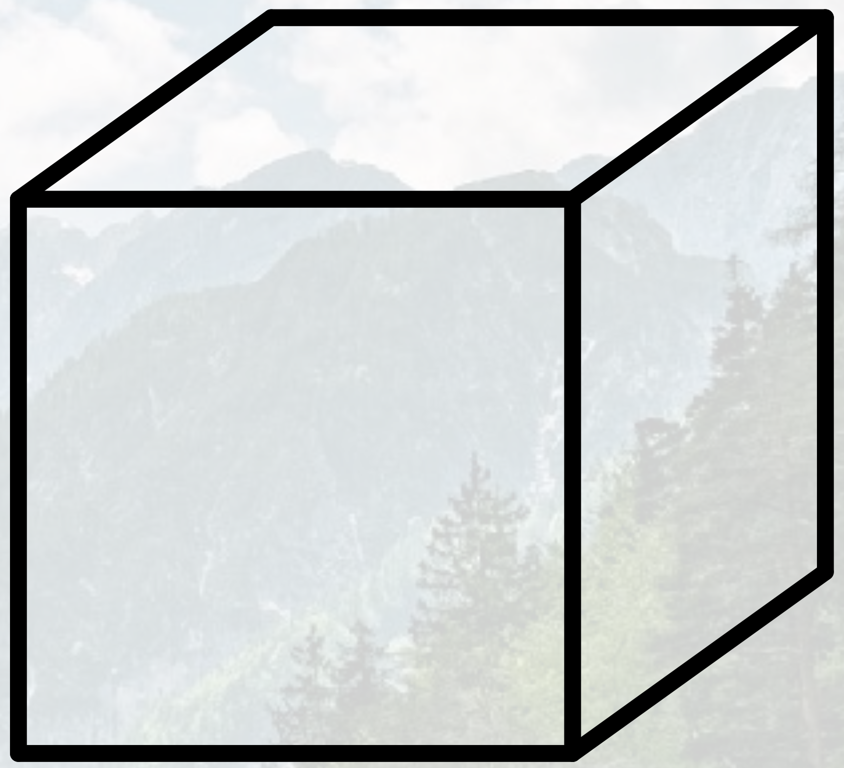
Faithfulness in maniplexes

Antonio Montero

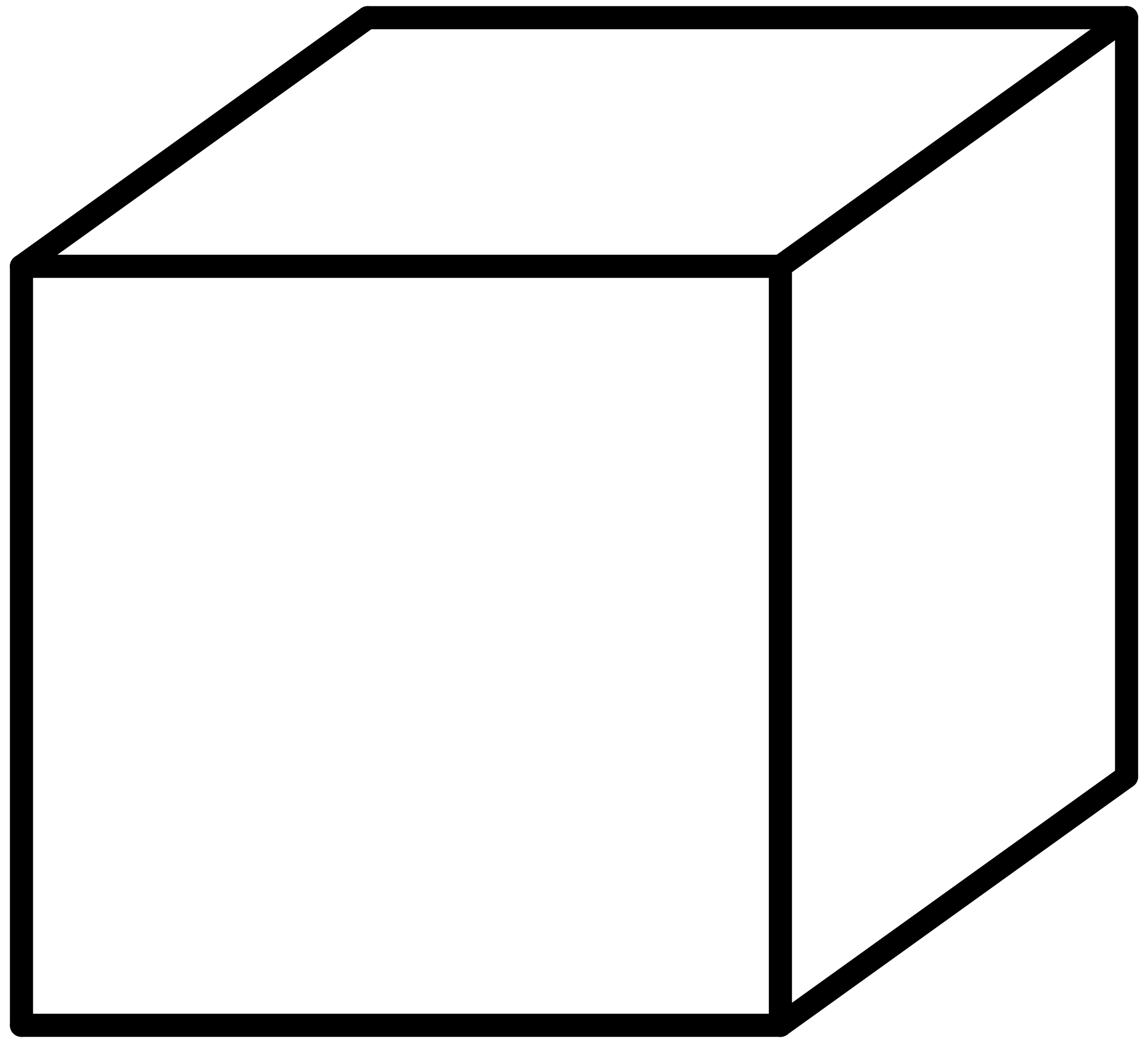
University of Ljubljana

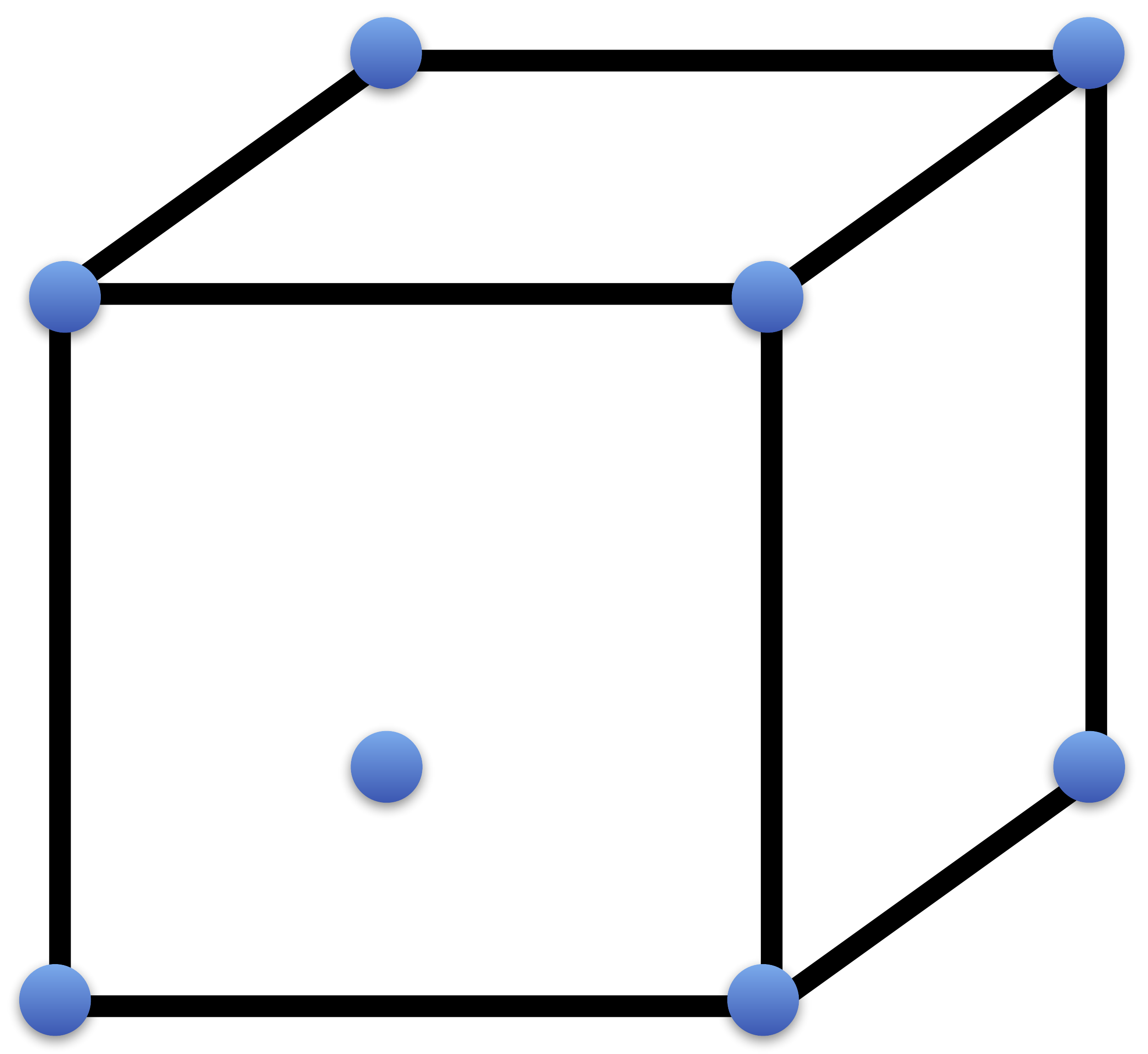
Based on joint work with
Gabe Cunningham, Elías Mochán, Primož Potočnik and Micael Toledo

