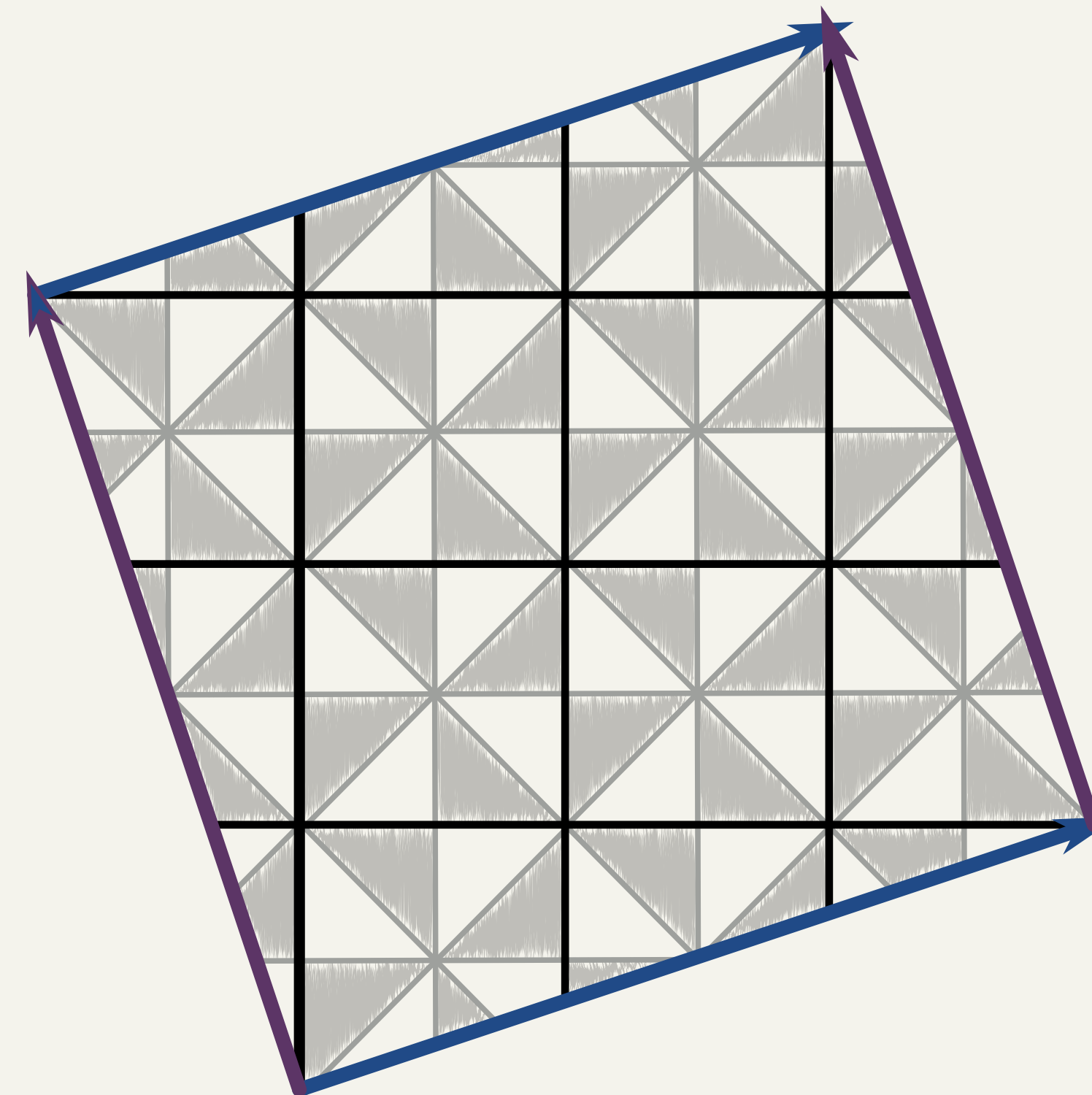
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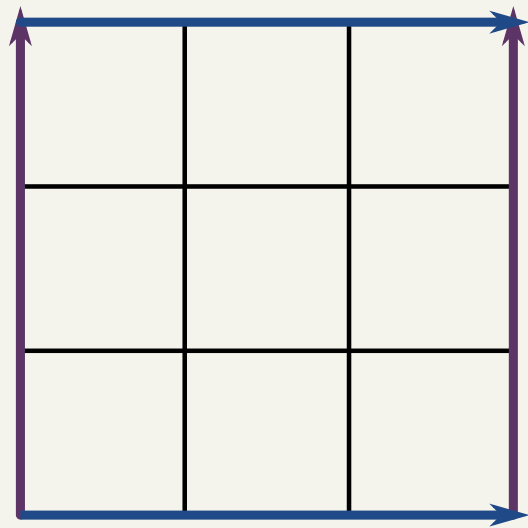
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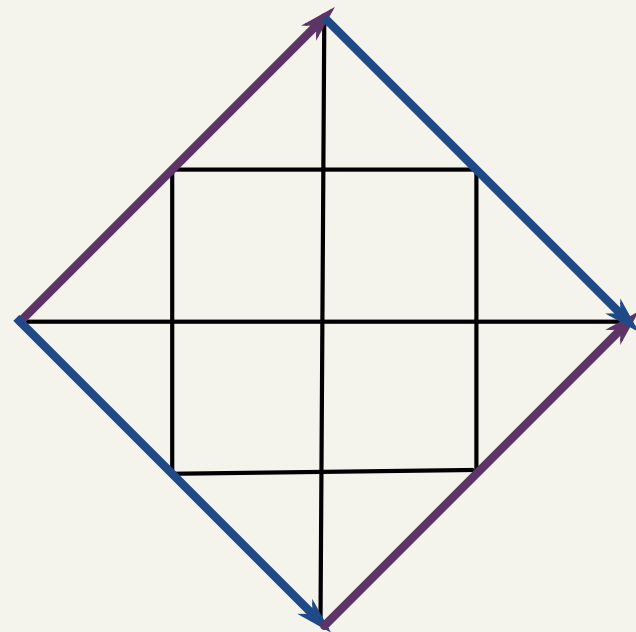
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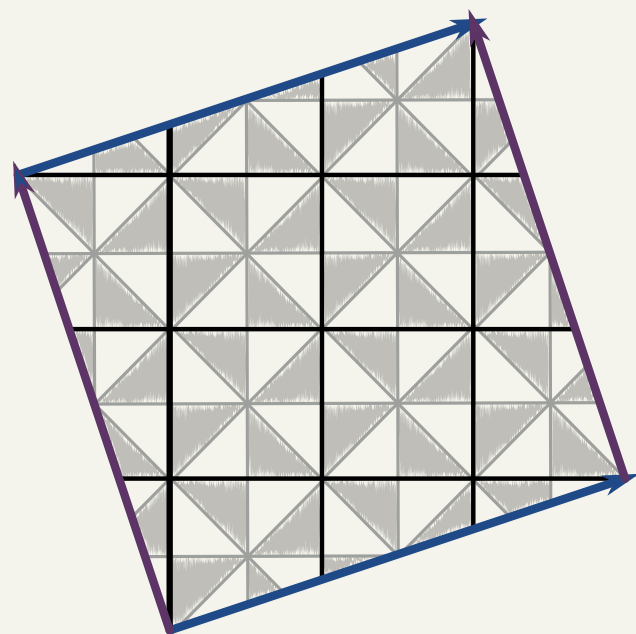
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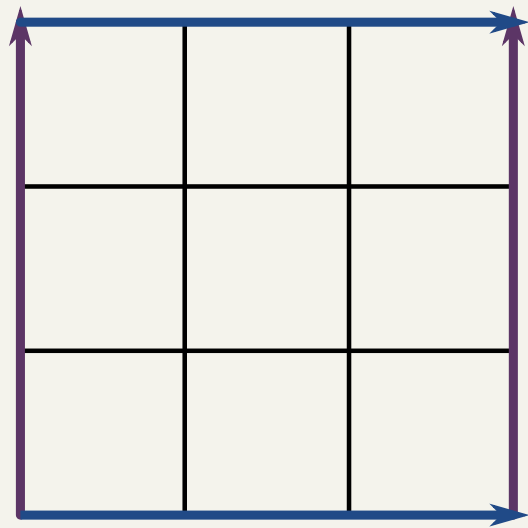
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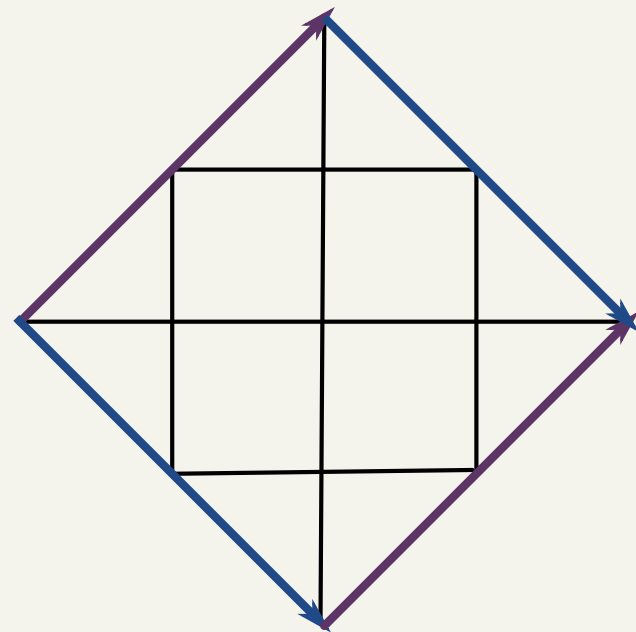
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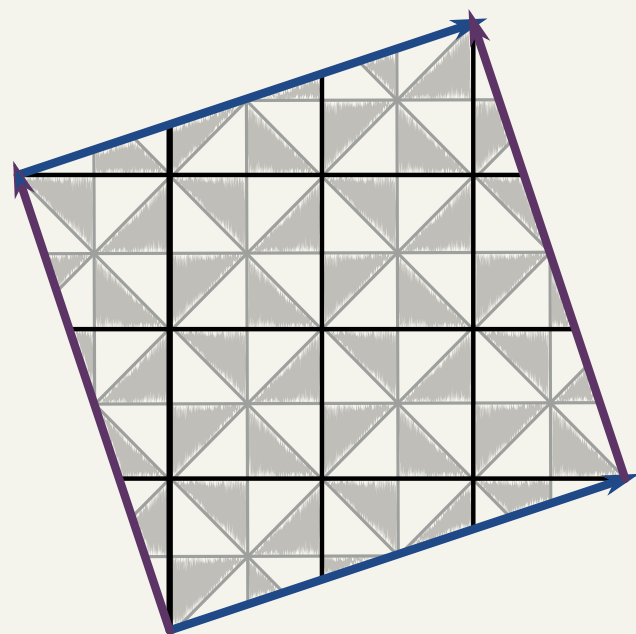
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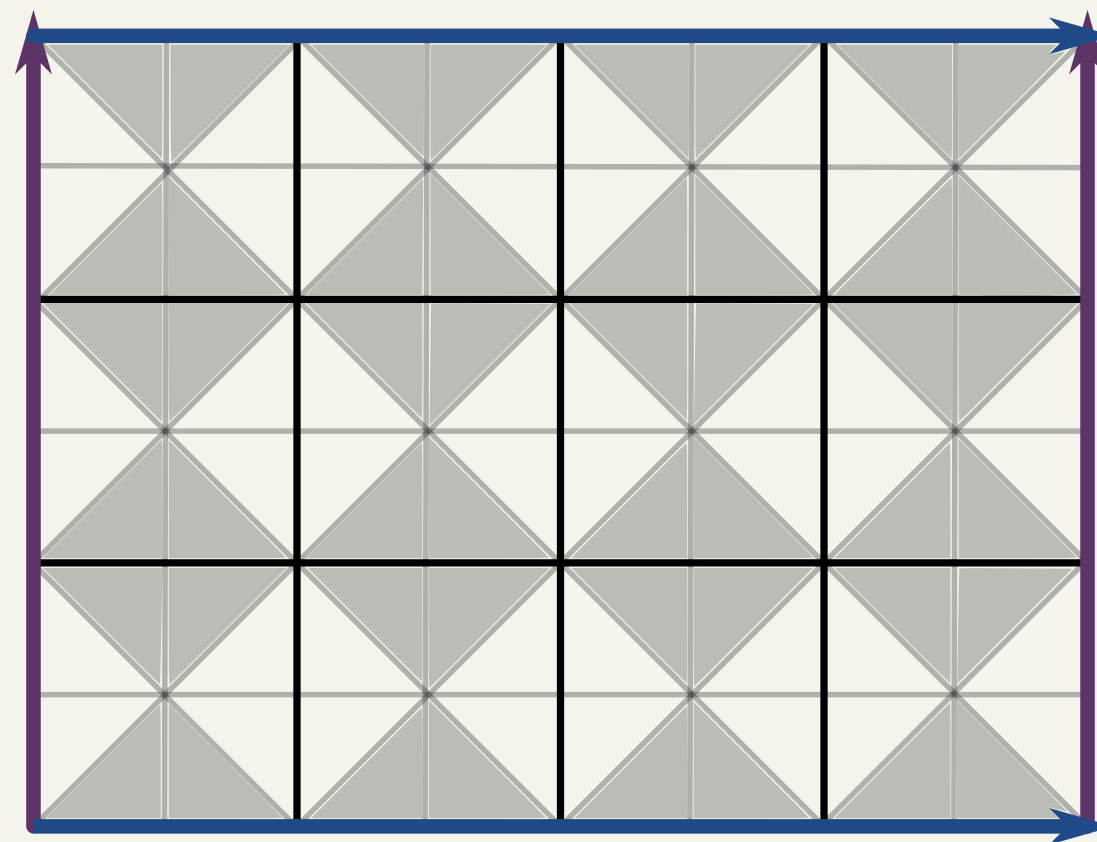
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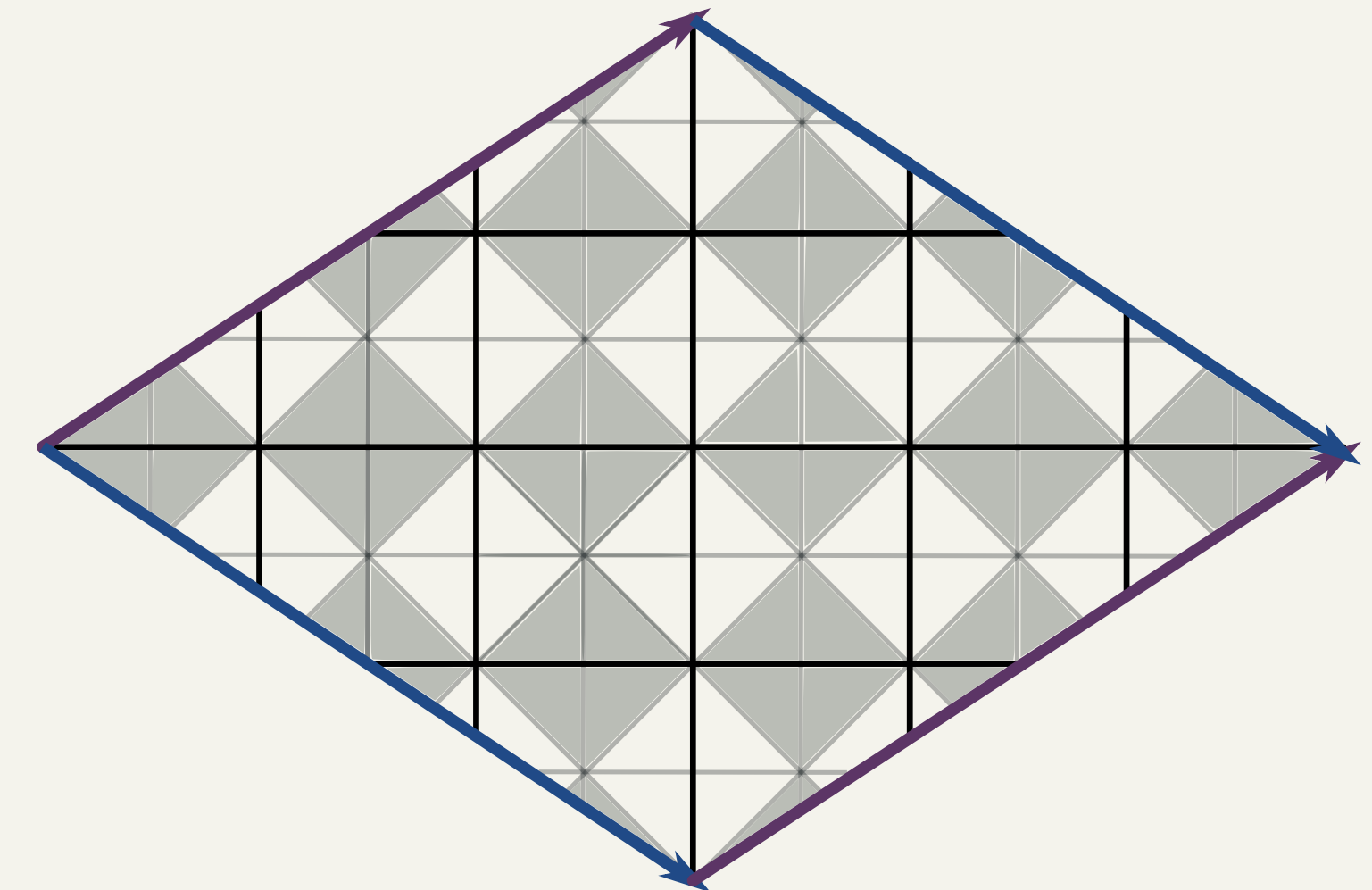
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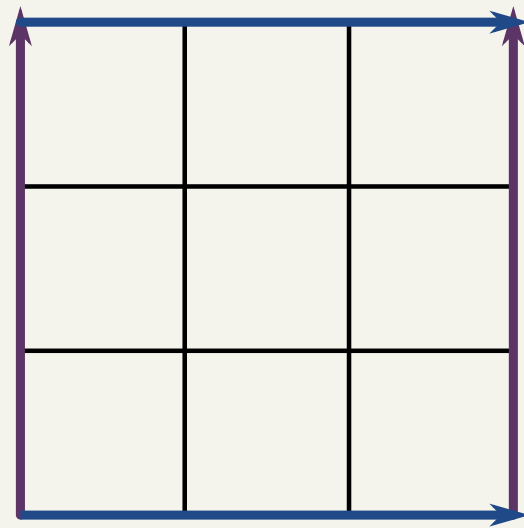
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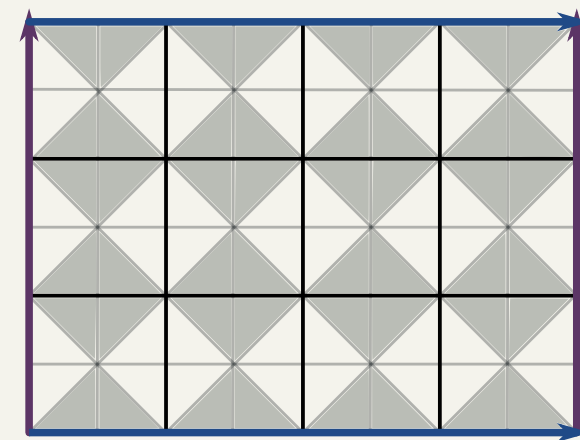
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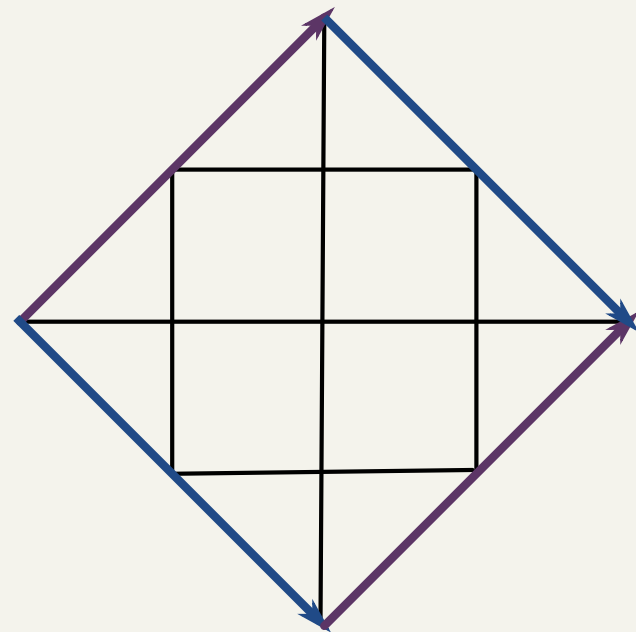