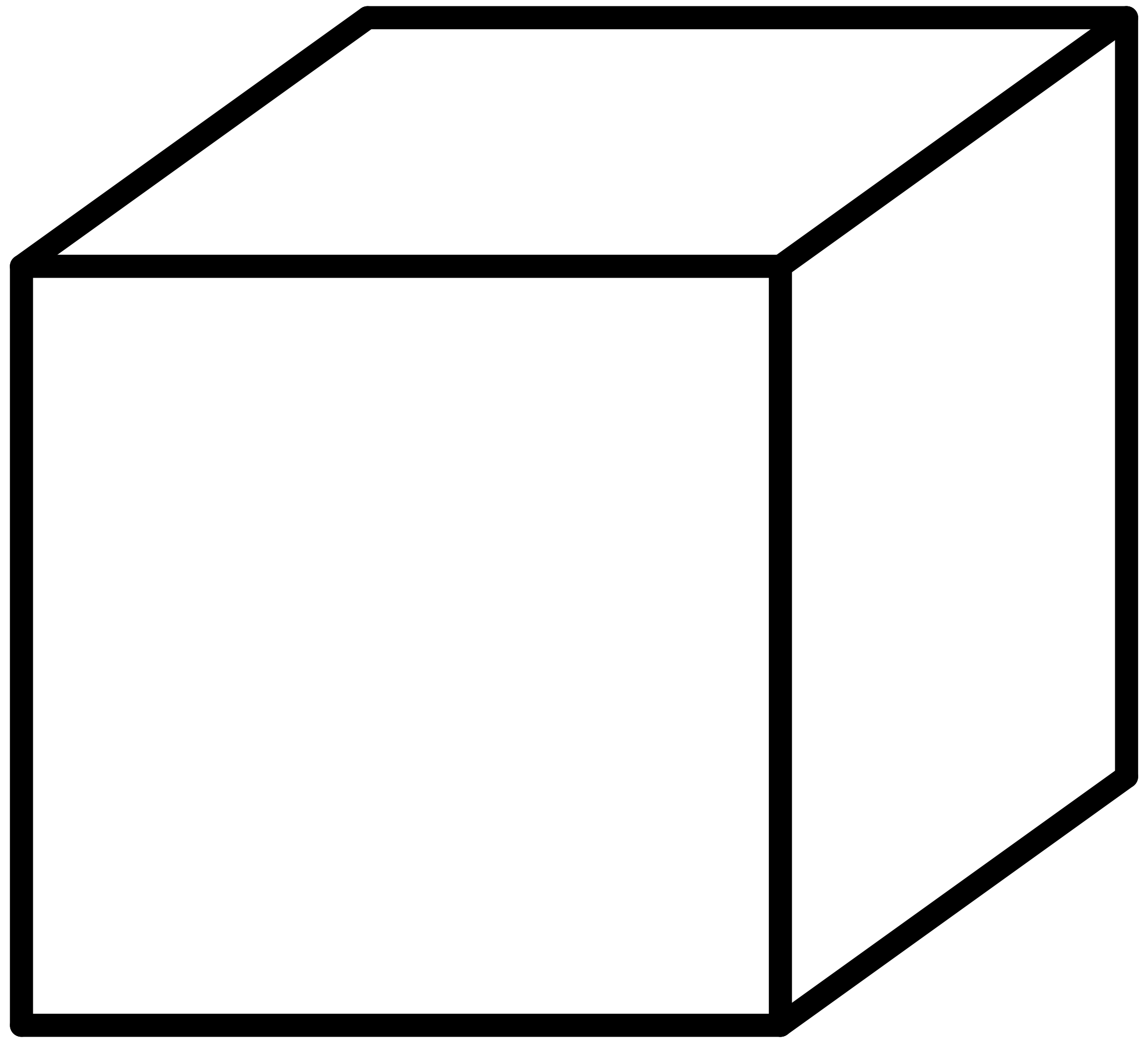


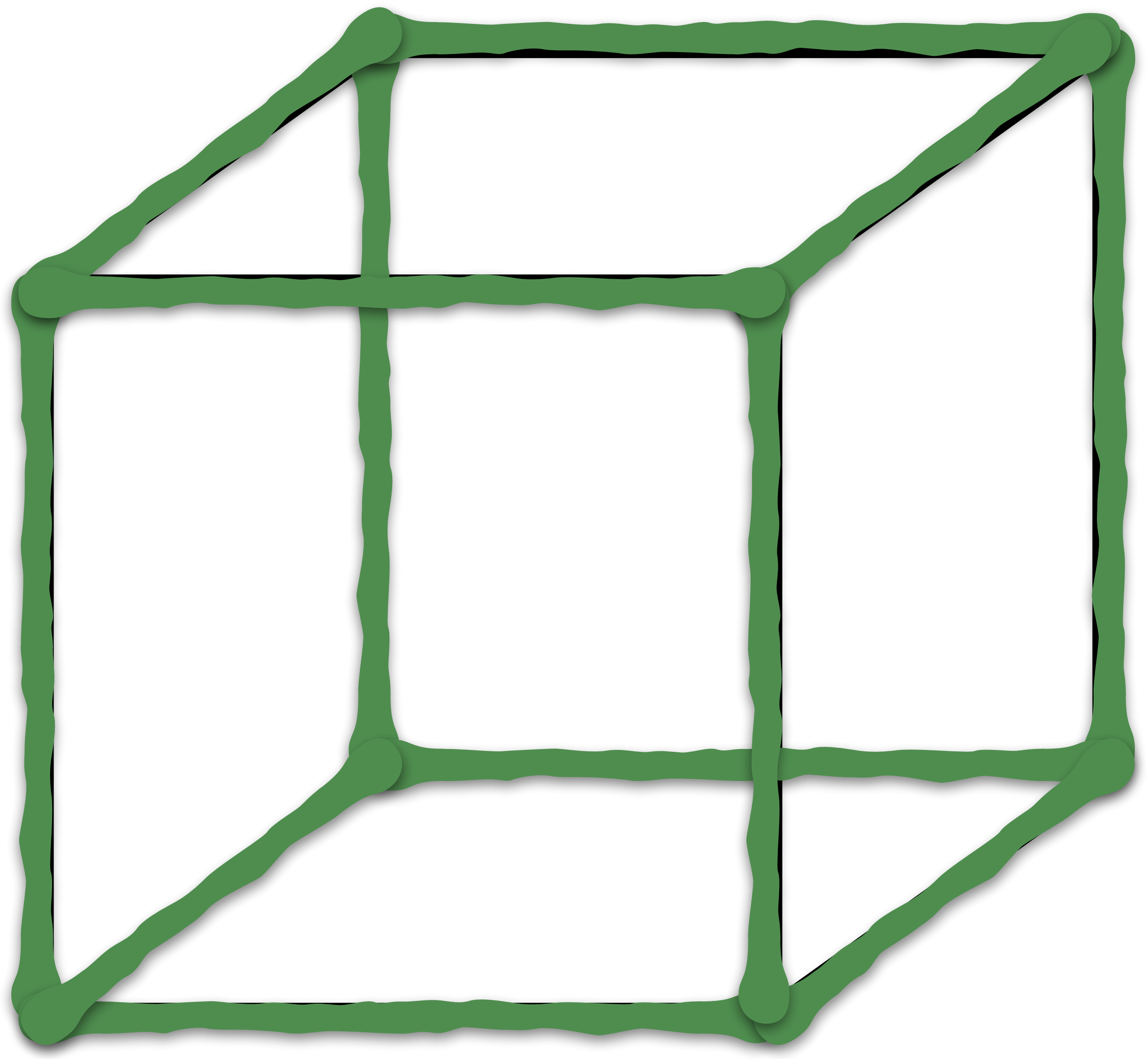
Computers in Scientific Discovery 11

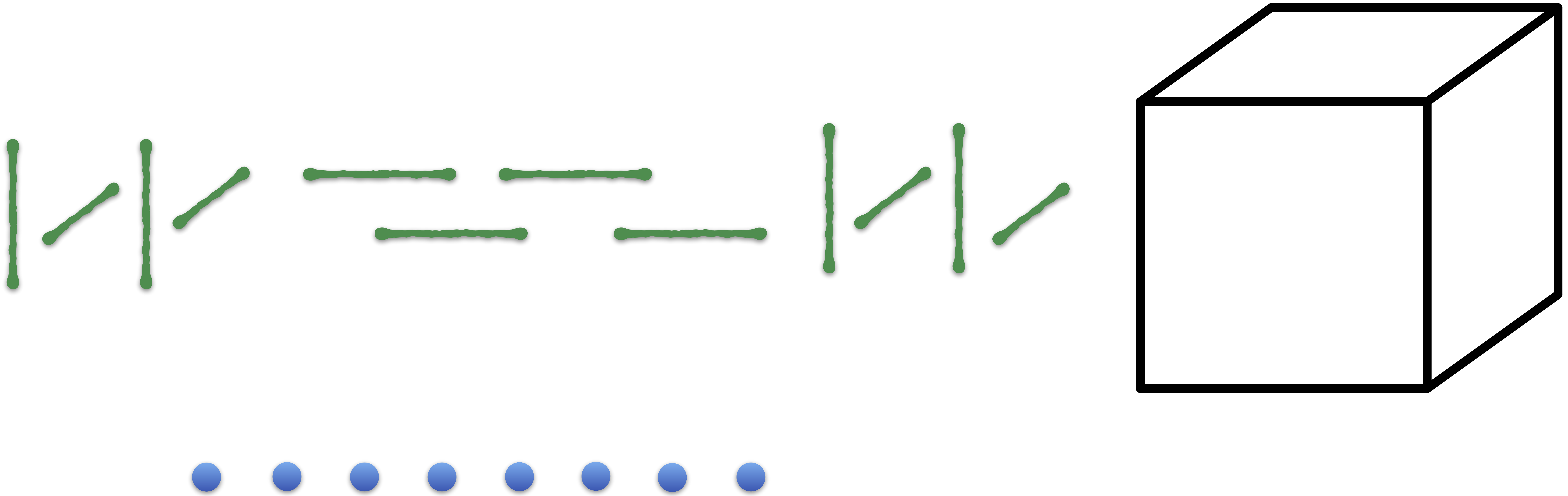
May 2026, Kranjska Gora, Slovenia.

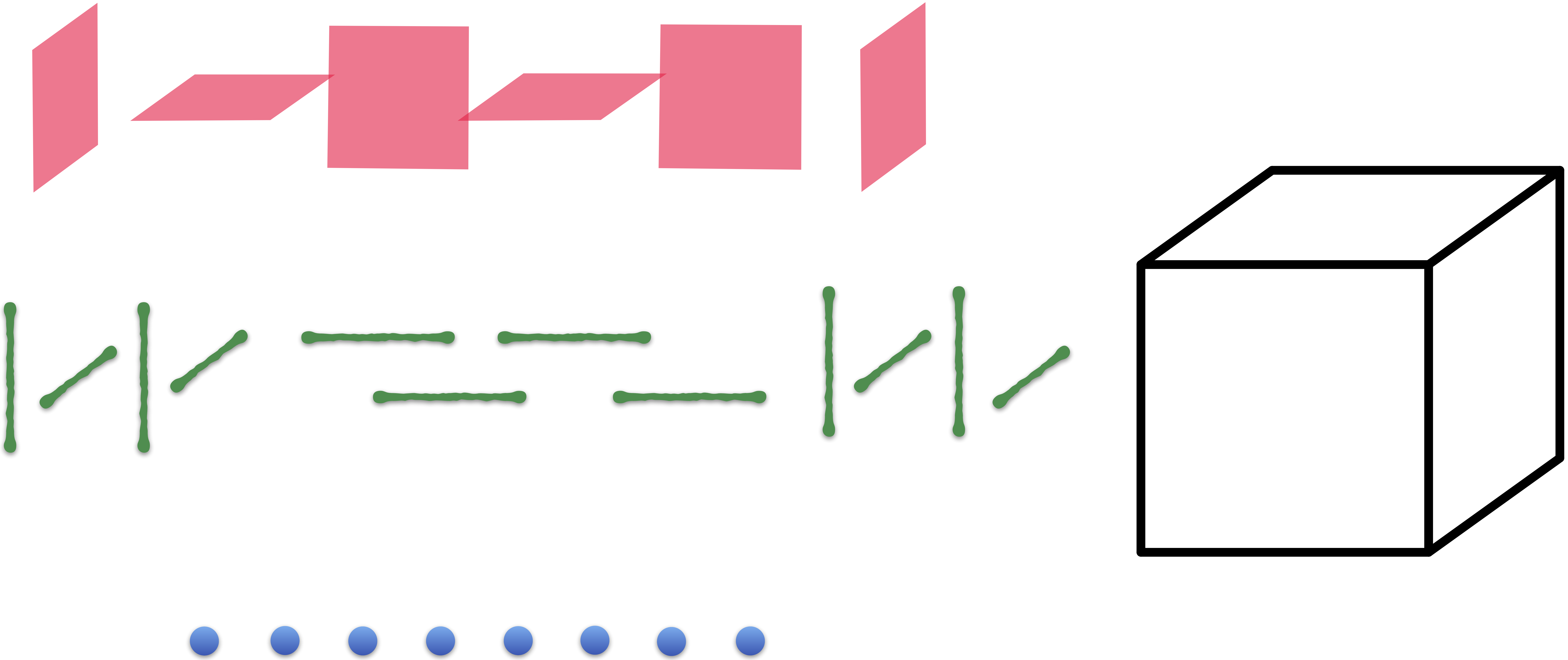


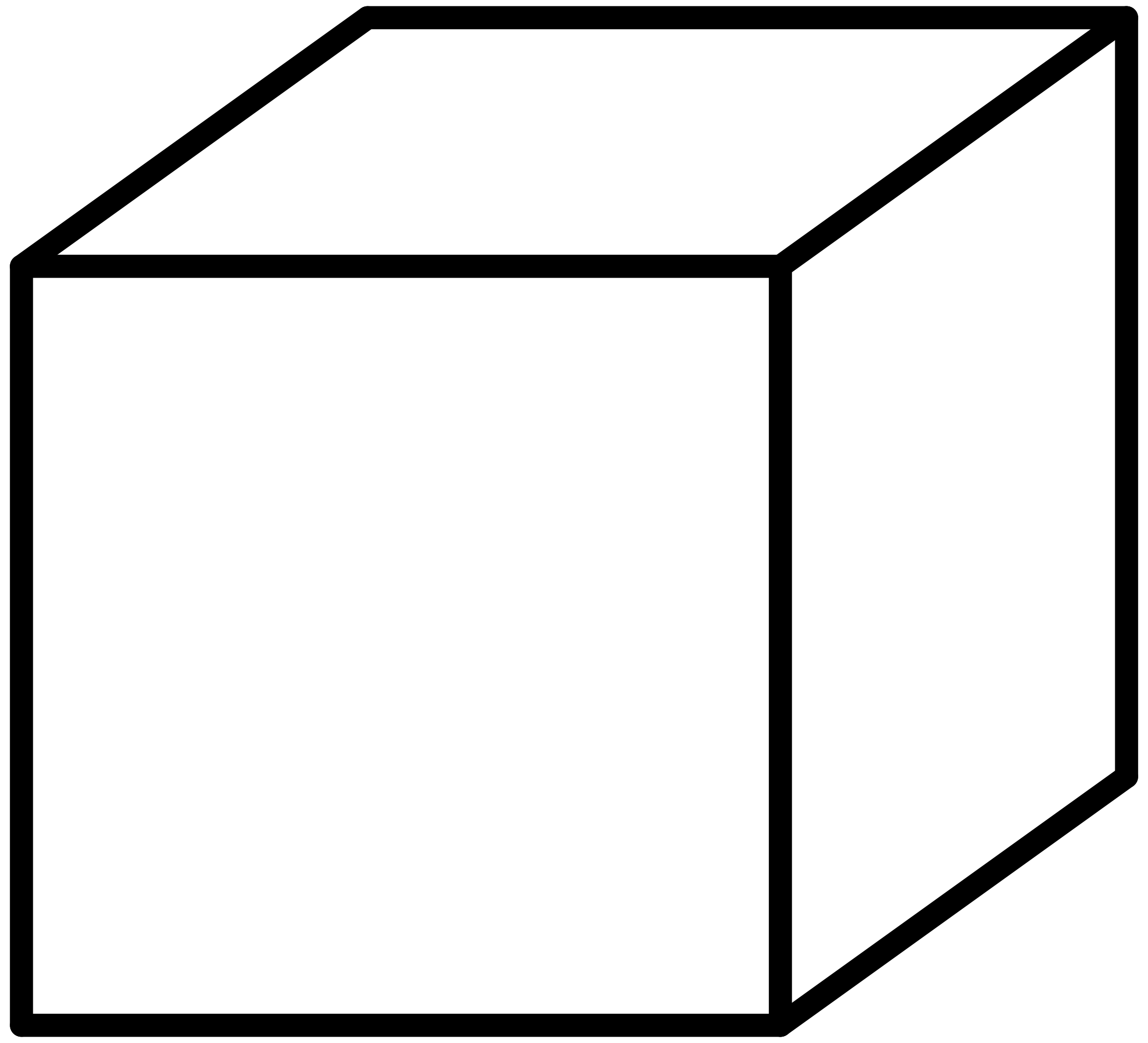








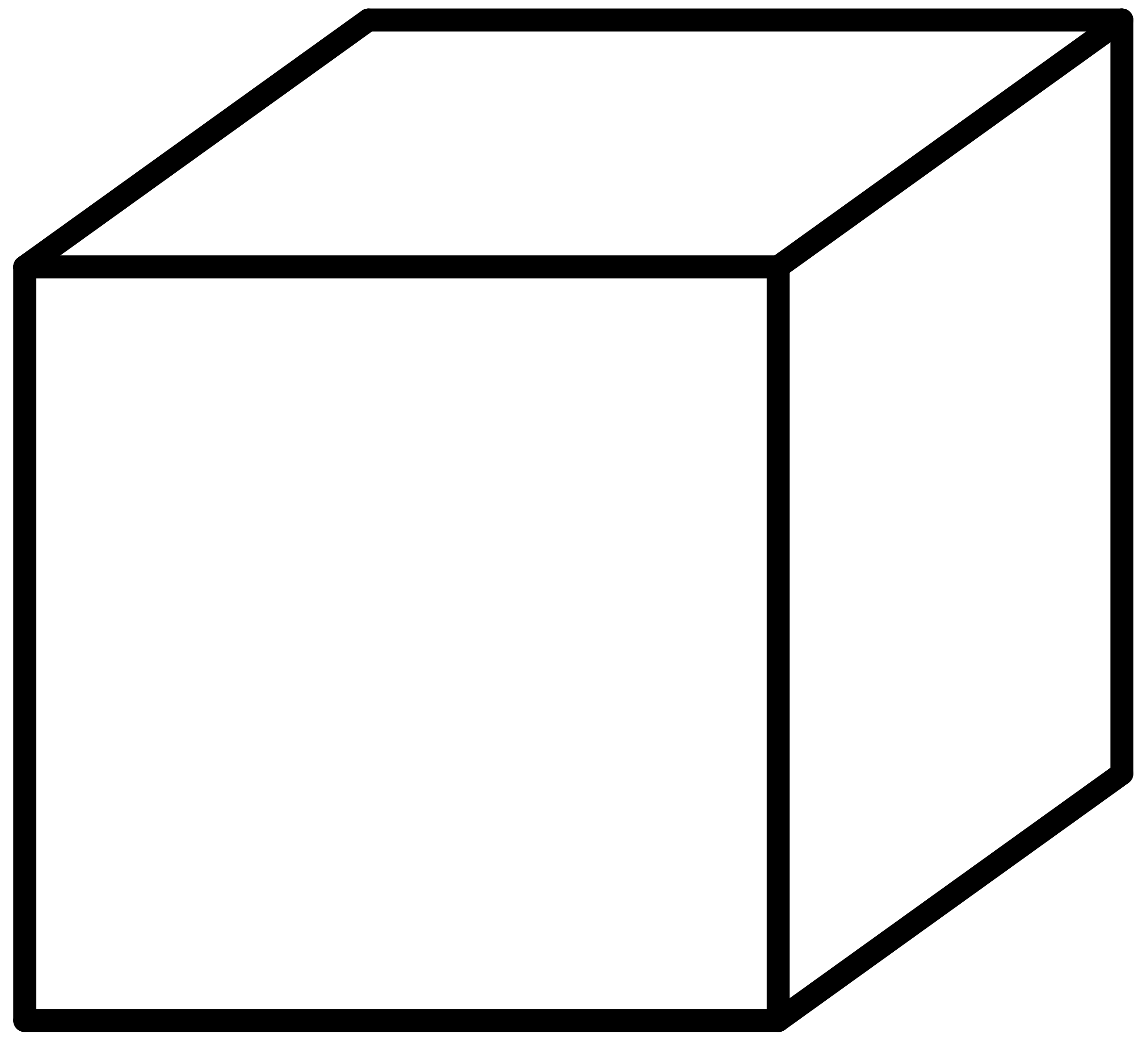