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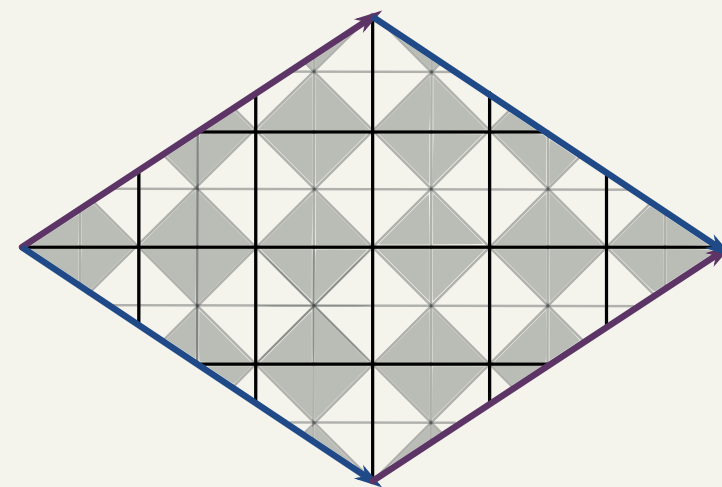
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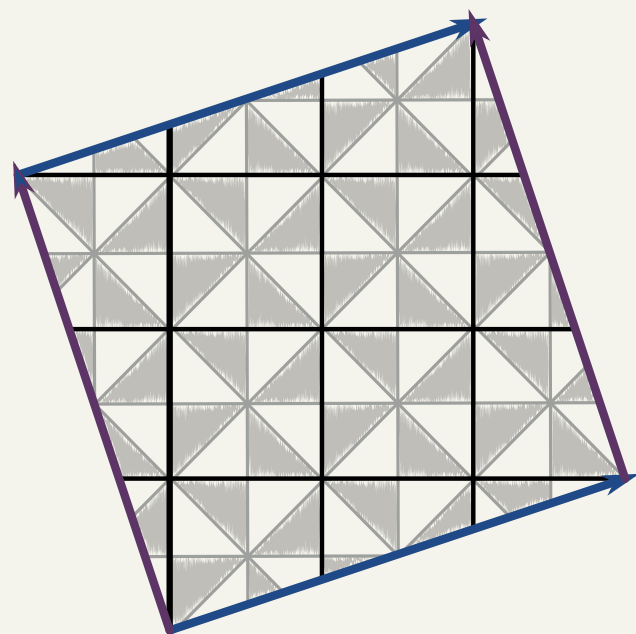
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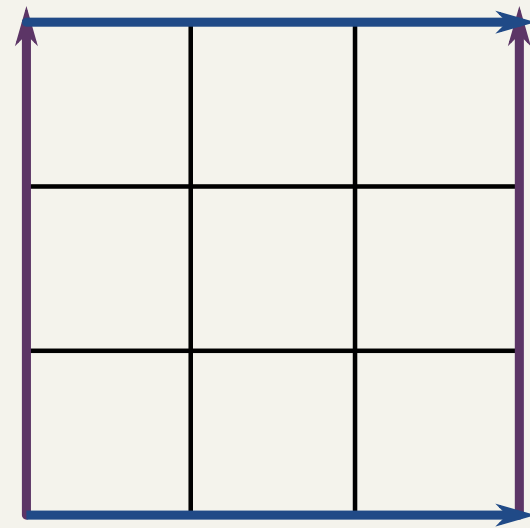
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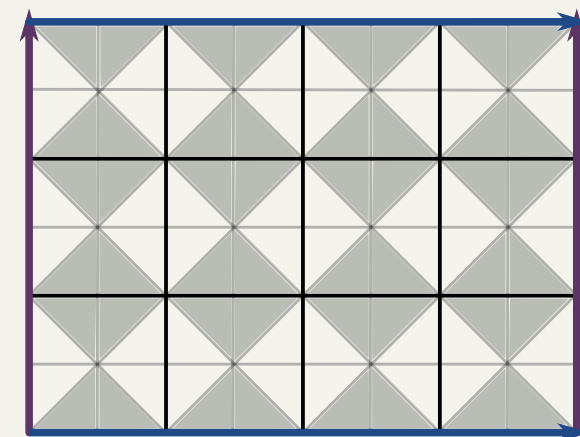
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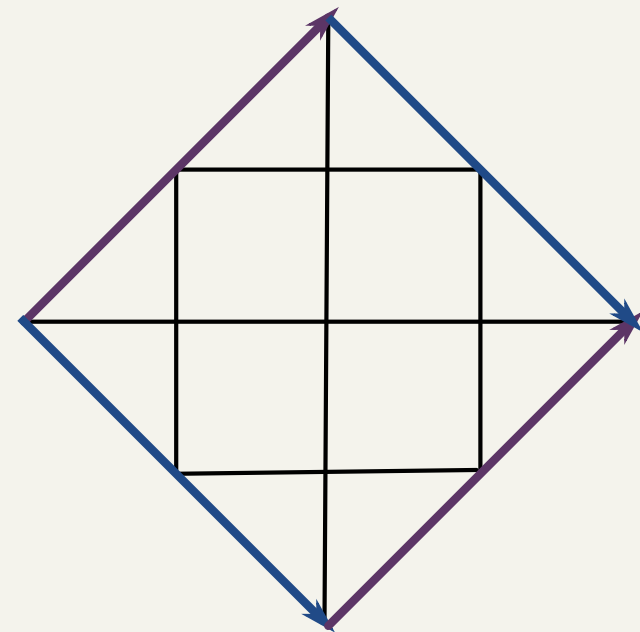
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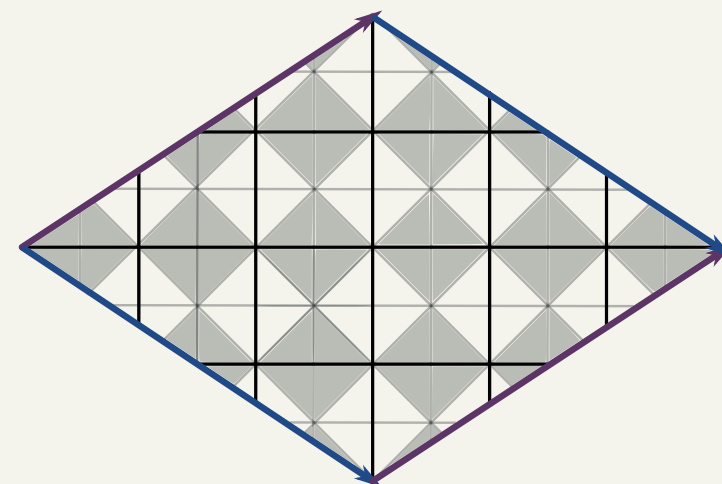
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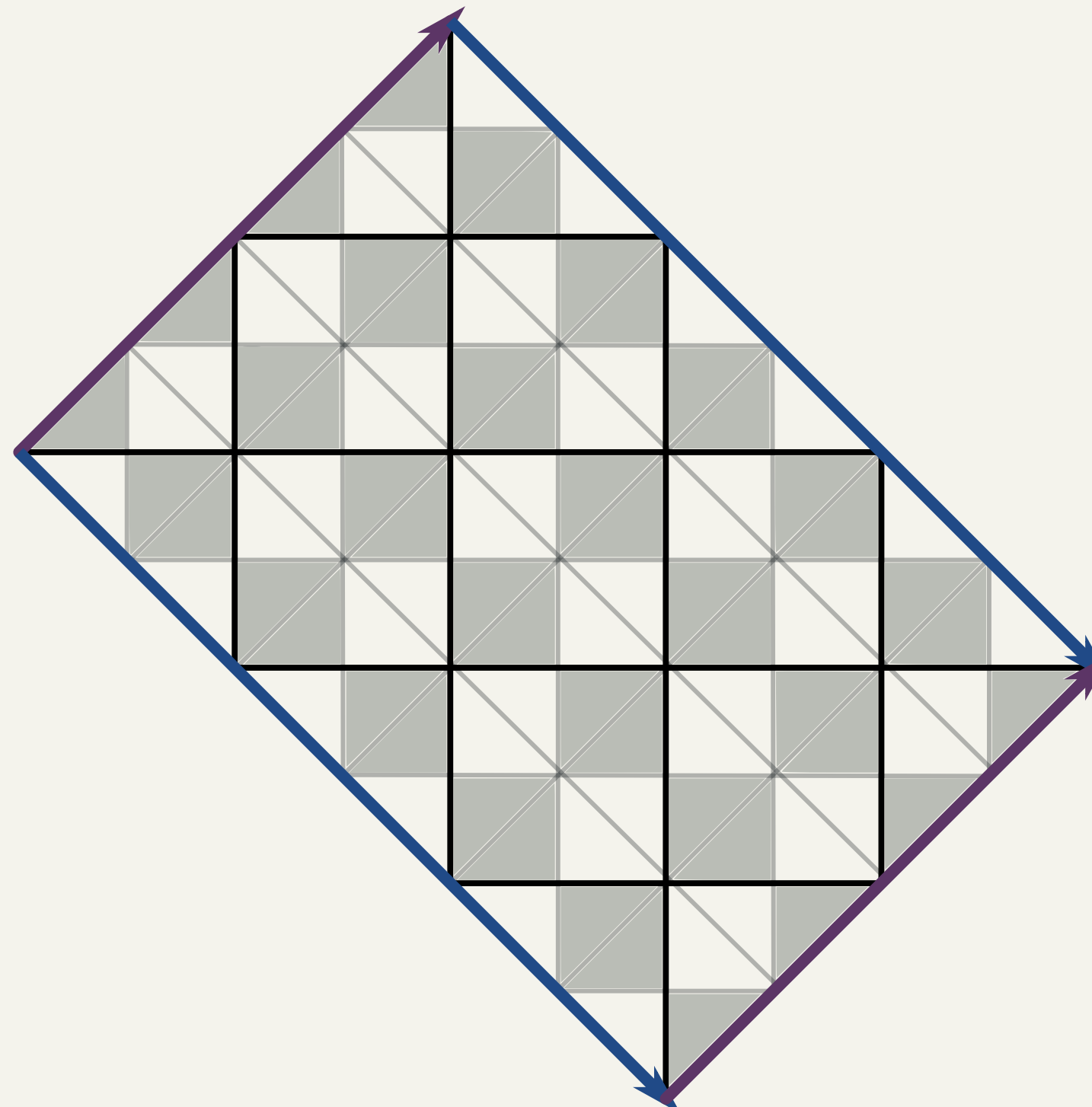
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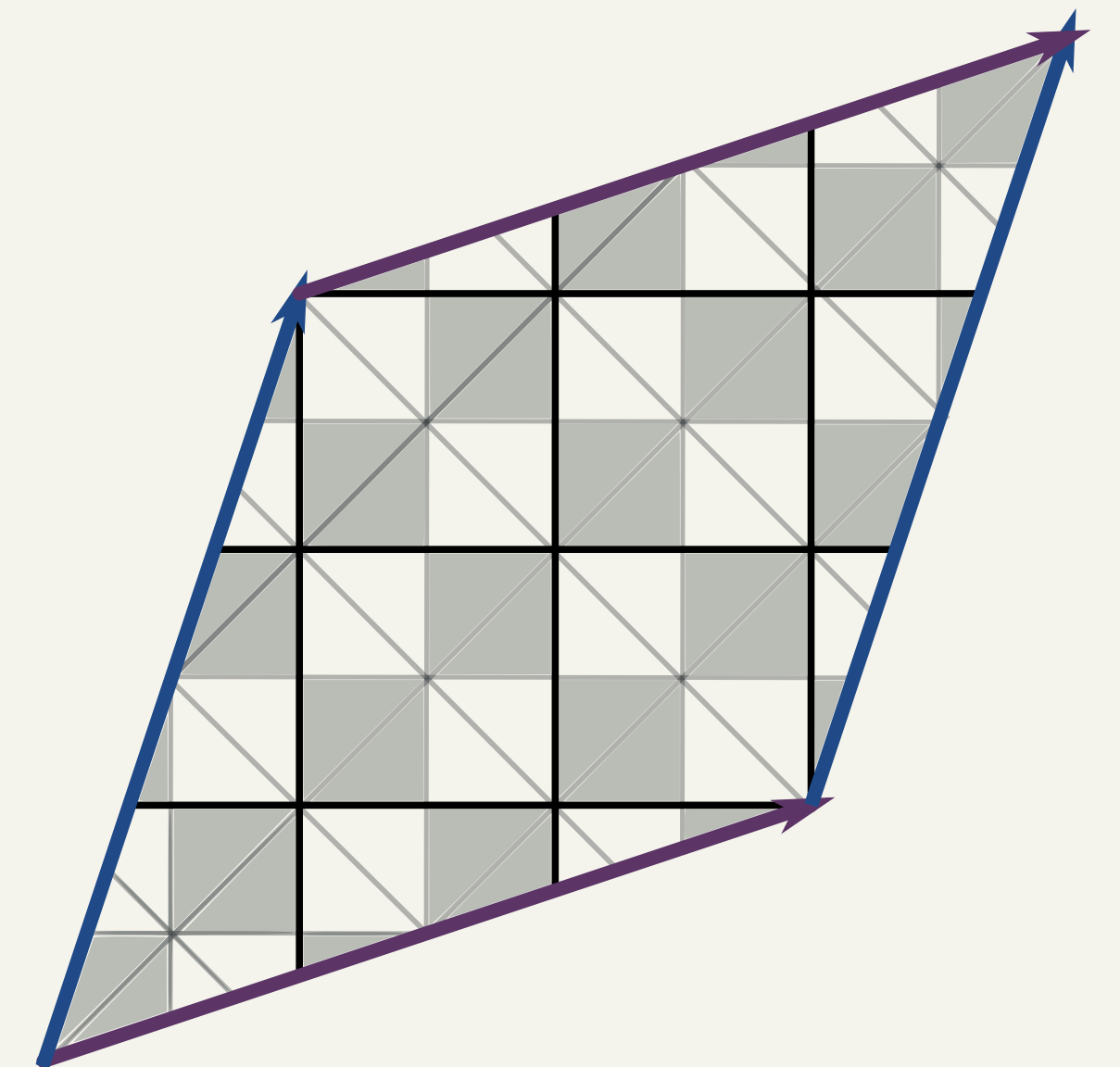
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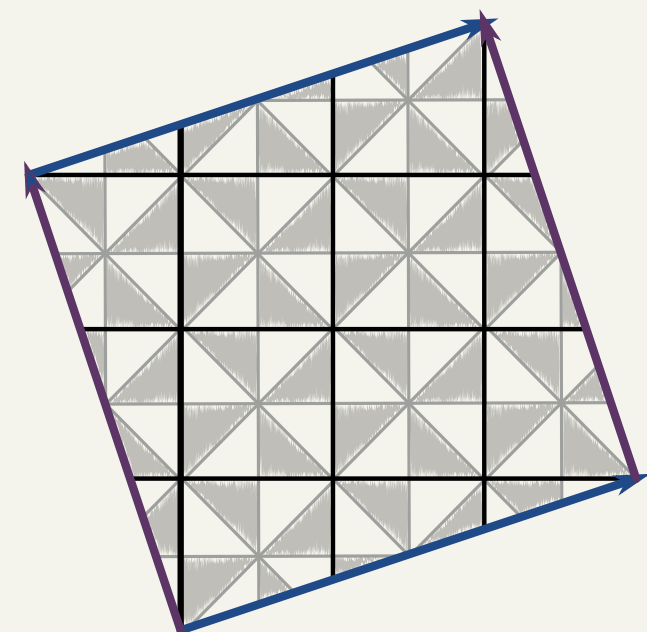
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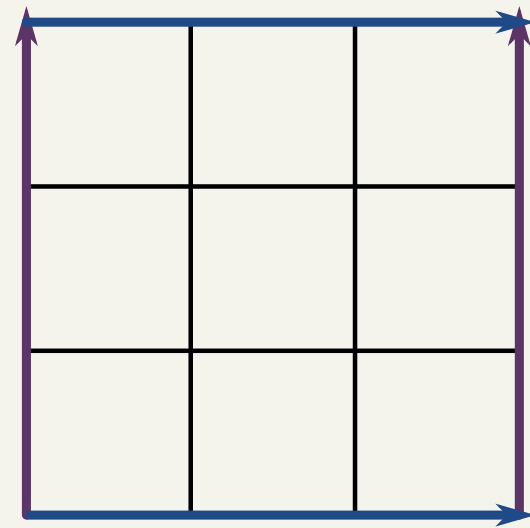
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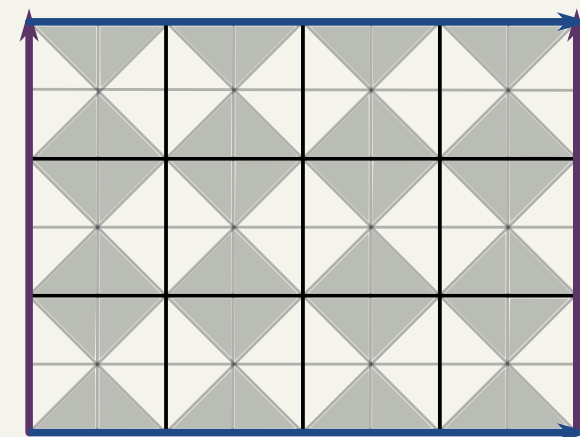