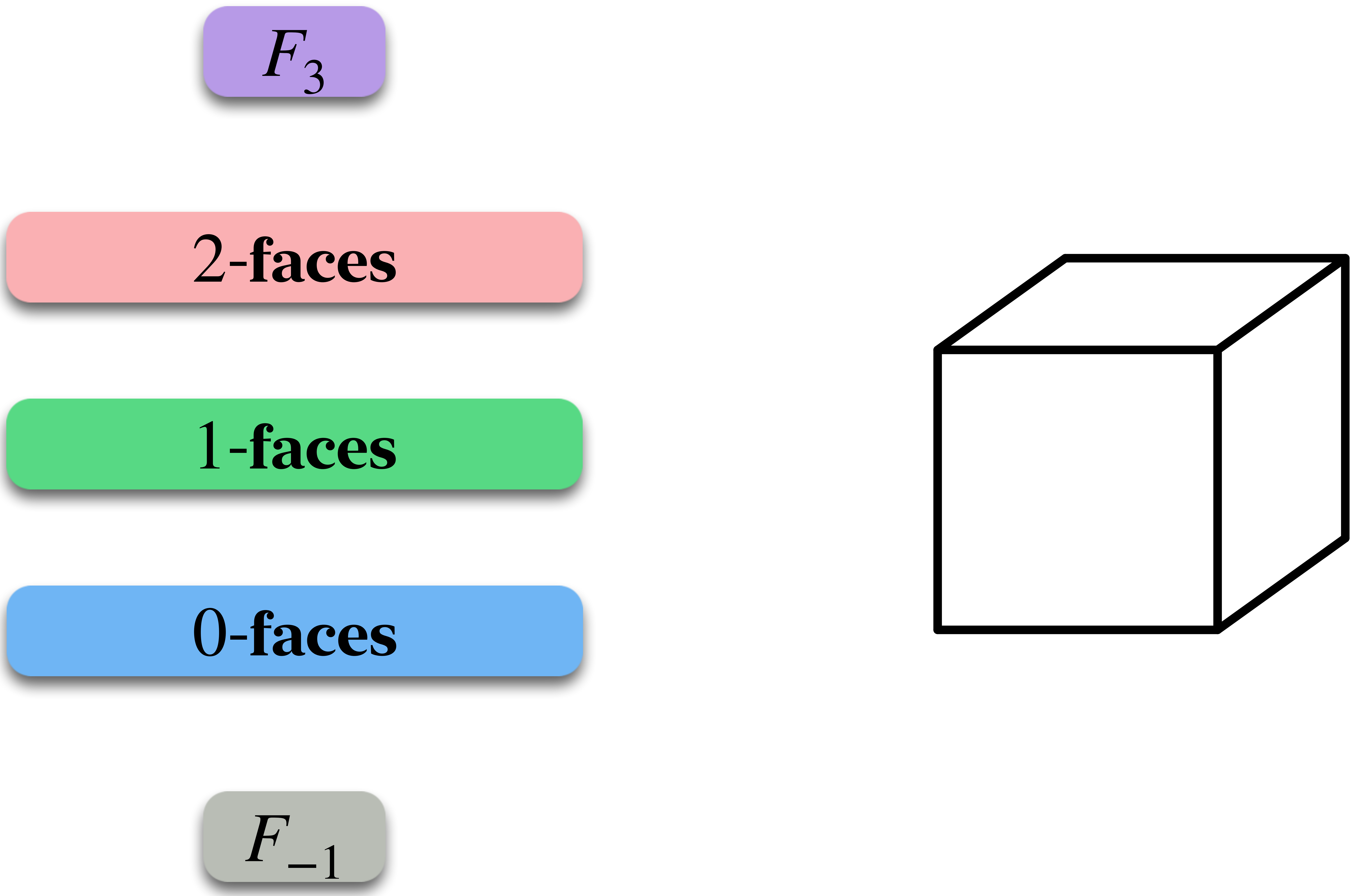


2-faces

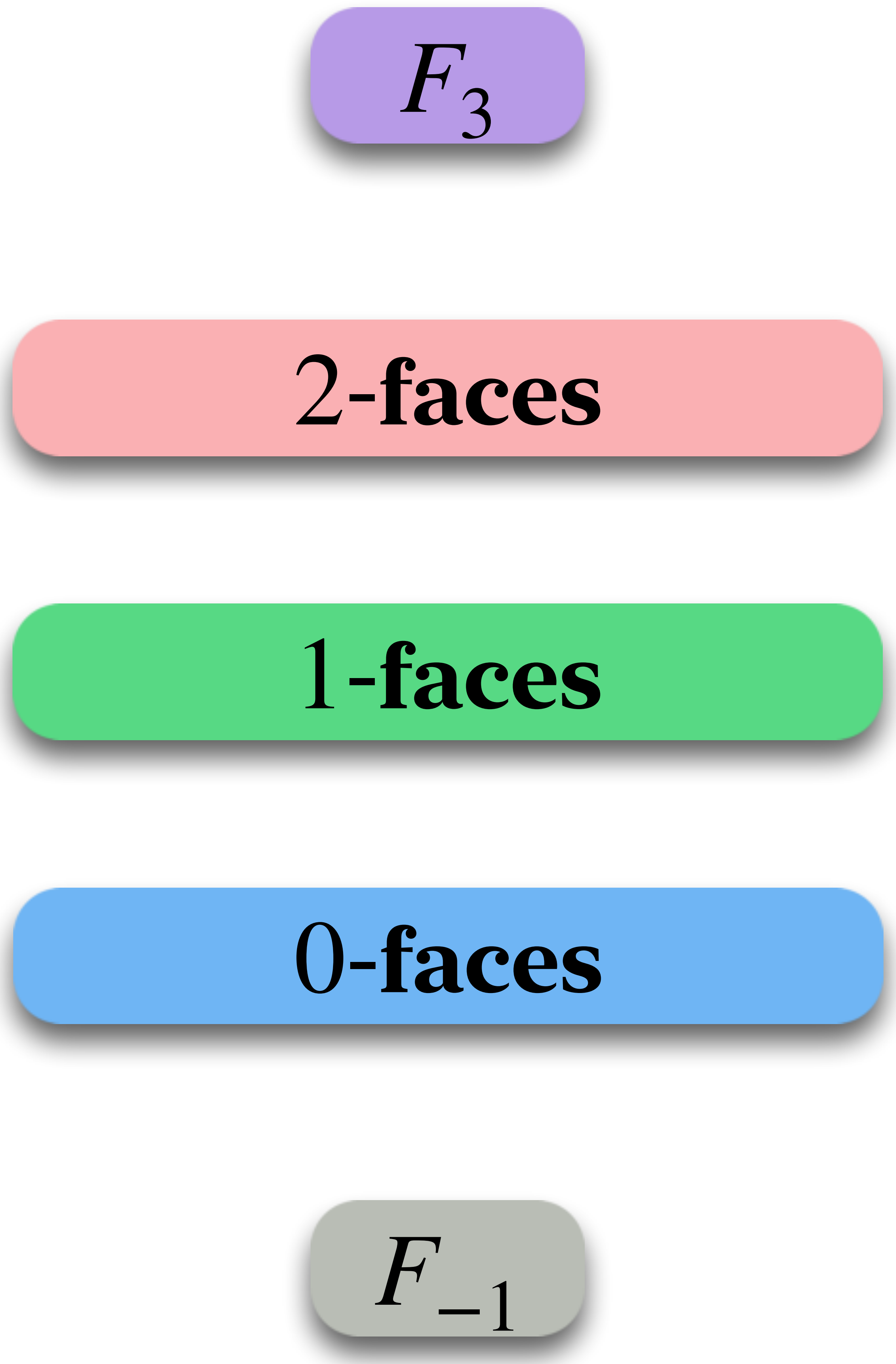
1-faces

0-faces




F_3
2-faces
1-faces