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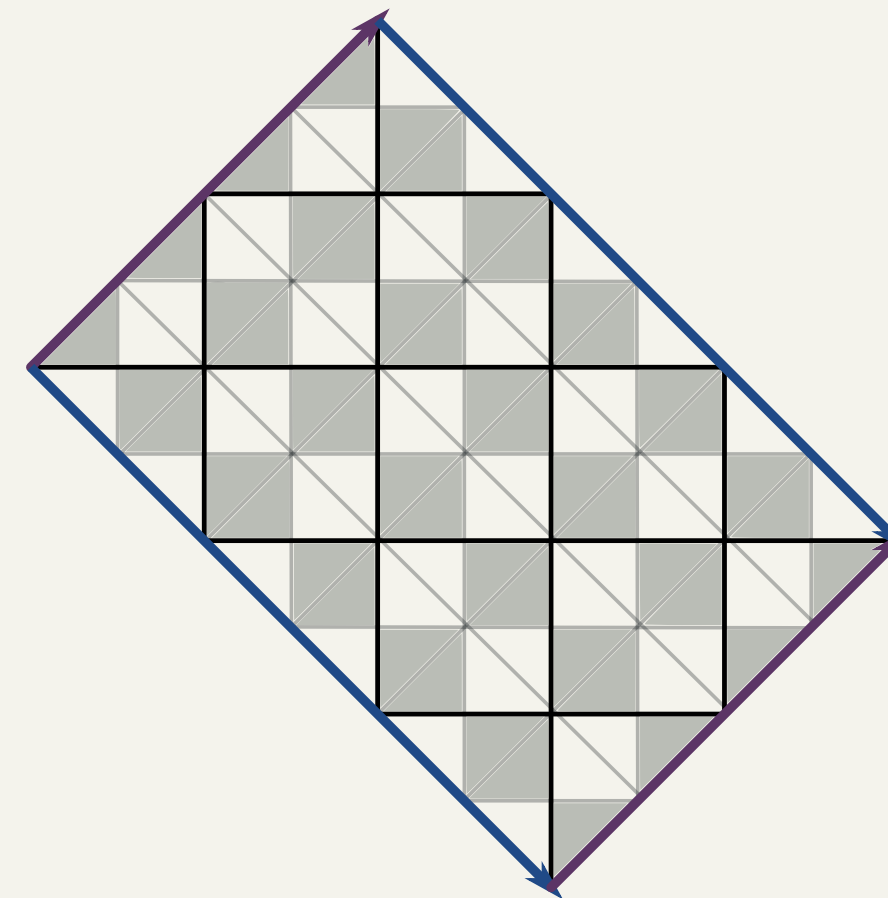
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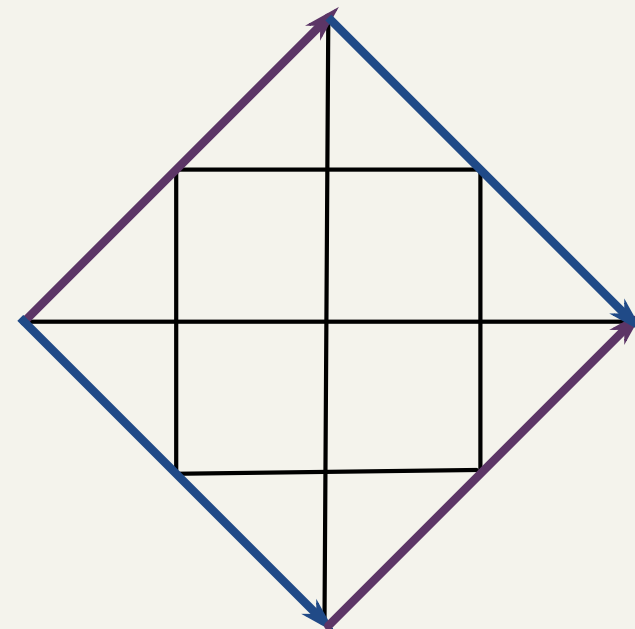
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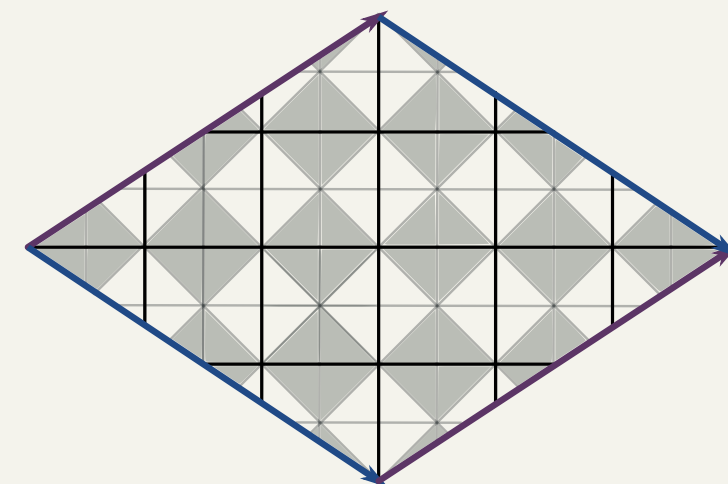
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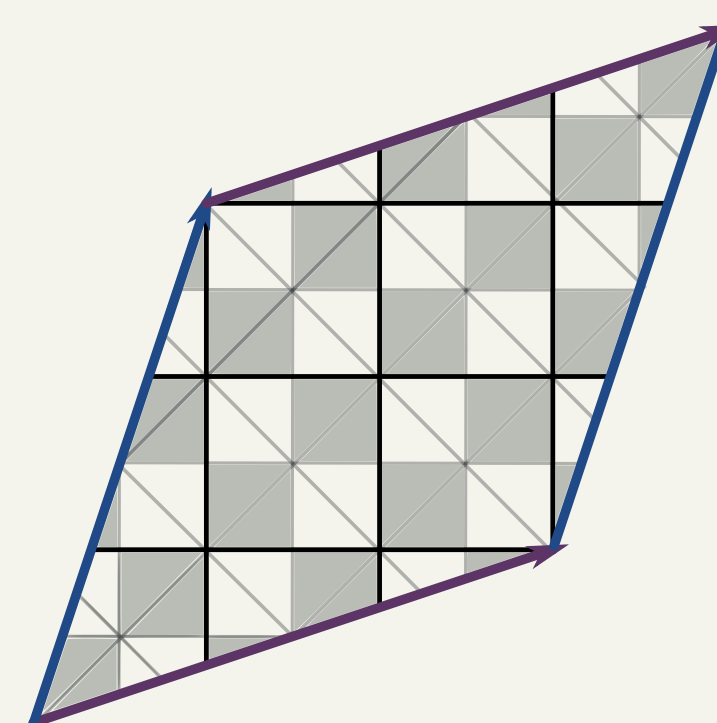
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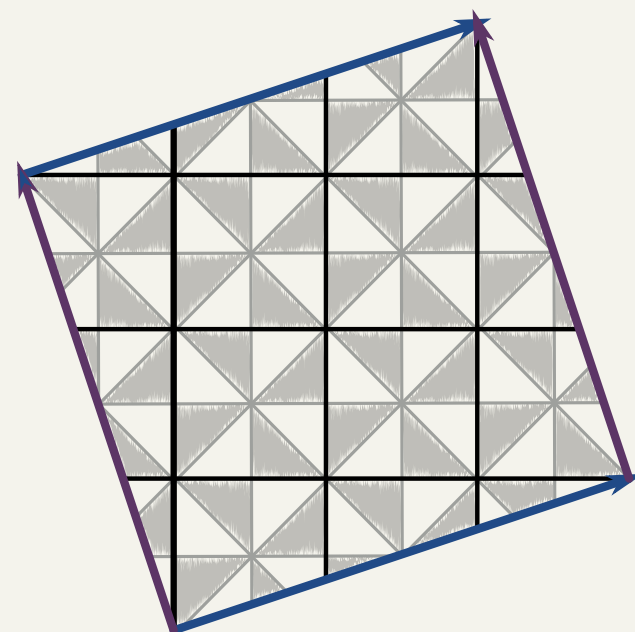
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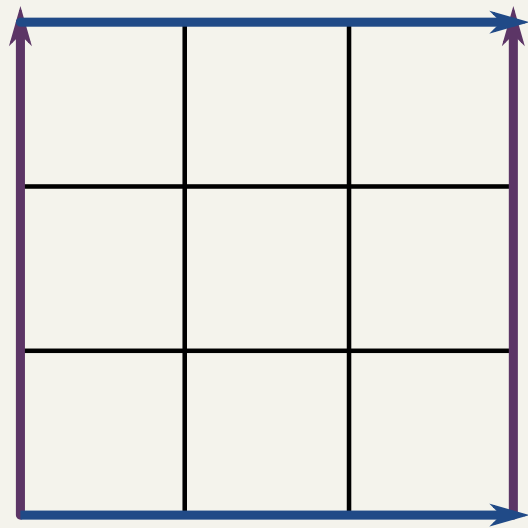
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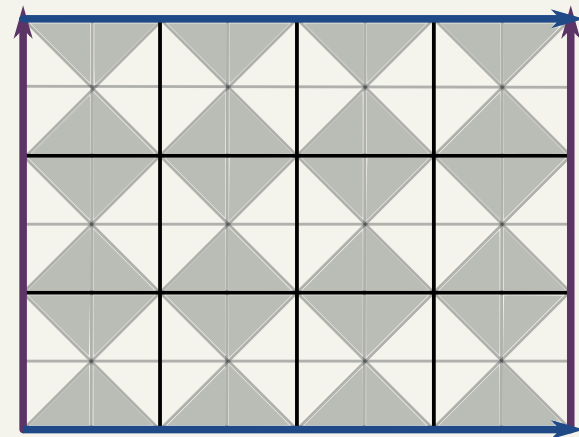
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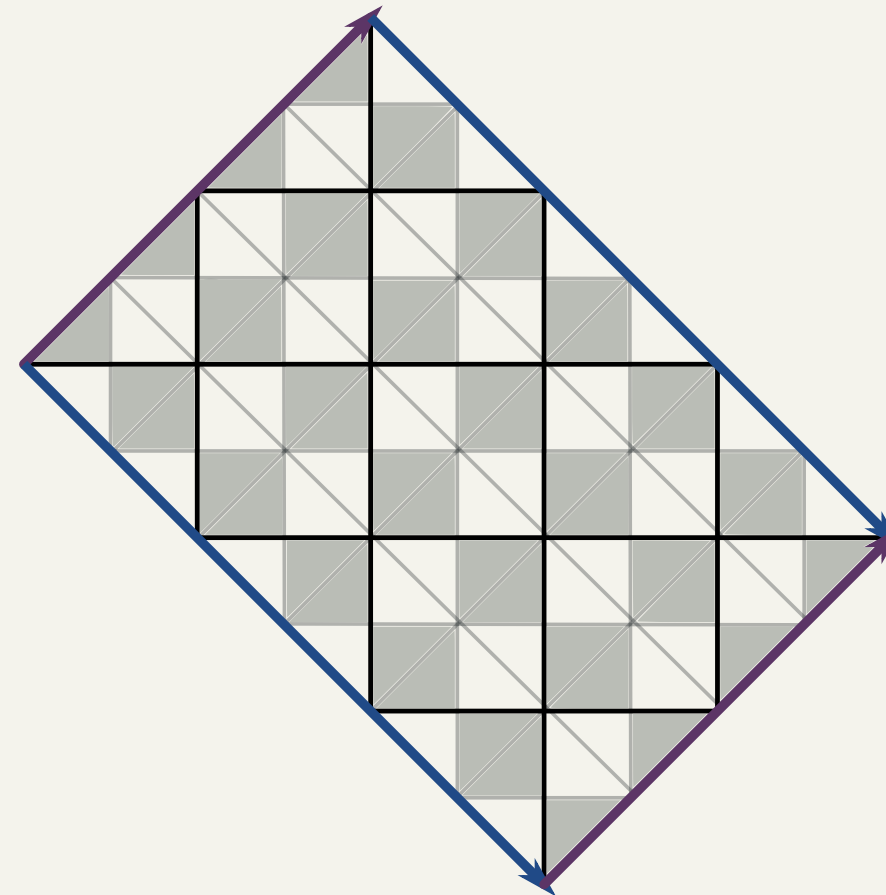
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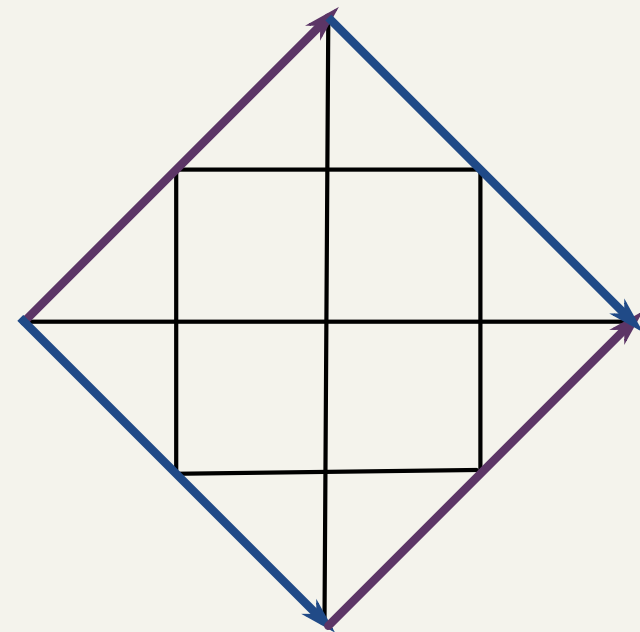
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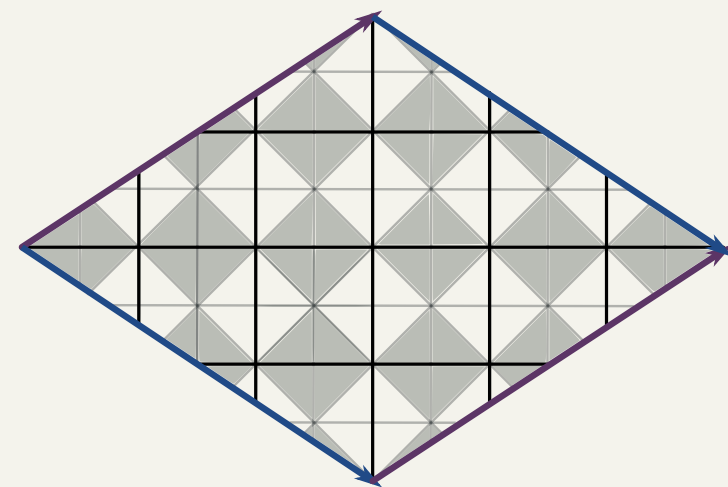
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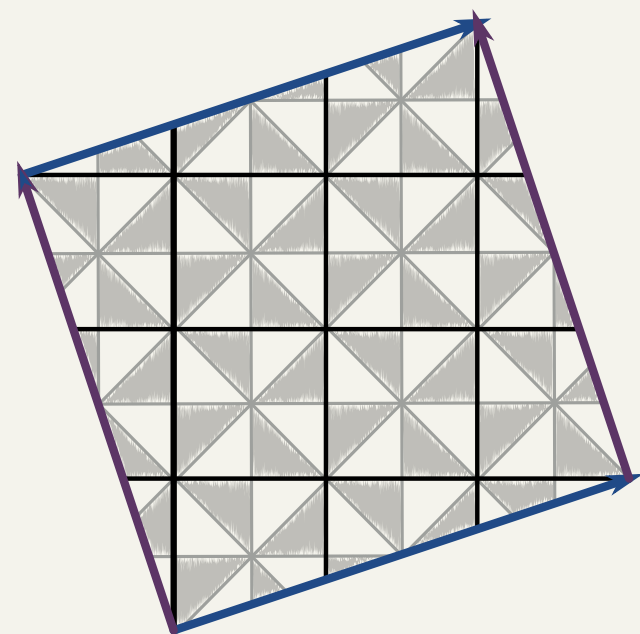
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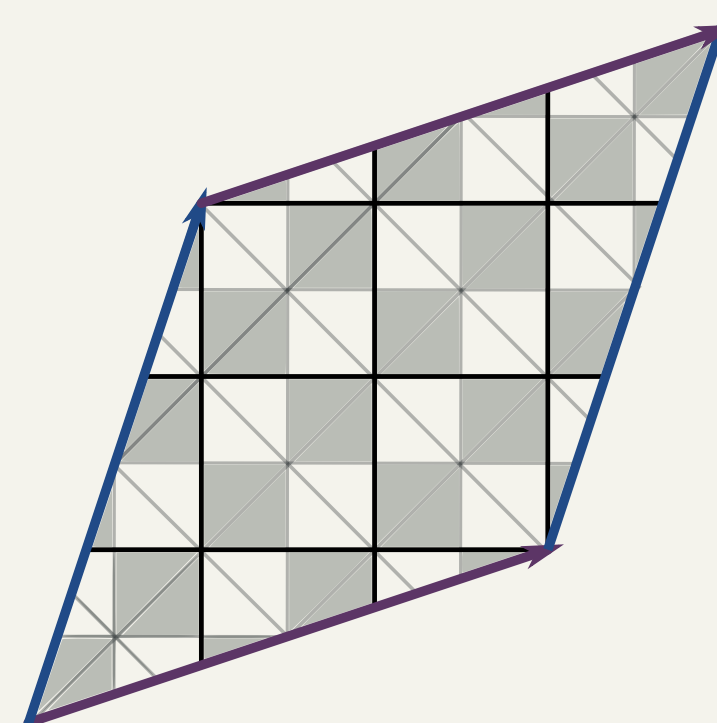
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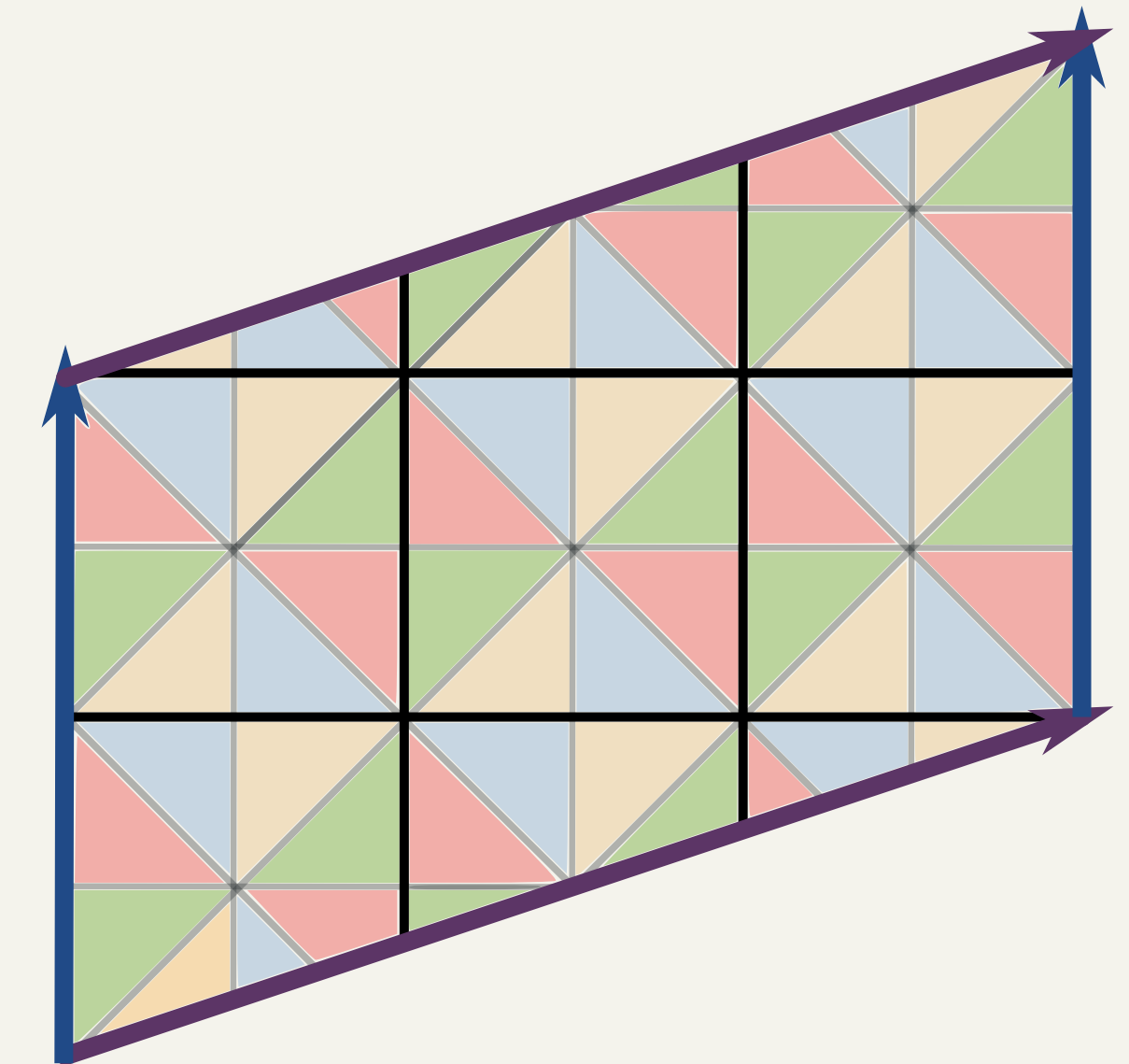
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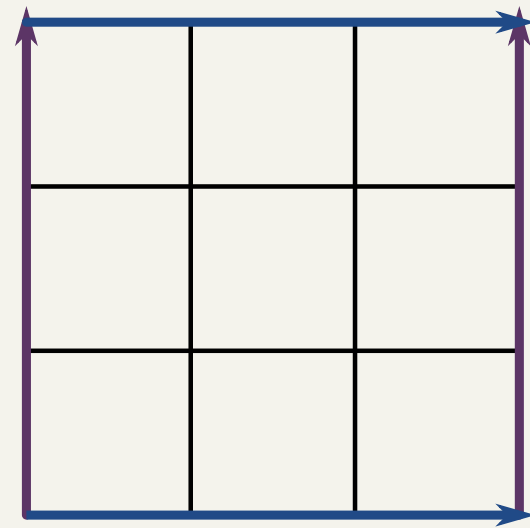
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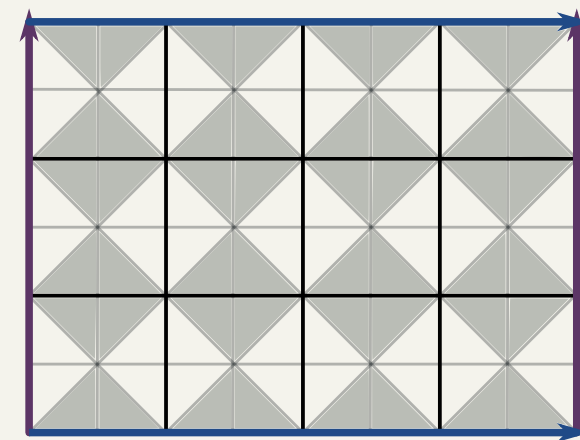
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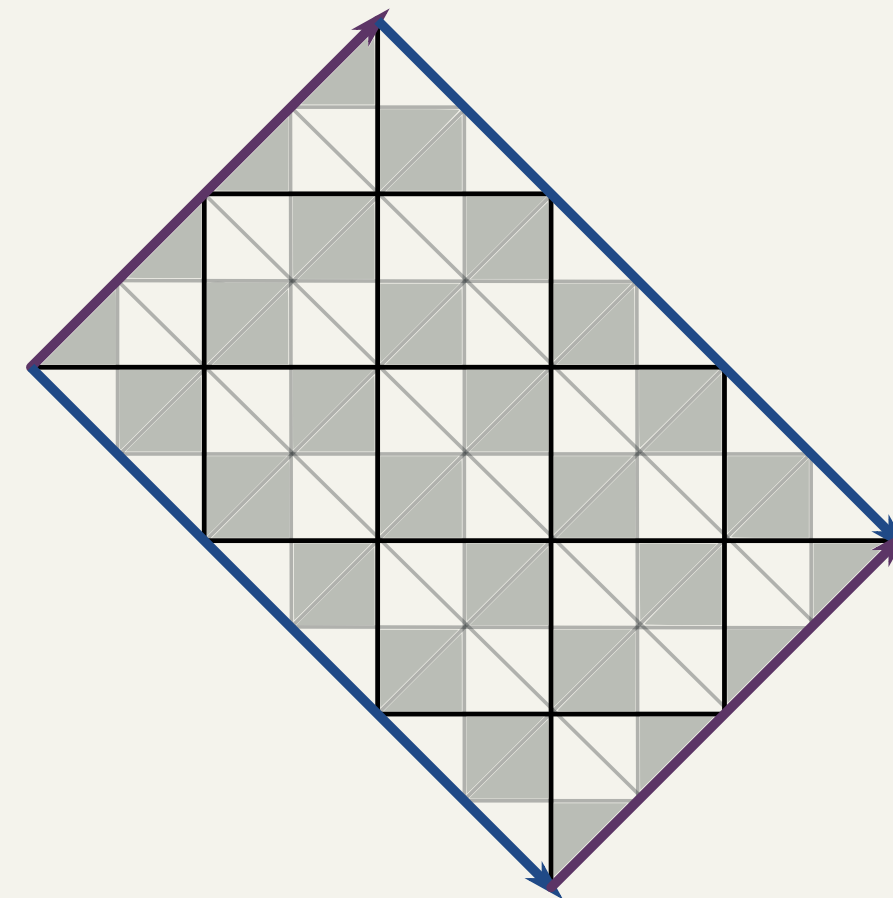
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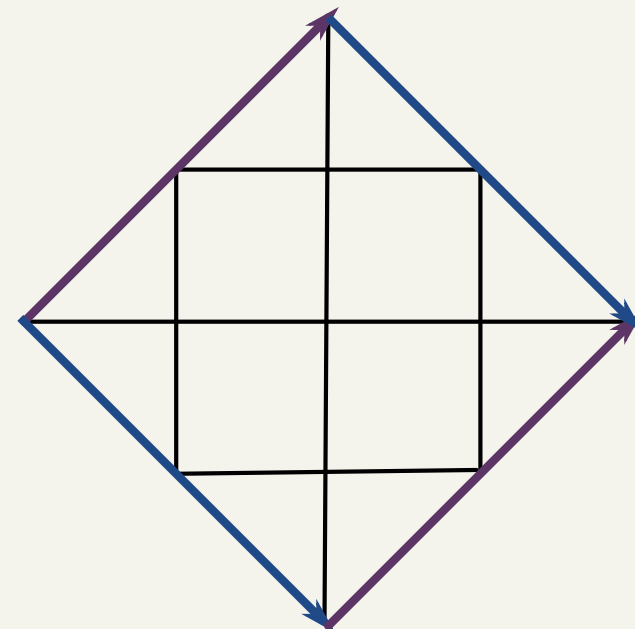
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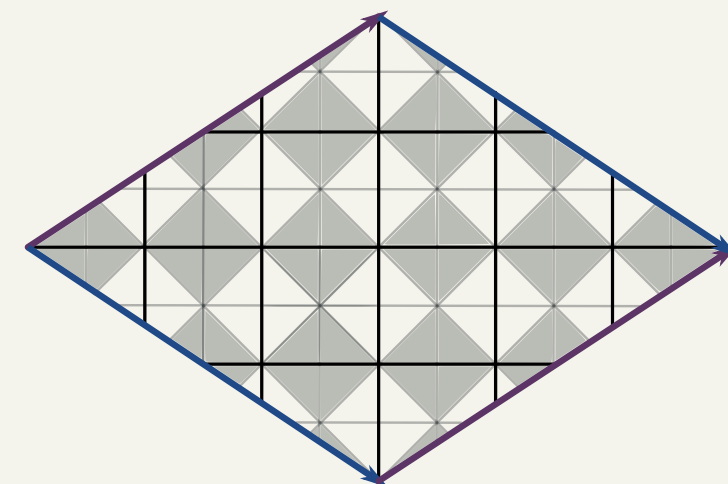
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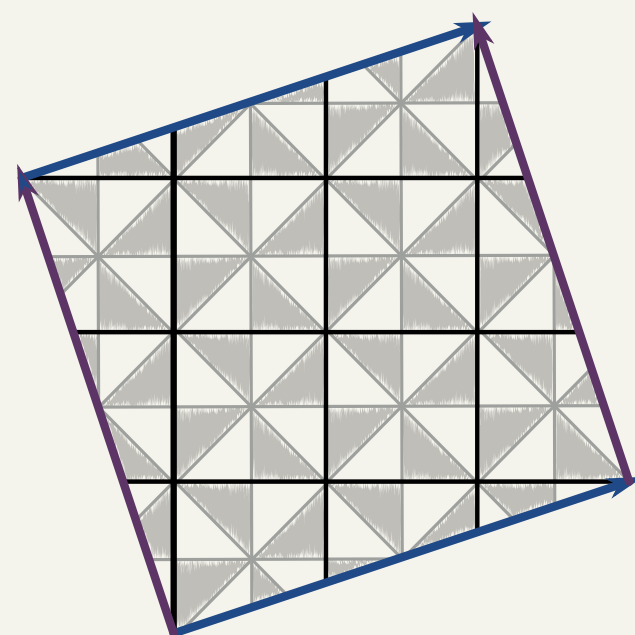
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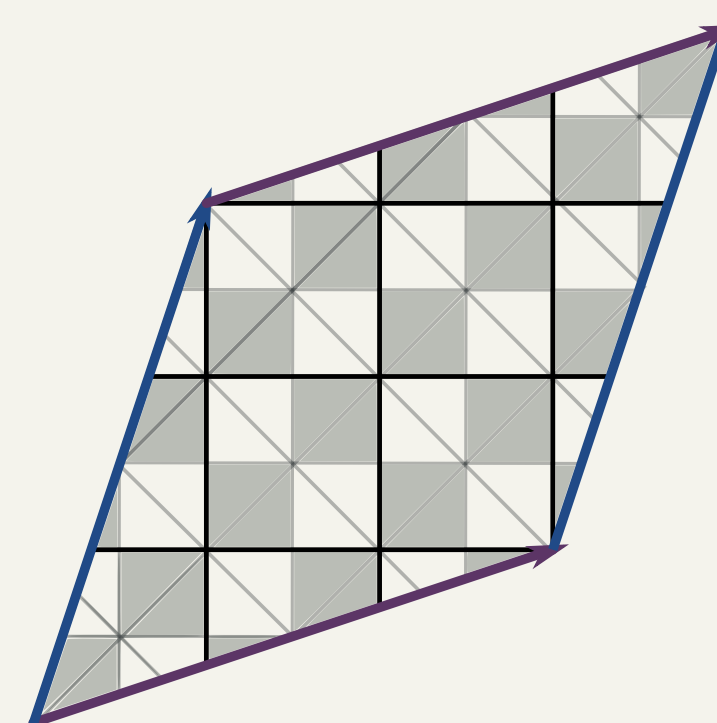
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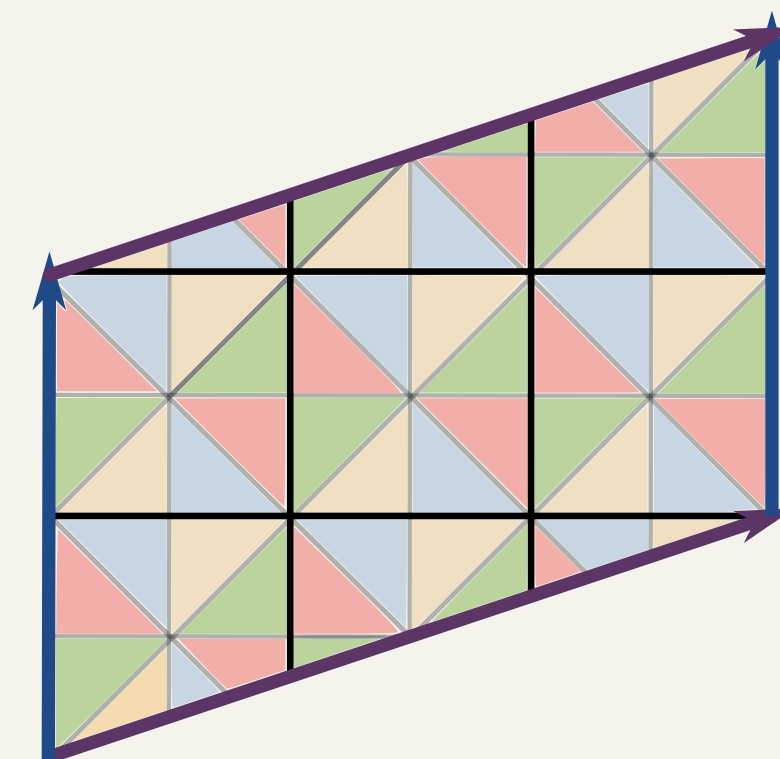
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- Duarte (2007)
- Brehem and Kühnel (2008)
- Hubbard, Pellicer, Orbanič, Weiss (2012)