0-faces
F_{-1}

Abstract polytopes



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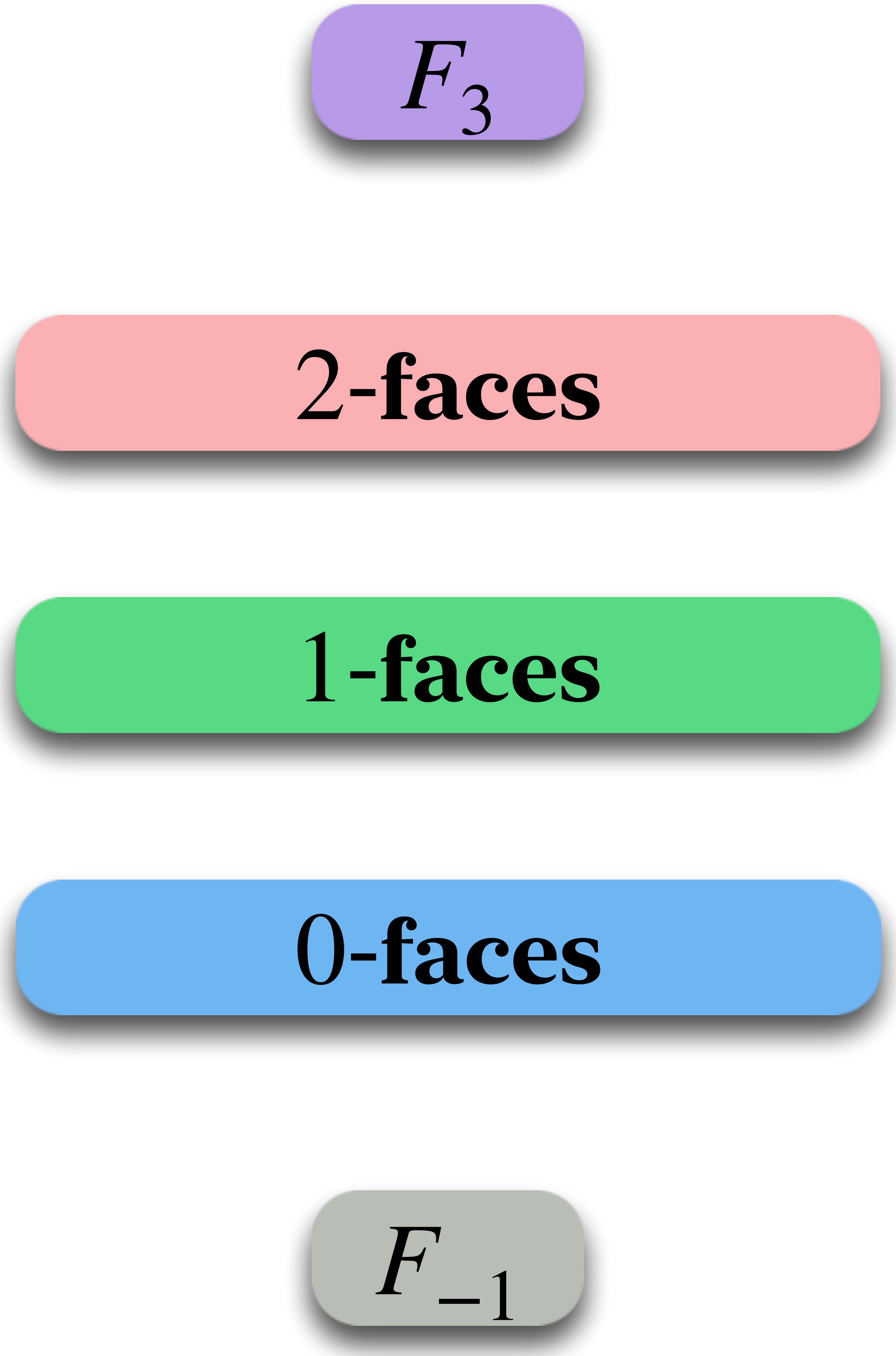