# Symmetries of $n$ -dimensional toroids

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Egon Schulte



Michael Hartley



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Classification of regular  $n$ -dimensional toroids



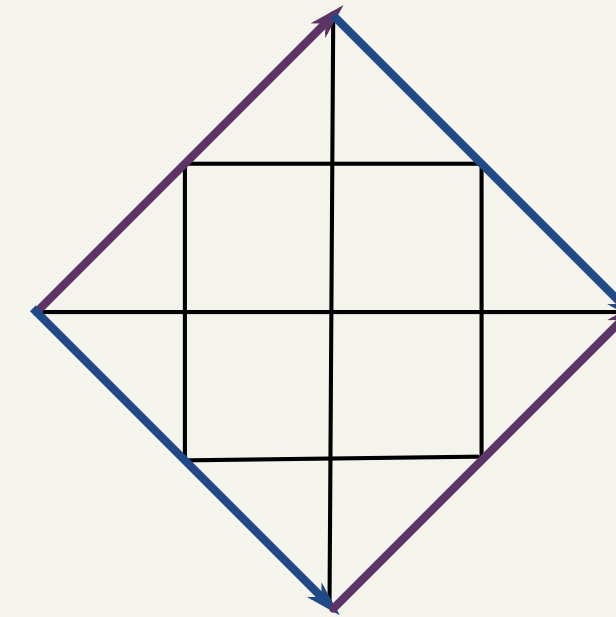
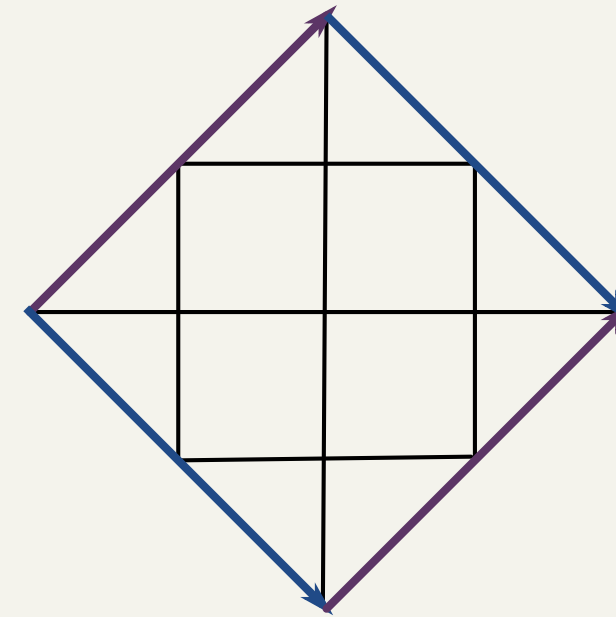
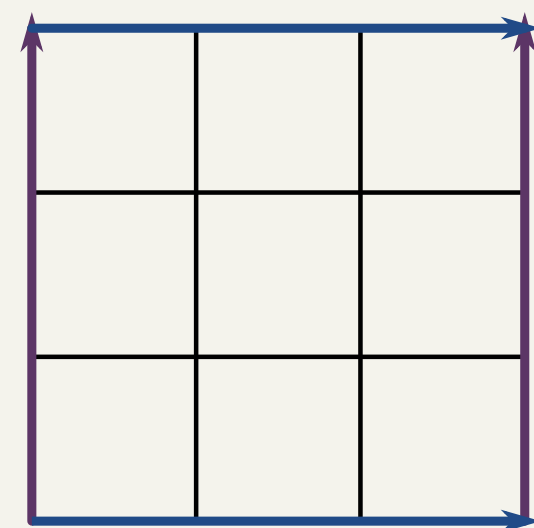
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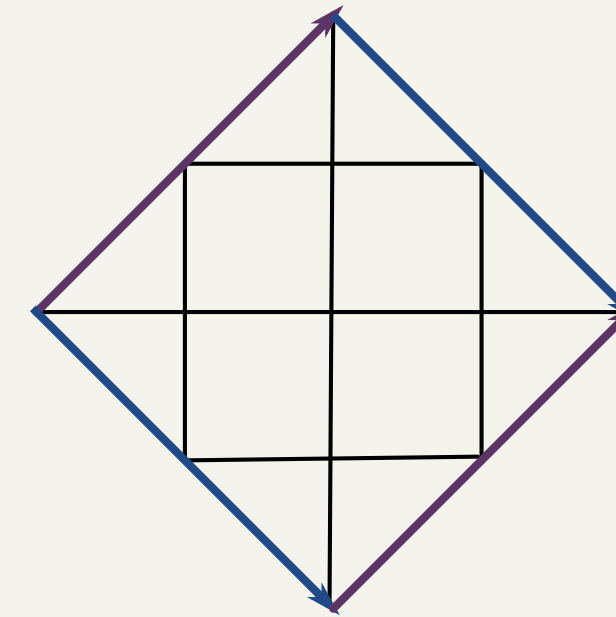
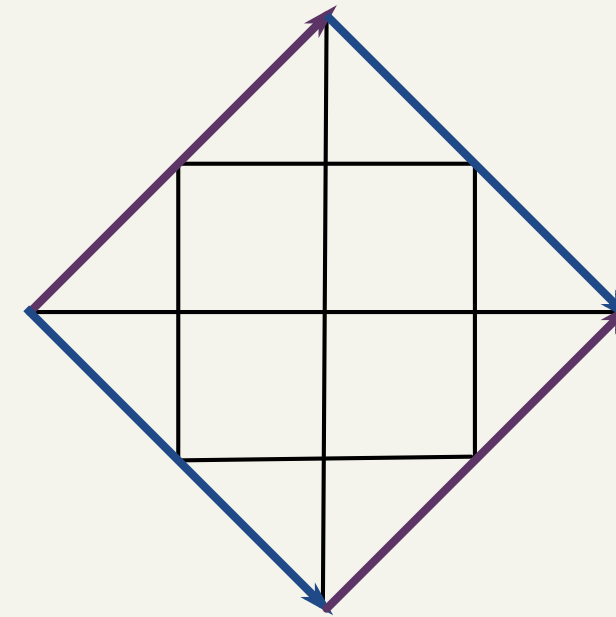
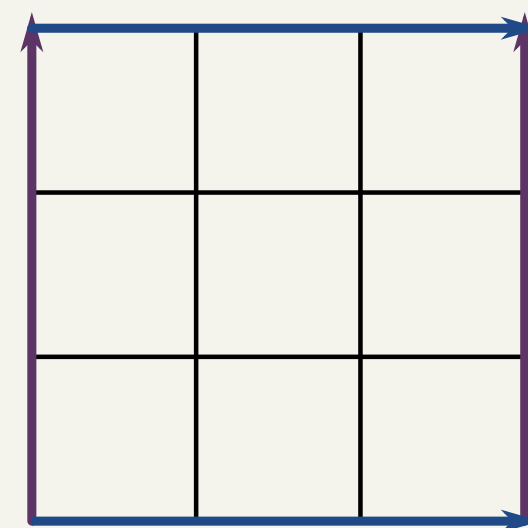
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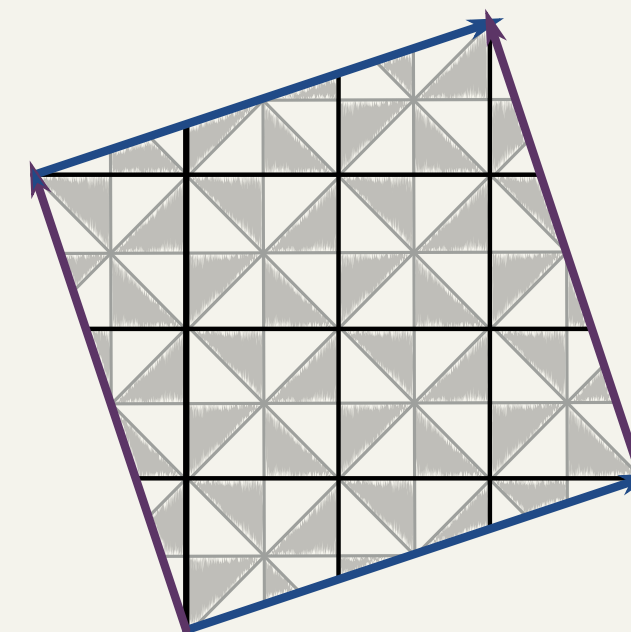
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There are no chiral  $n$ -dimensional toroids for  $n \geq 3$



Peter McMullen

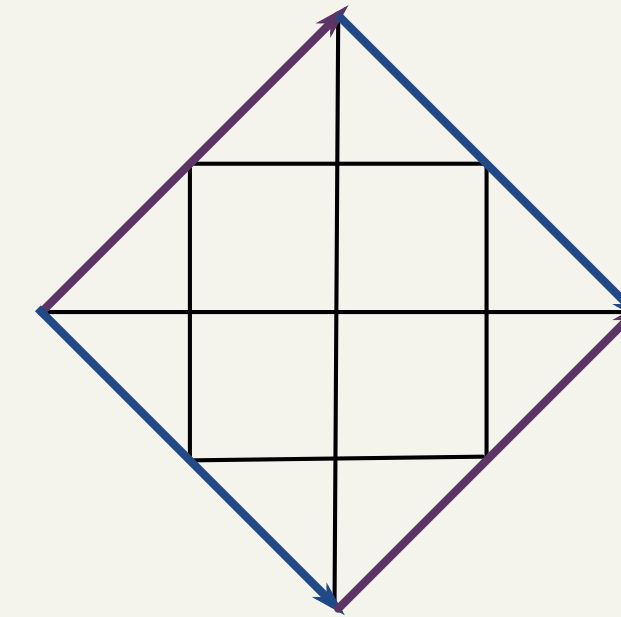
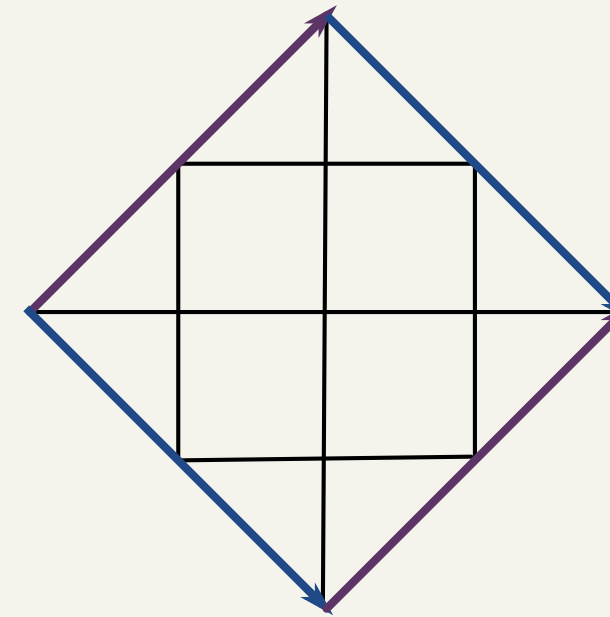
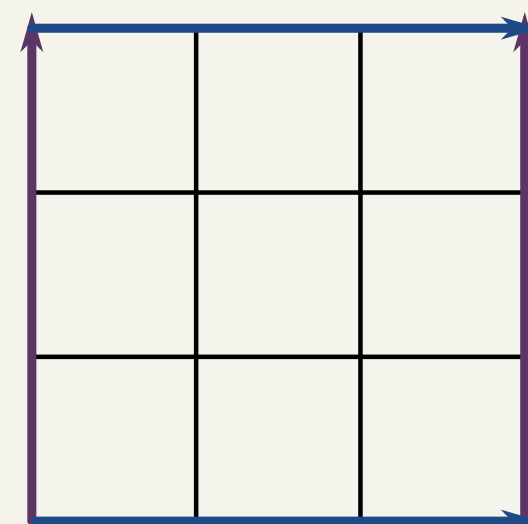
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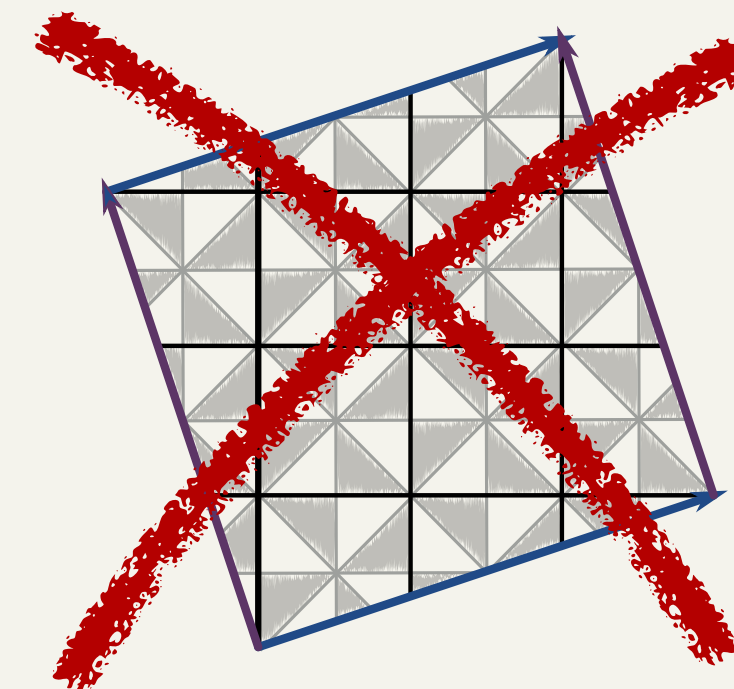
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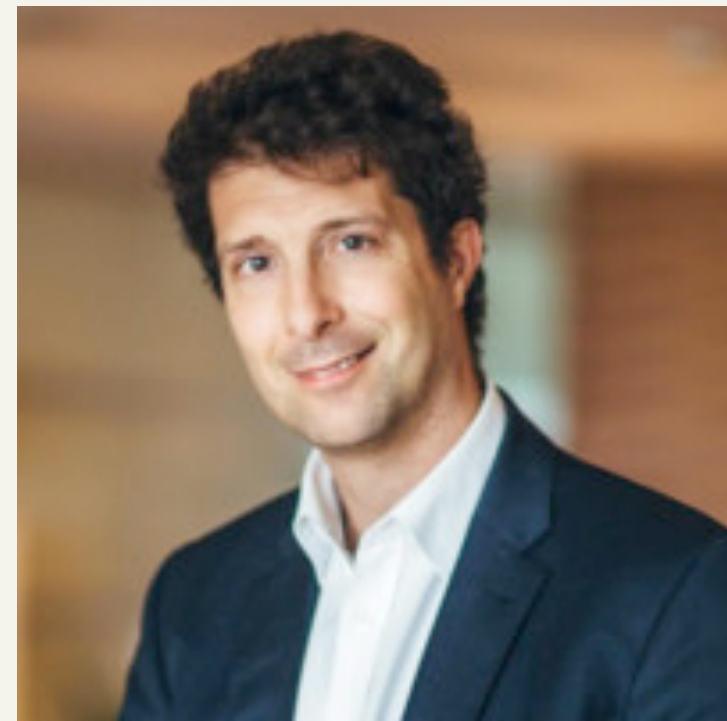
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Isabel Hubard



Alan Orbanič



Daniel Pellicer



Asia Ivić Weiss

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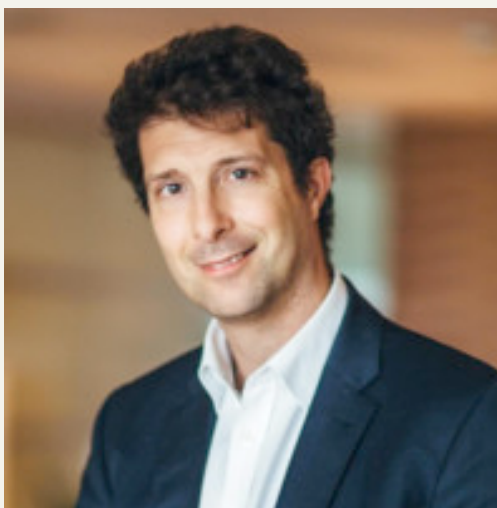


# Symmetries of $n$ -dimensional toroids

What are the possibilities for the symmetry type of an equivelar tessellation of the 3-torus?



- Three families of regular tessellations;
- Four families of 3-orbit tessellations;
- Sixteen families of 6-orbit toroids with three different symmetry type (7+7+2);
- Three families of 8-orbit;
- Twelve families of 12-orbit with two different symmetry types (7+5);
- The rest, with 24 flag-orbits.



# Symmetries of $n$ -dimensional toroids

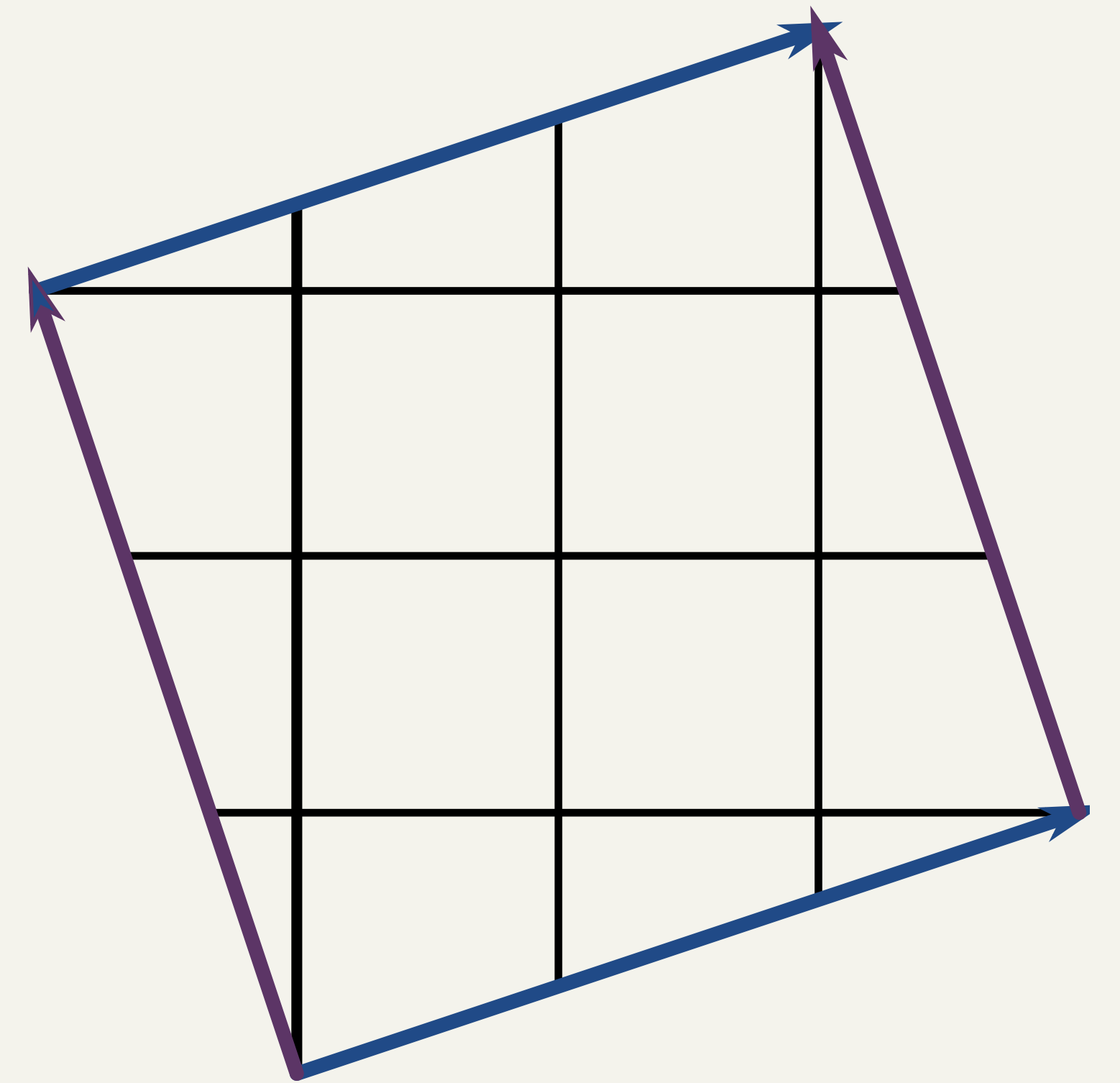
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The secret... we solve geometric problems without geometry.