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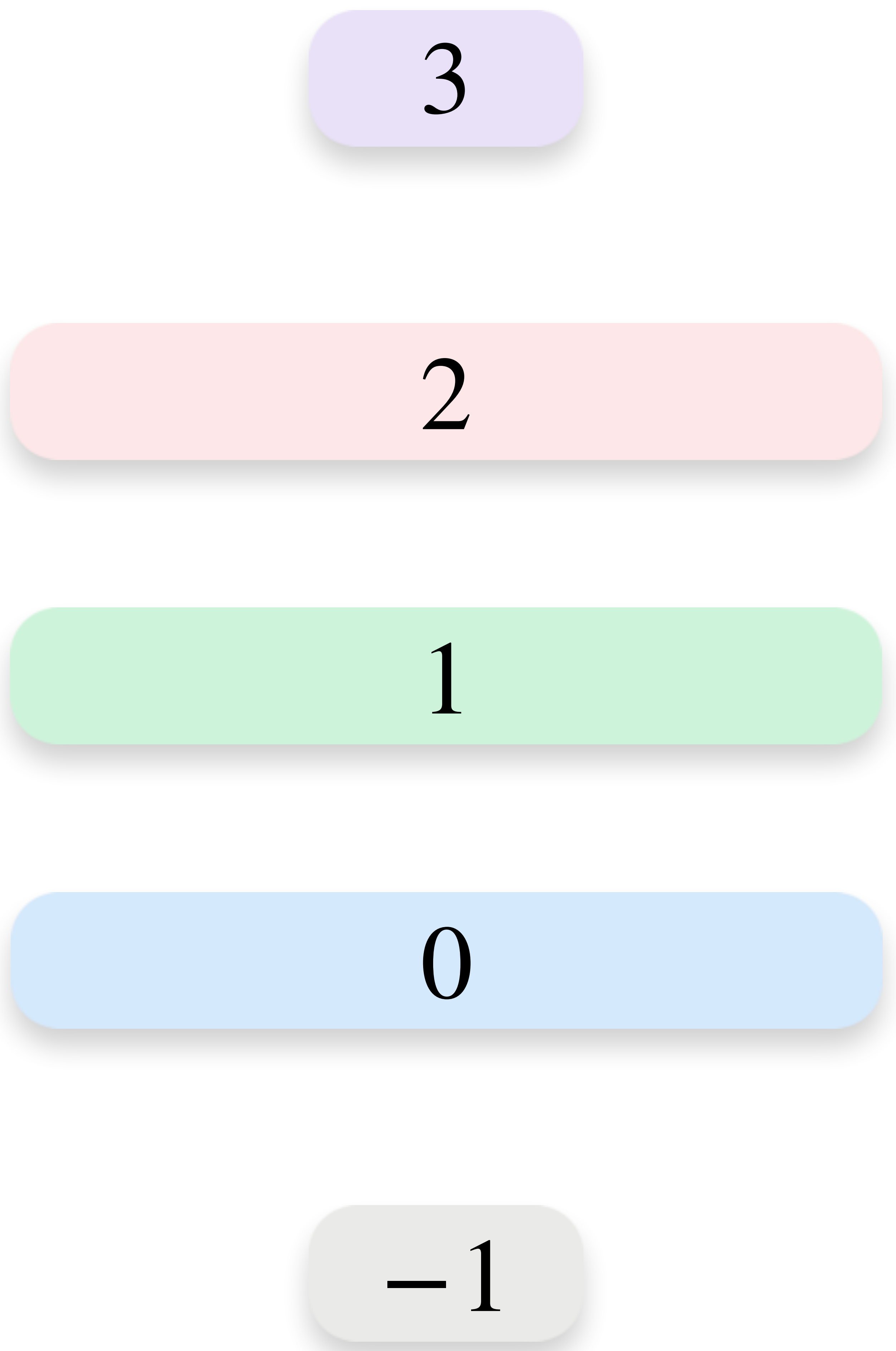

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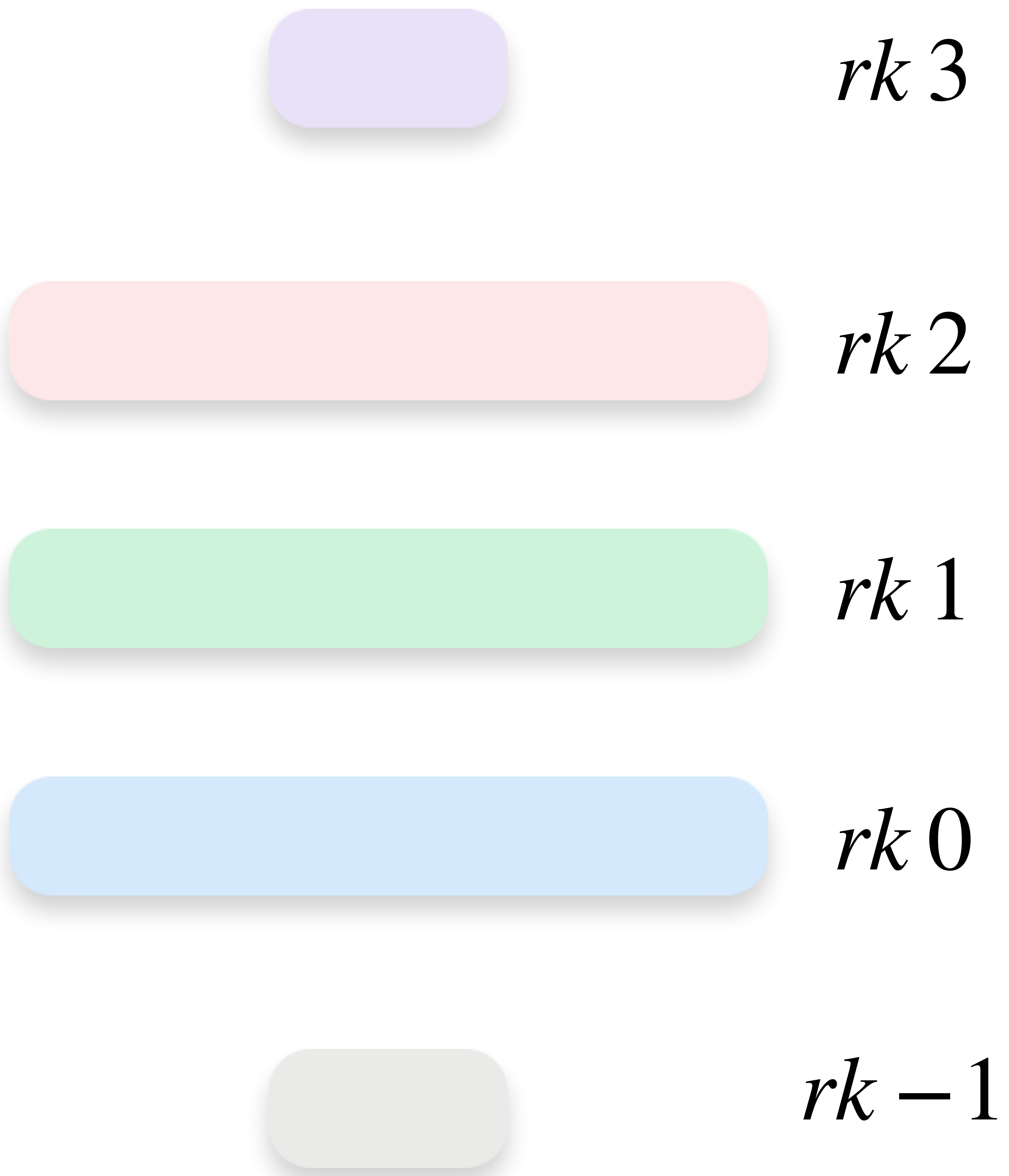
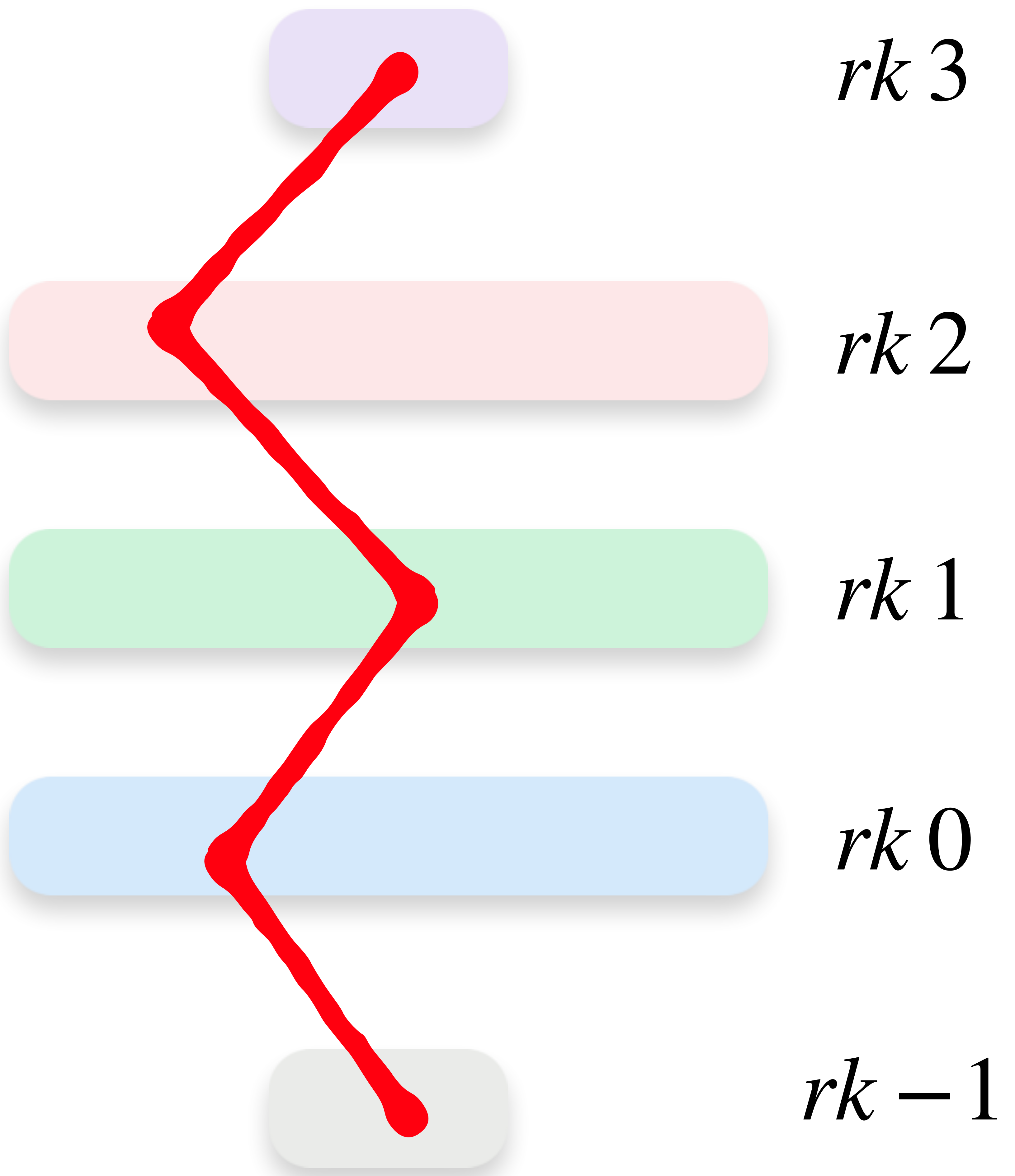


Abstract polytopes

- Ranked poset (each maximal chain contains $(n+2)$ elements);



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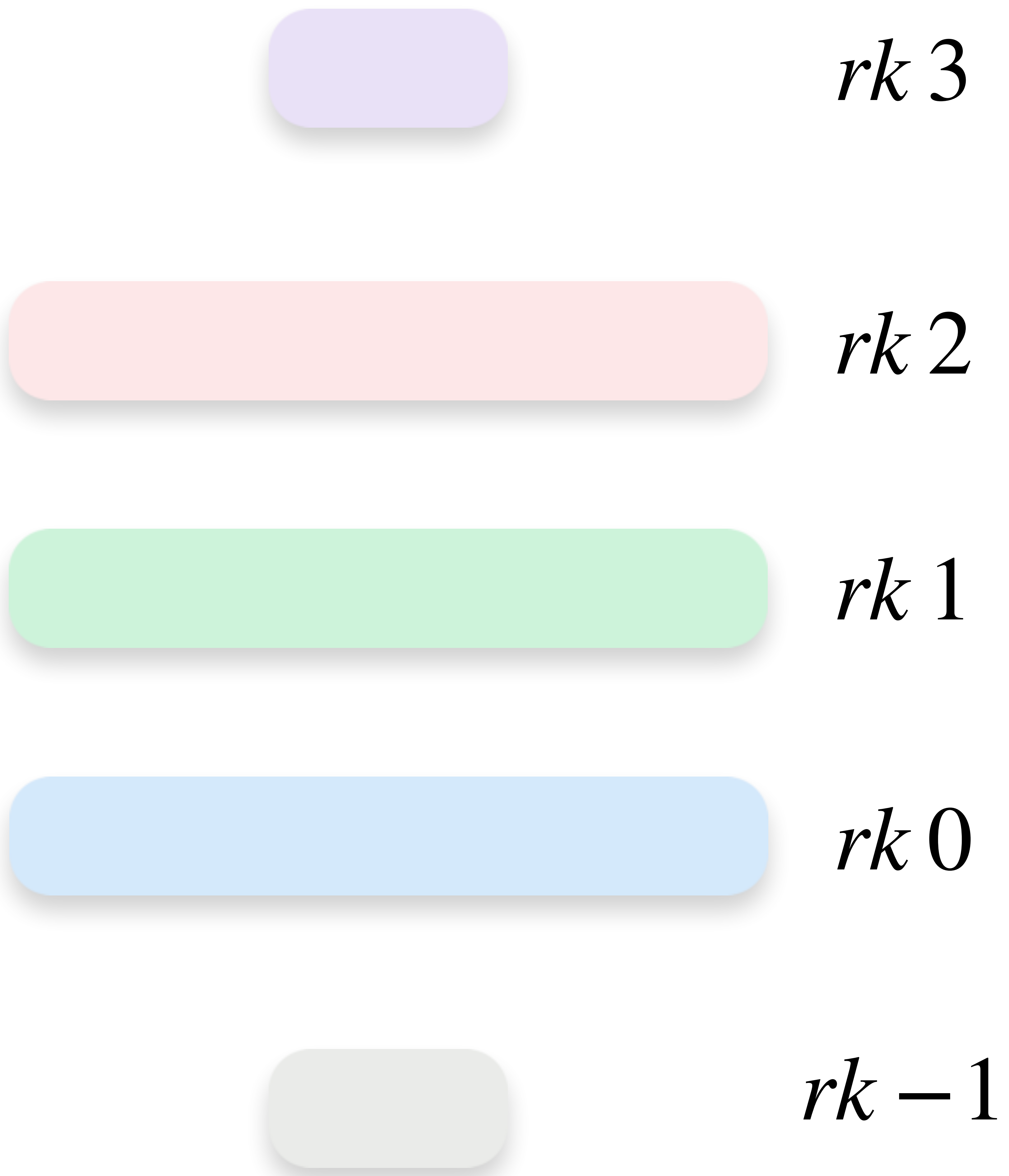

Abstract polytopes

- Ranked poset (each maximal chain contains $(n+2)$ elements);

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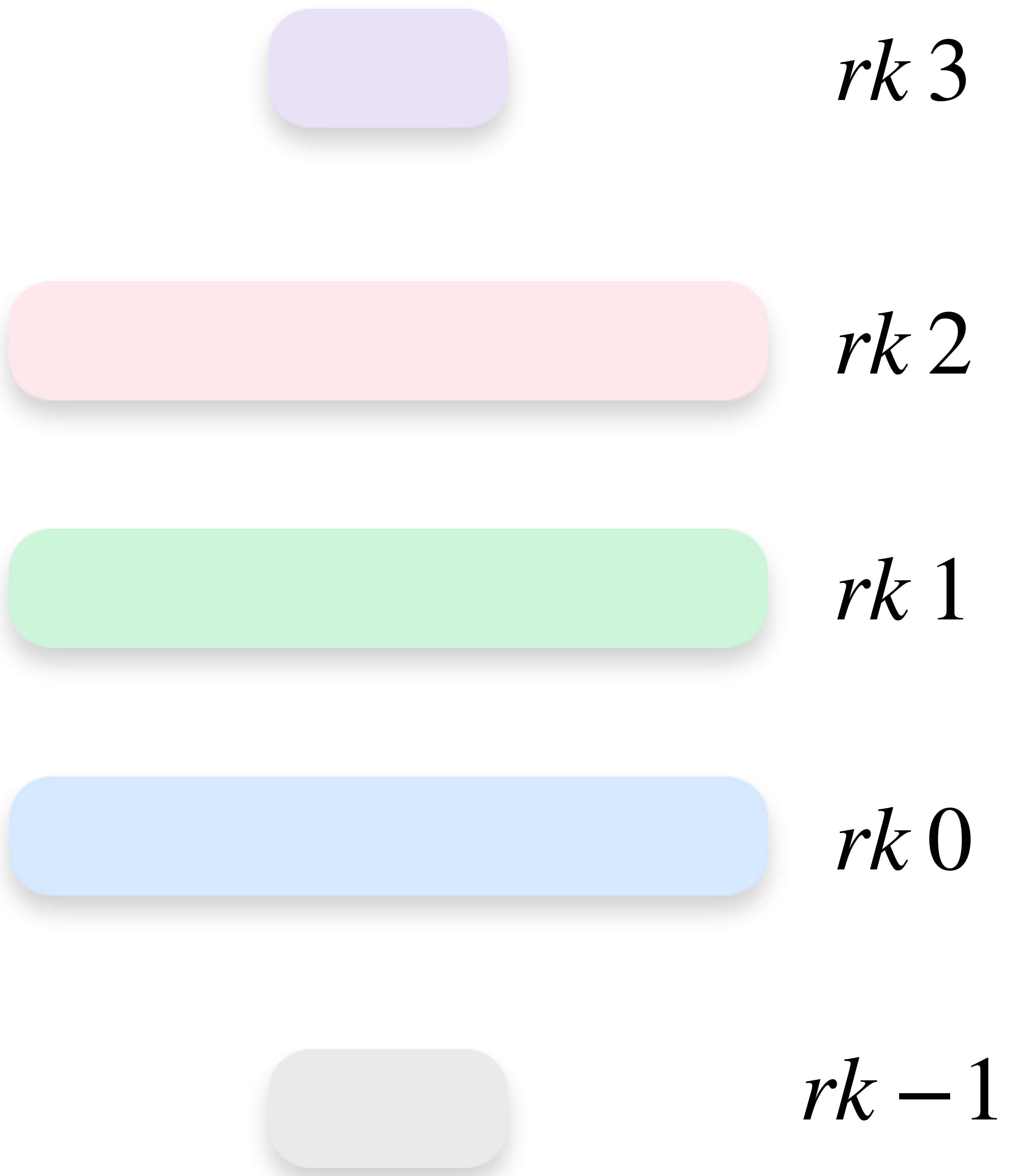