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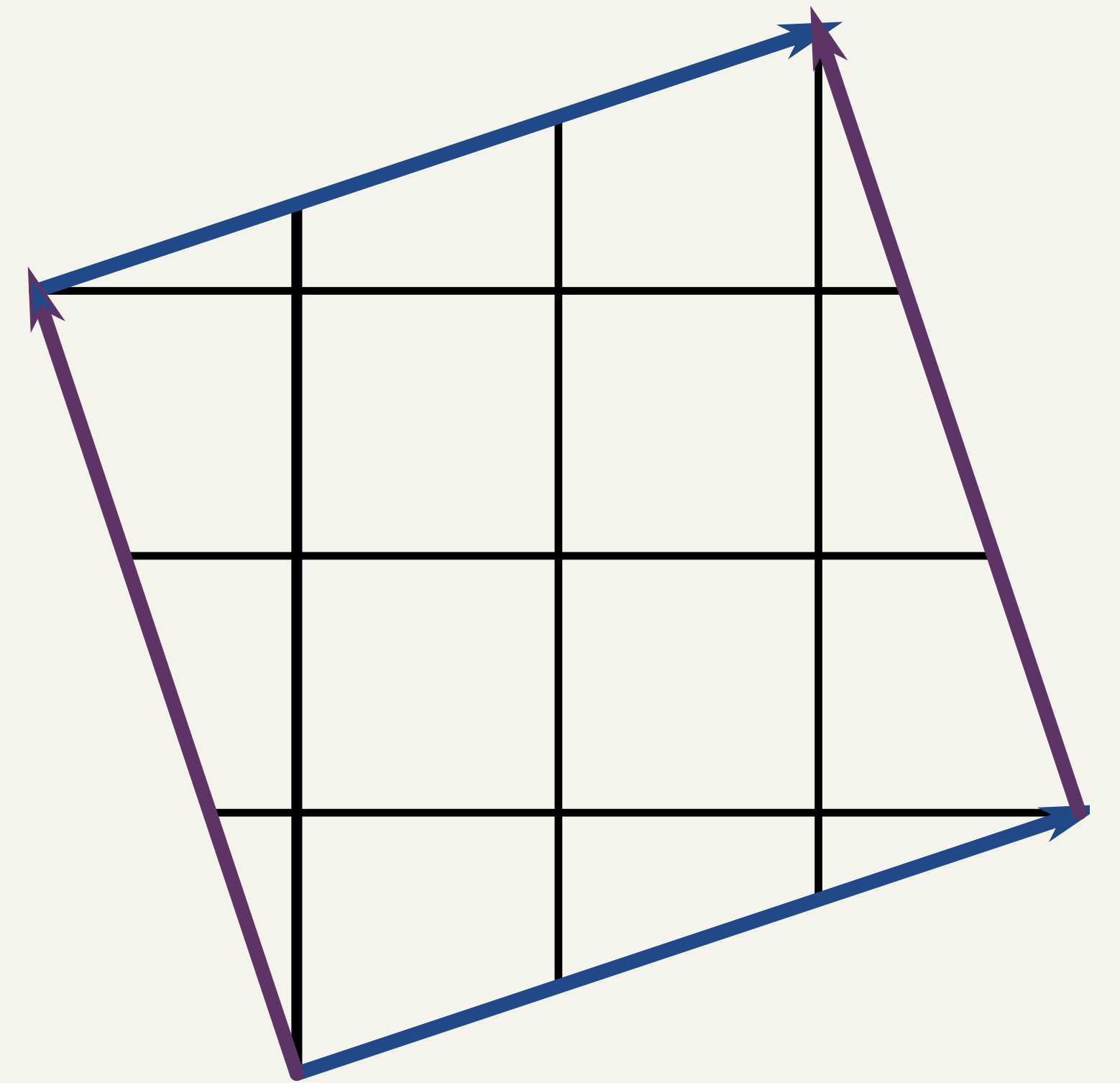
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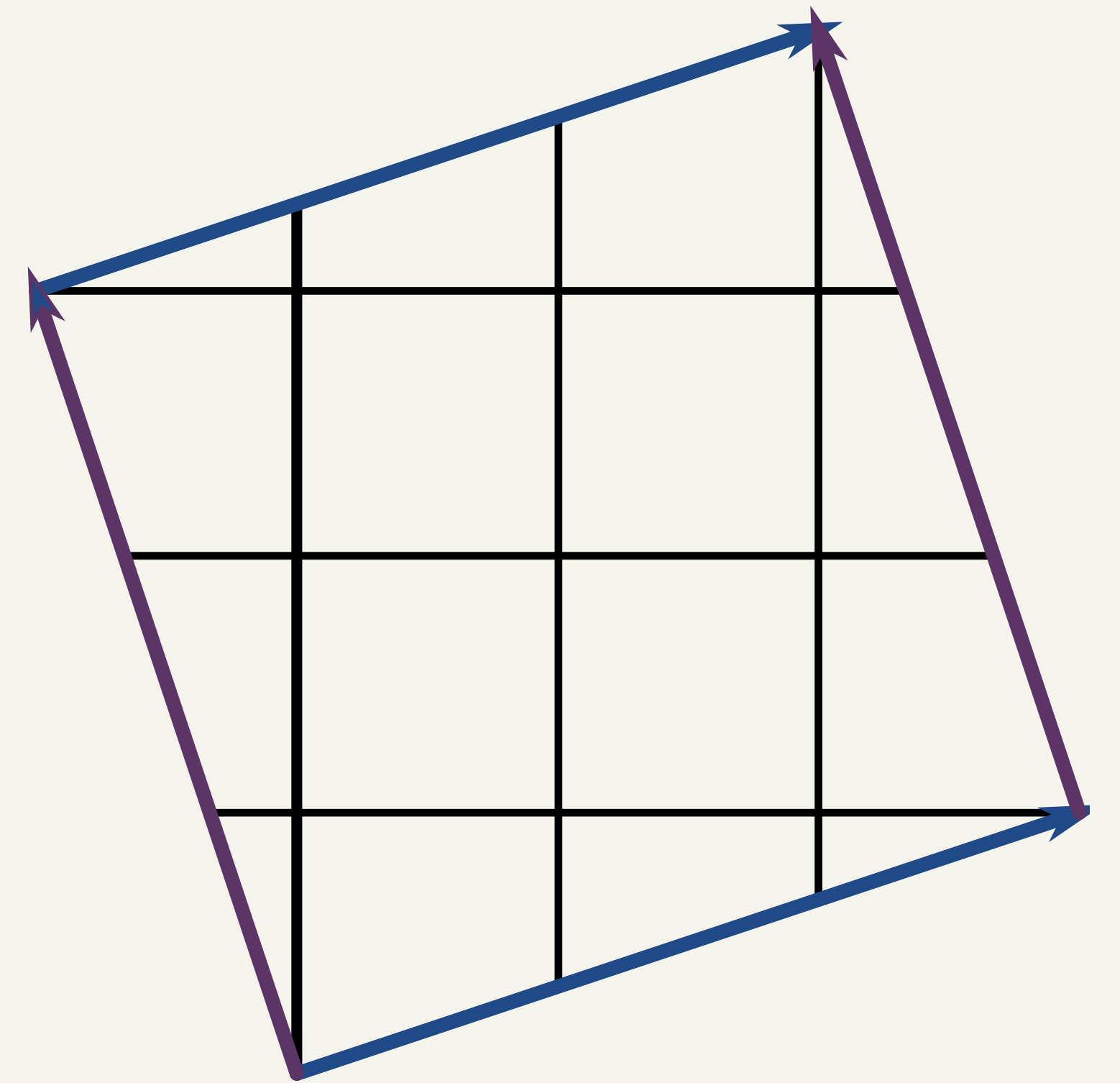
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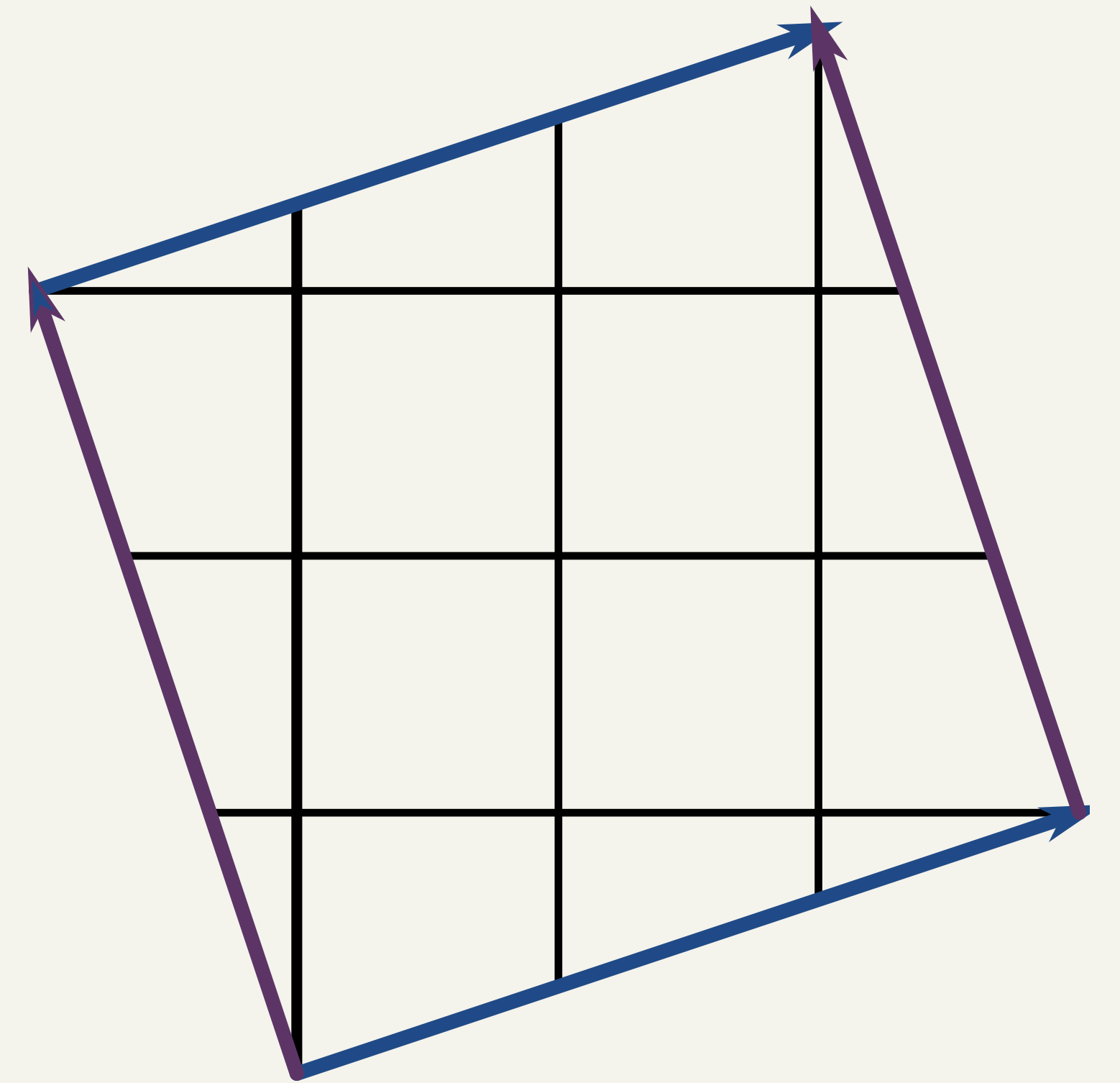
- Why not all of them are regular?
  - Not all the symmetries of  $\mathcal{U}$  are **compatible** with the group  $\Lambda$
- Which of them are?



# Symmetries of $n$ -dimensional toroids

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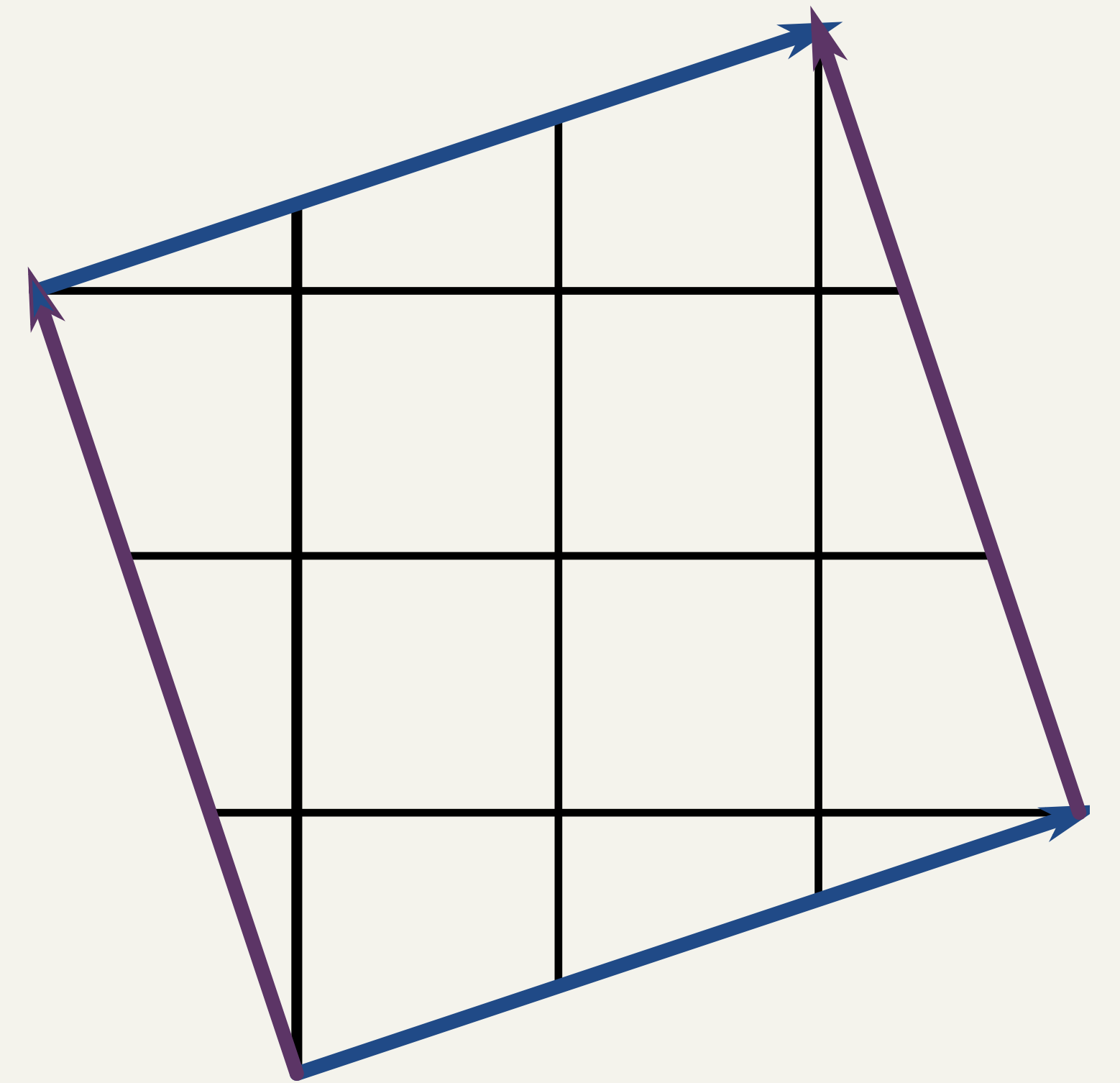
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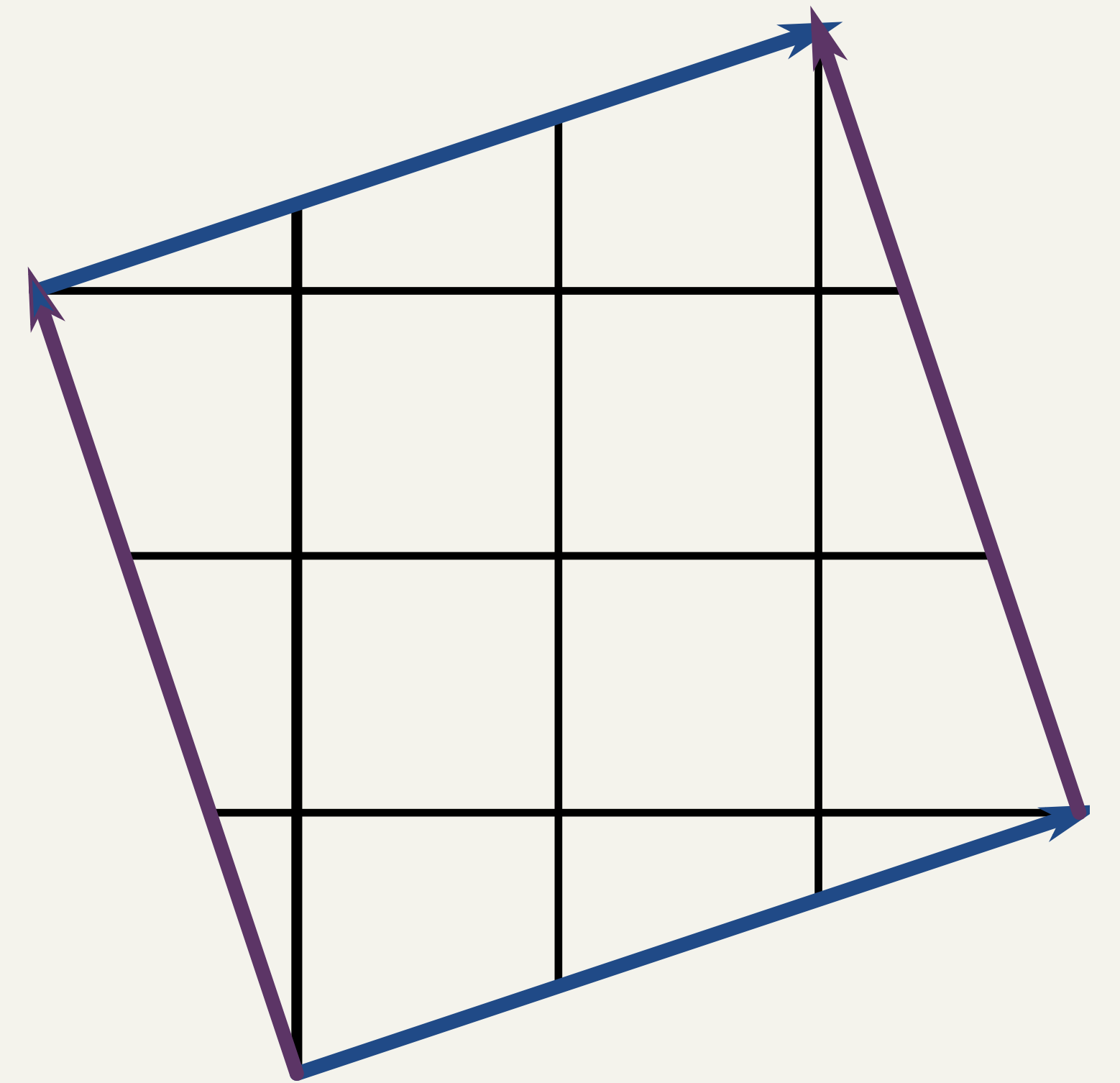
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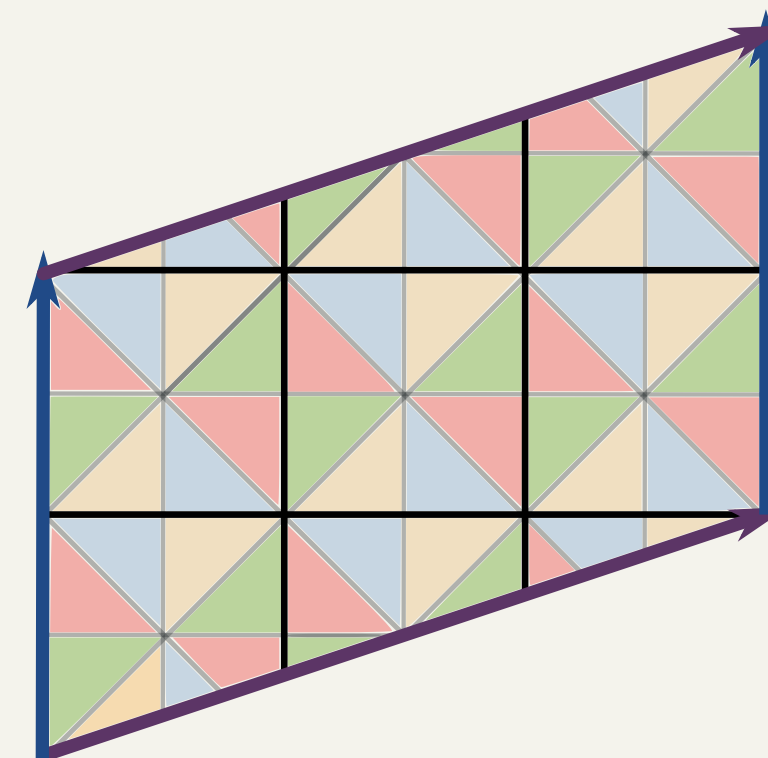
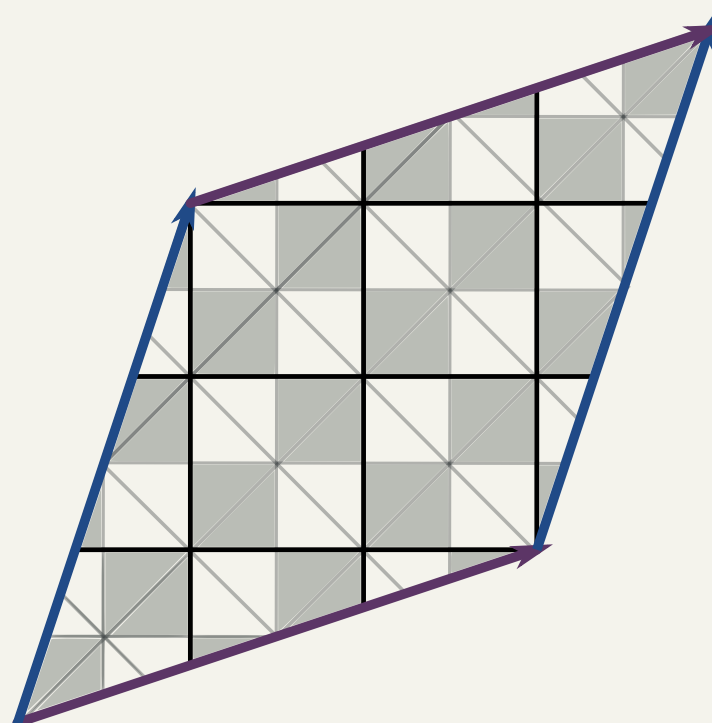
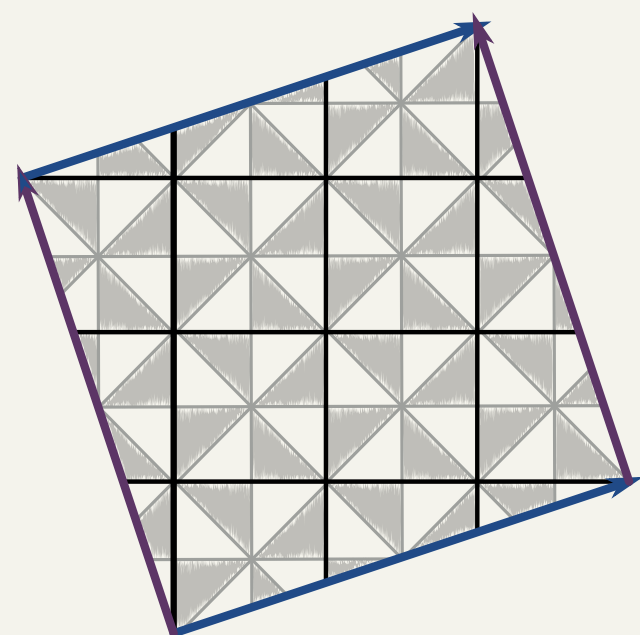
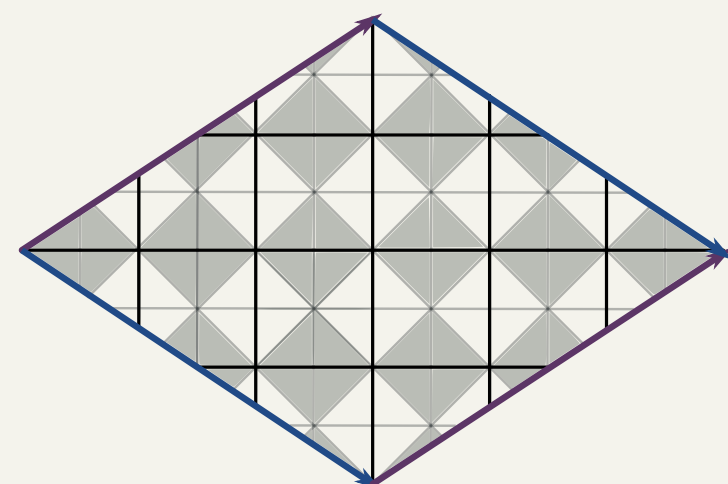
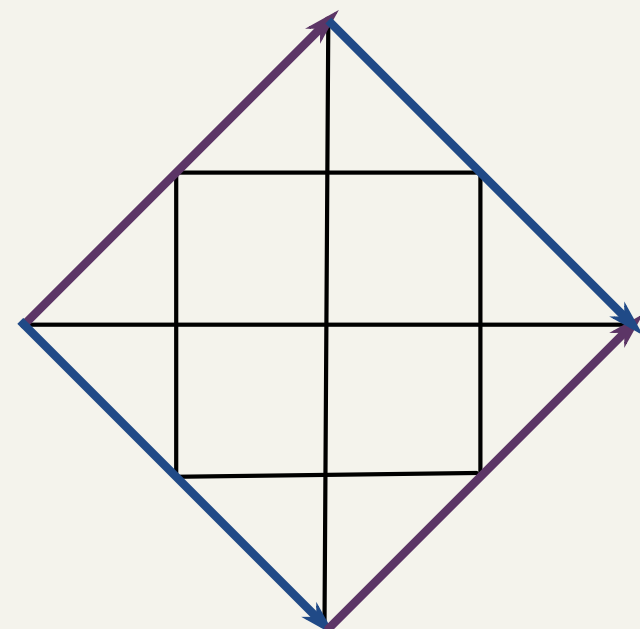
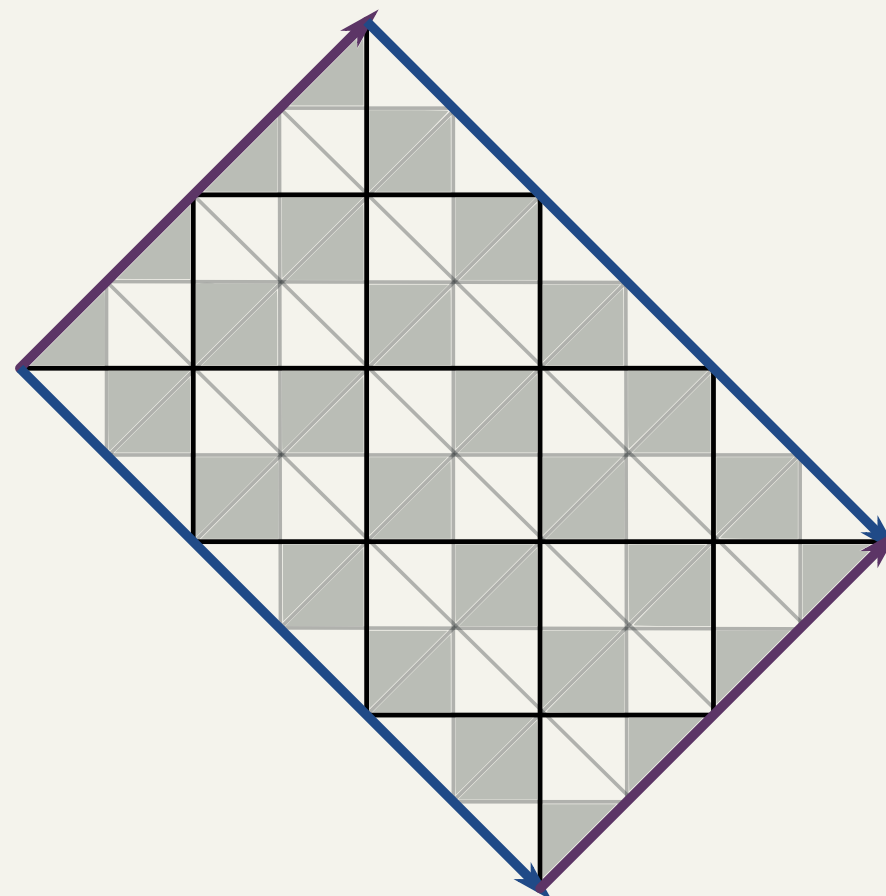
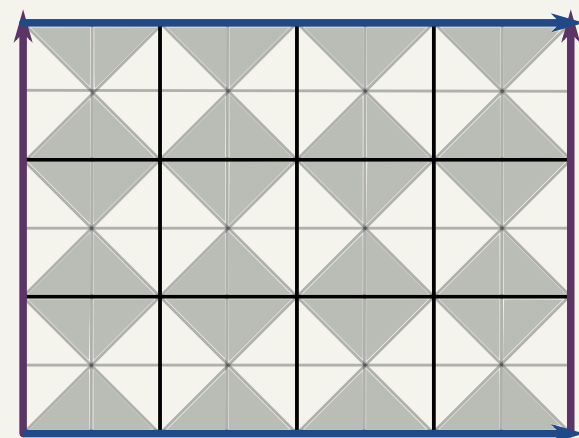
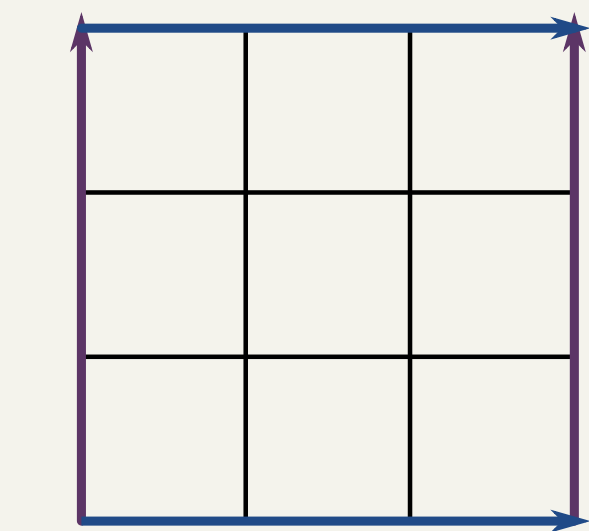
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Sym Type

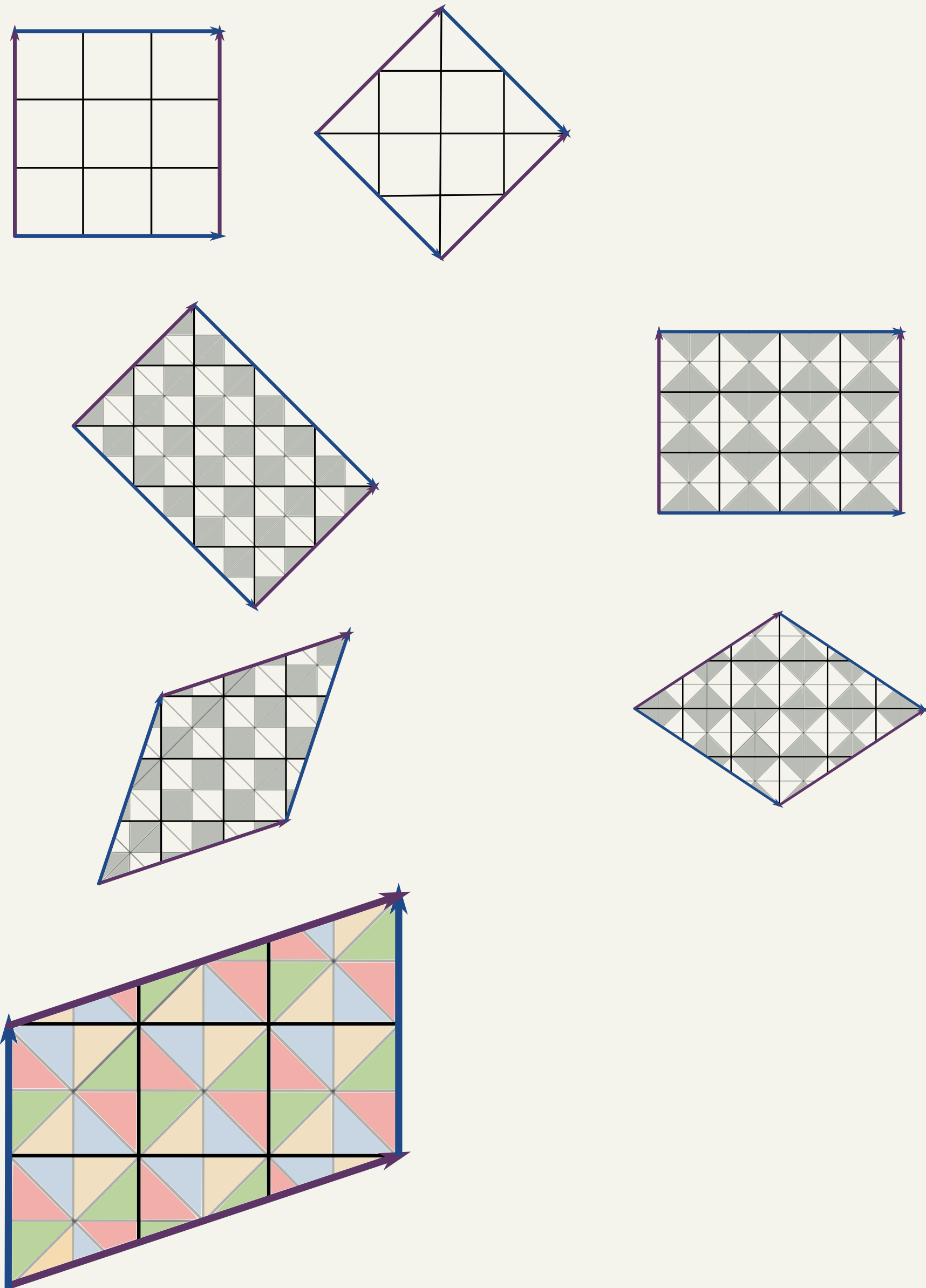
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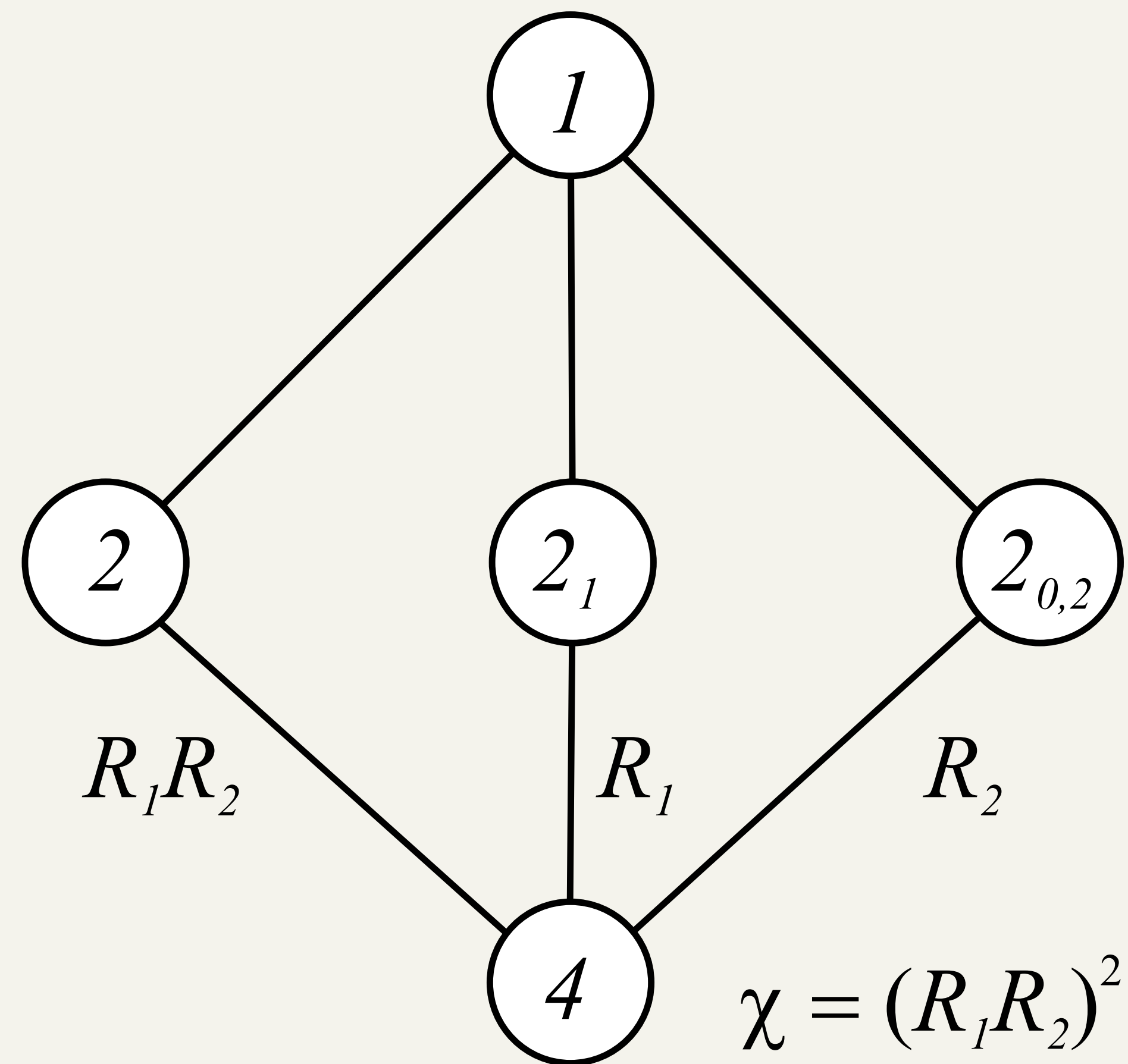
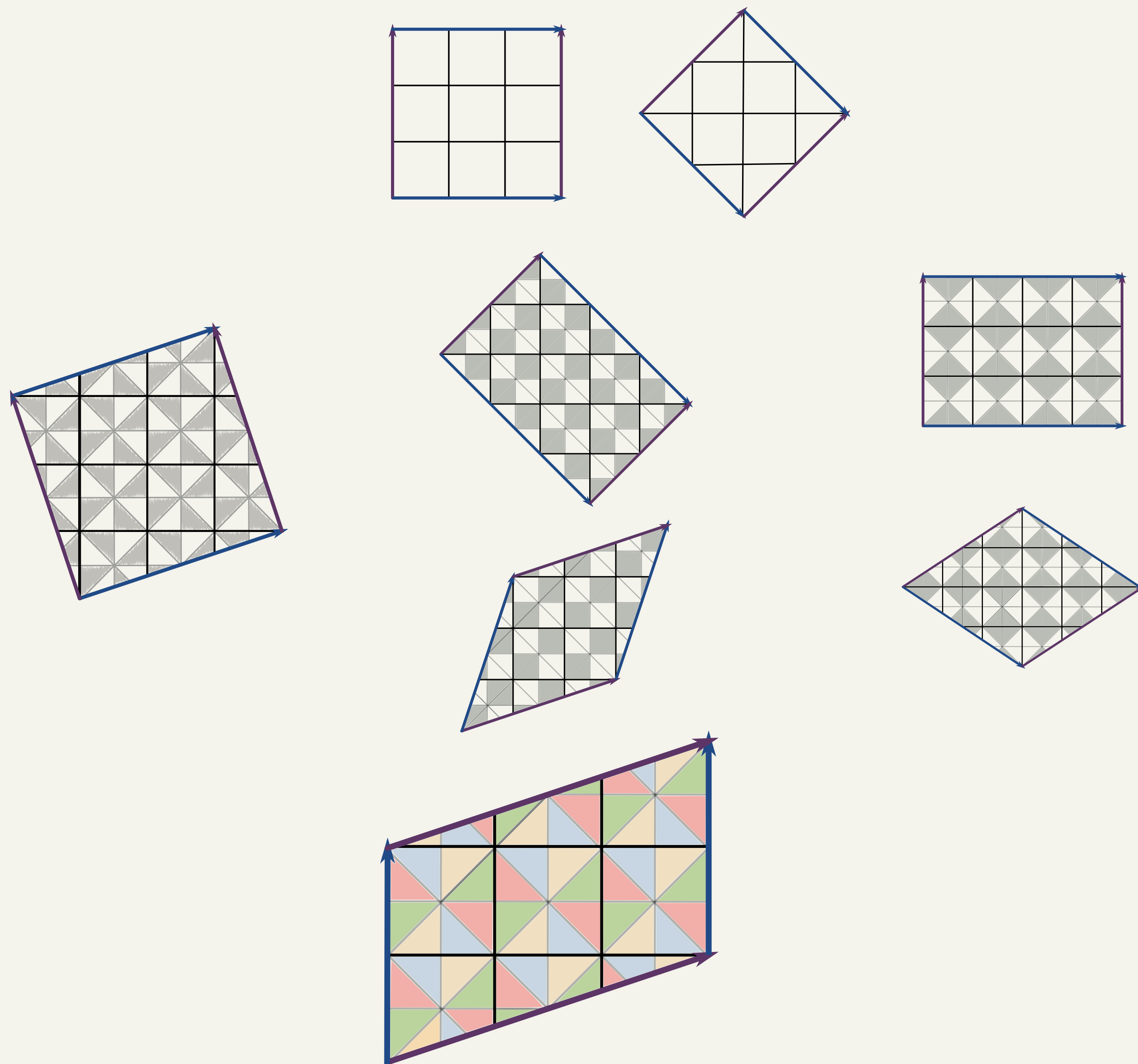
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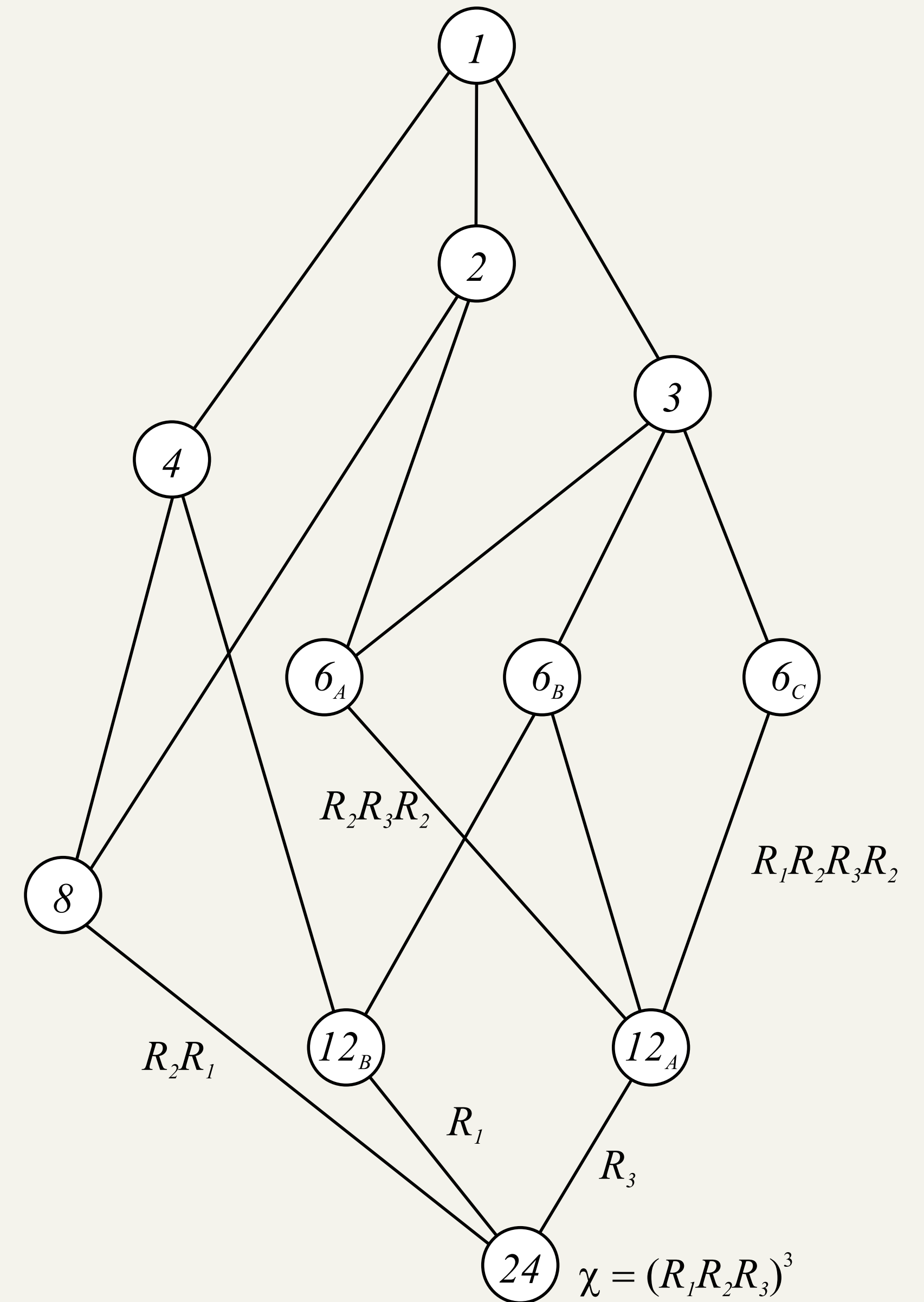


# Symmetries of $n$ -dimensional toroids

- Three families of regular tessellations;
- Four families of 3-orbit tessellations;
- Sixteen families of 6-orbit toroids with three different symmetry type (7+7+2);
- Three families of 8-orbit;
- Twelve families of 12-orbit with two different symmetry types (7+5);
- The rest, with 24 flag-orbits.

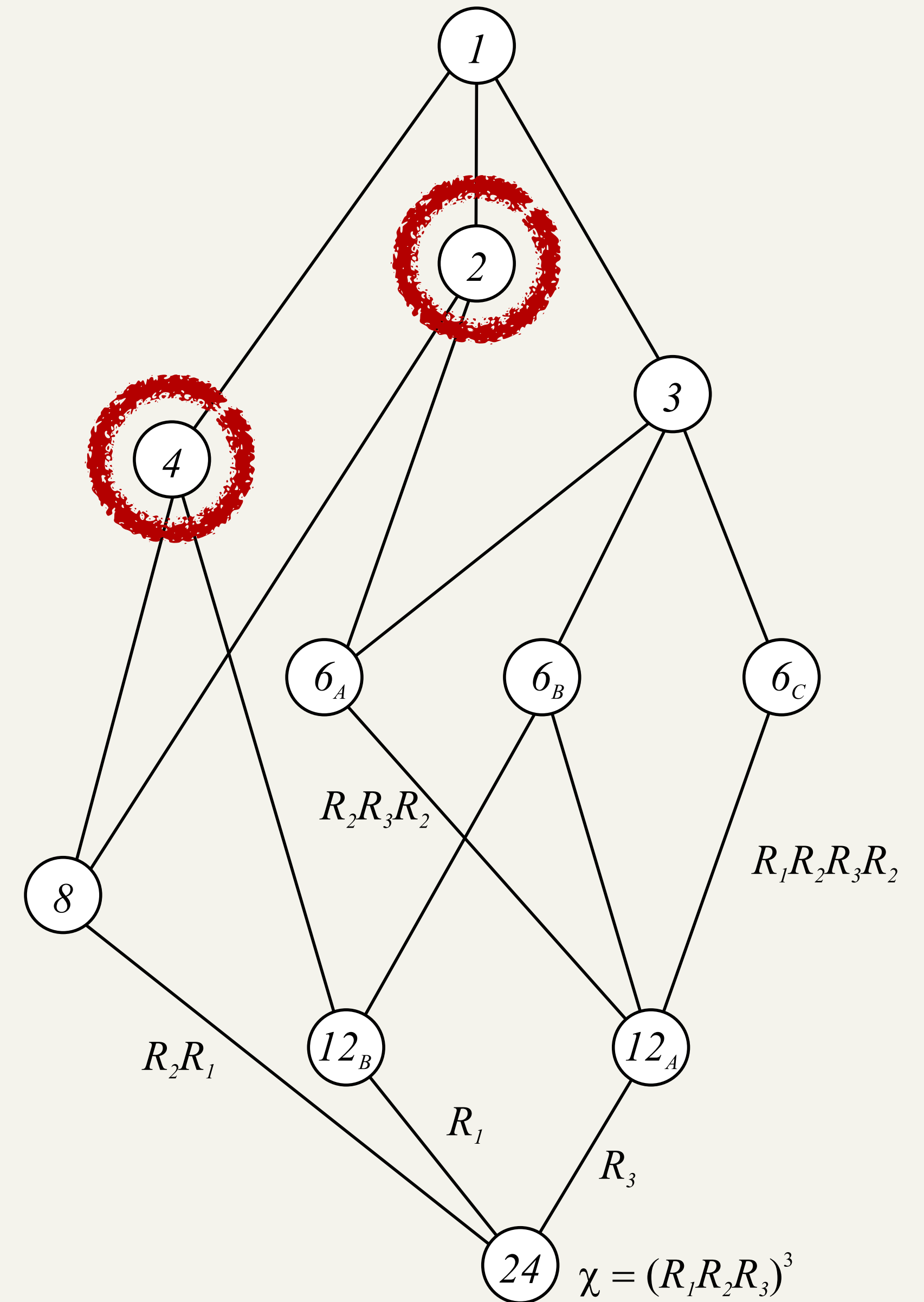
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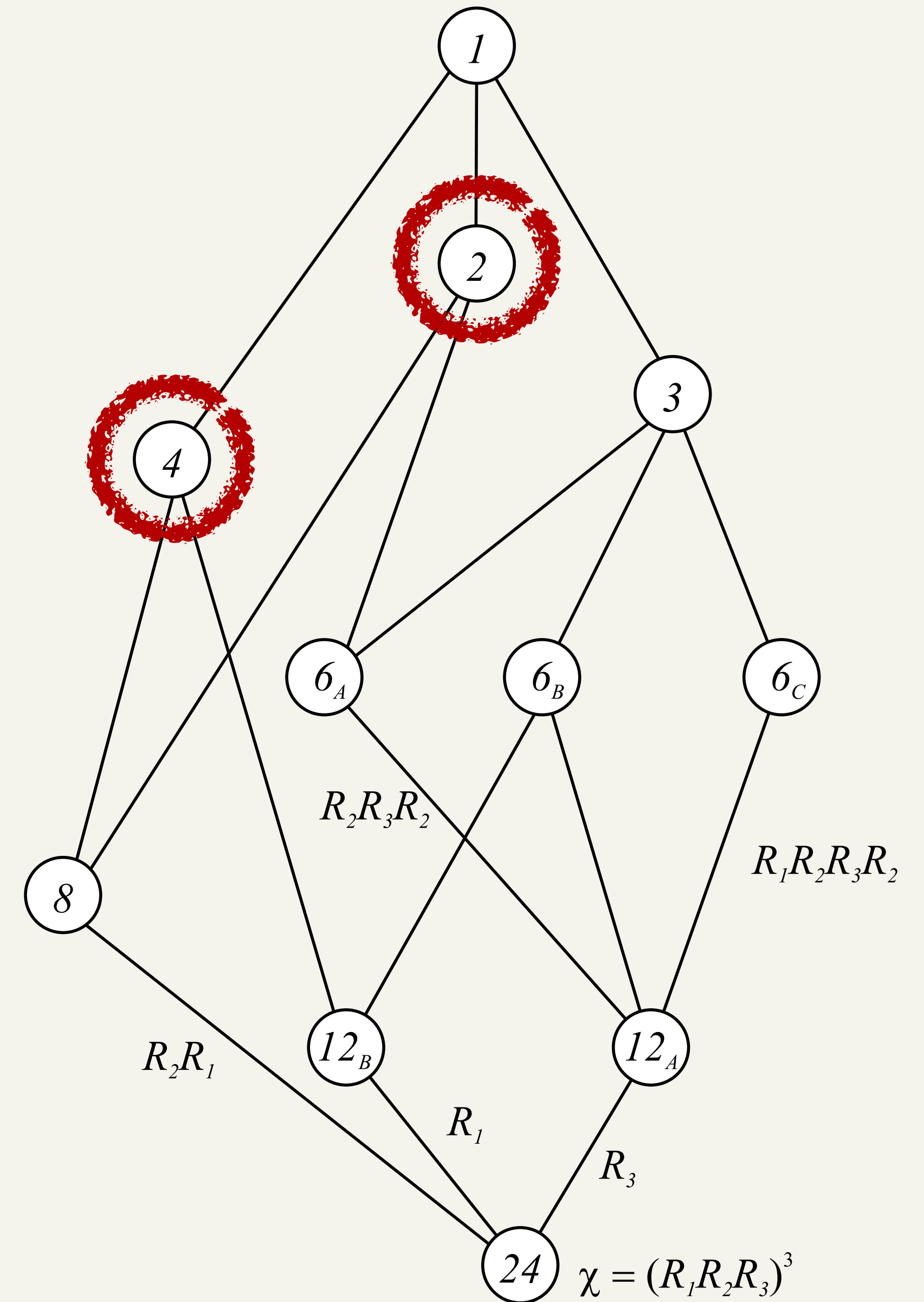
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## Collins, M., 2021

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Complete classification of equivelar tessellations of the  $n$ -torus with at most  $n$  flag-orbits.

Thank you

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Slides



References