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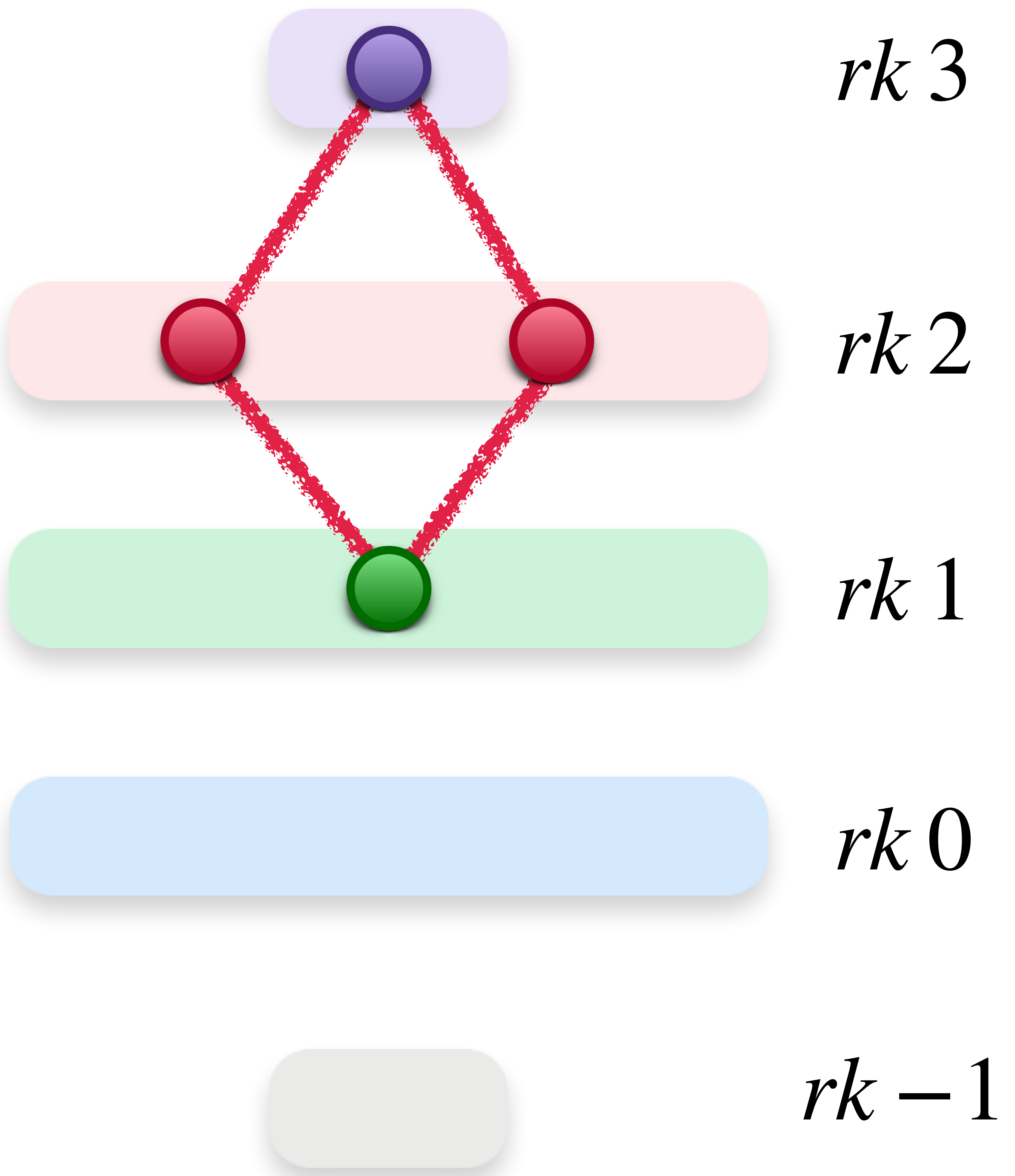

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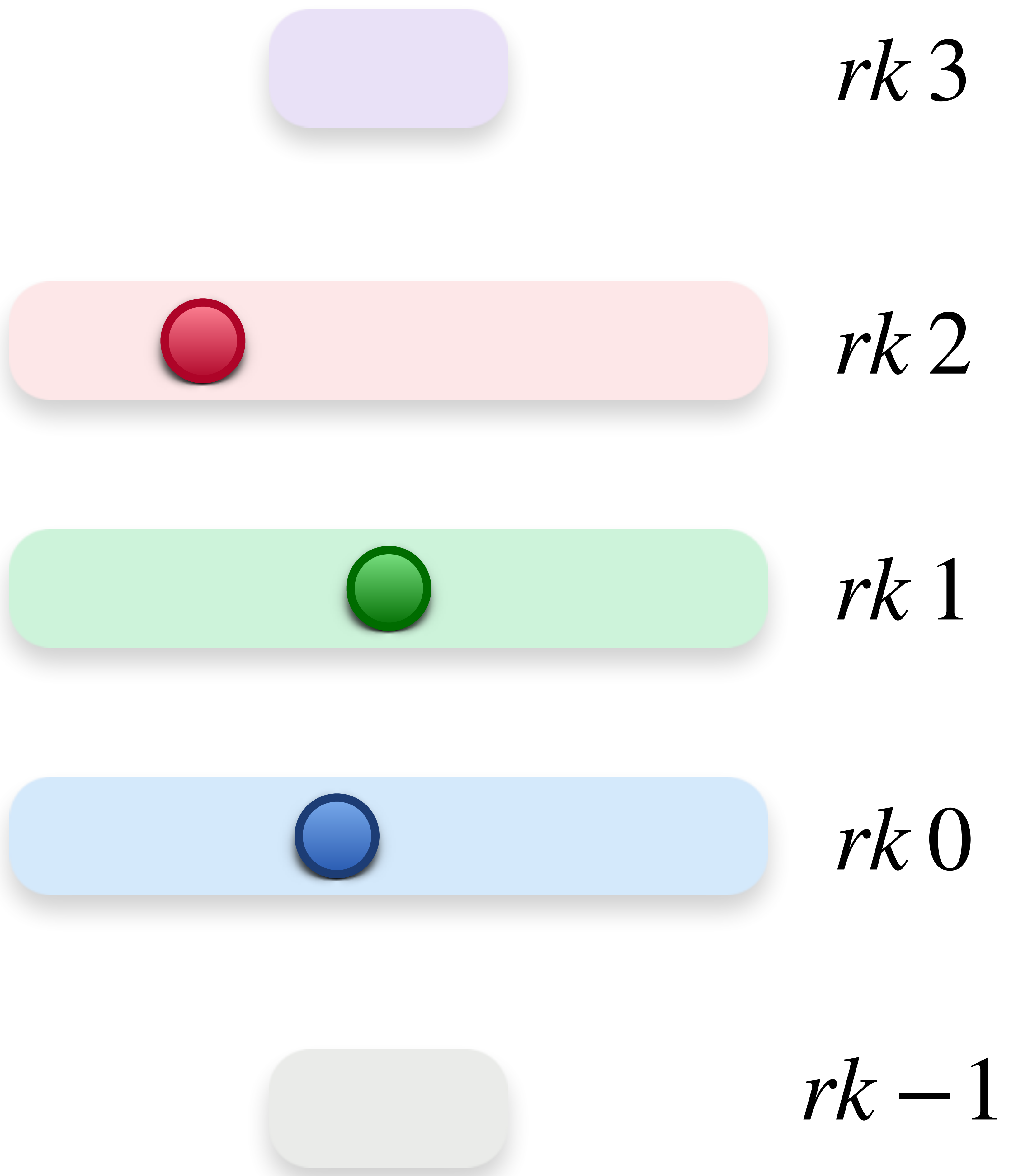

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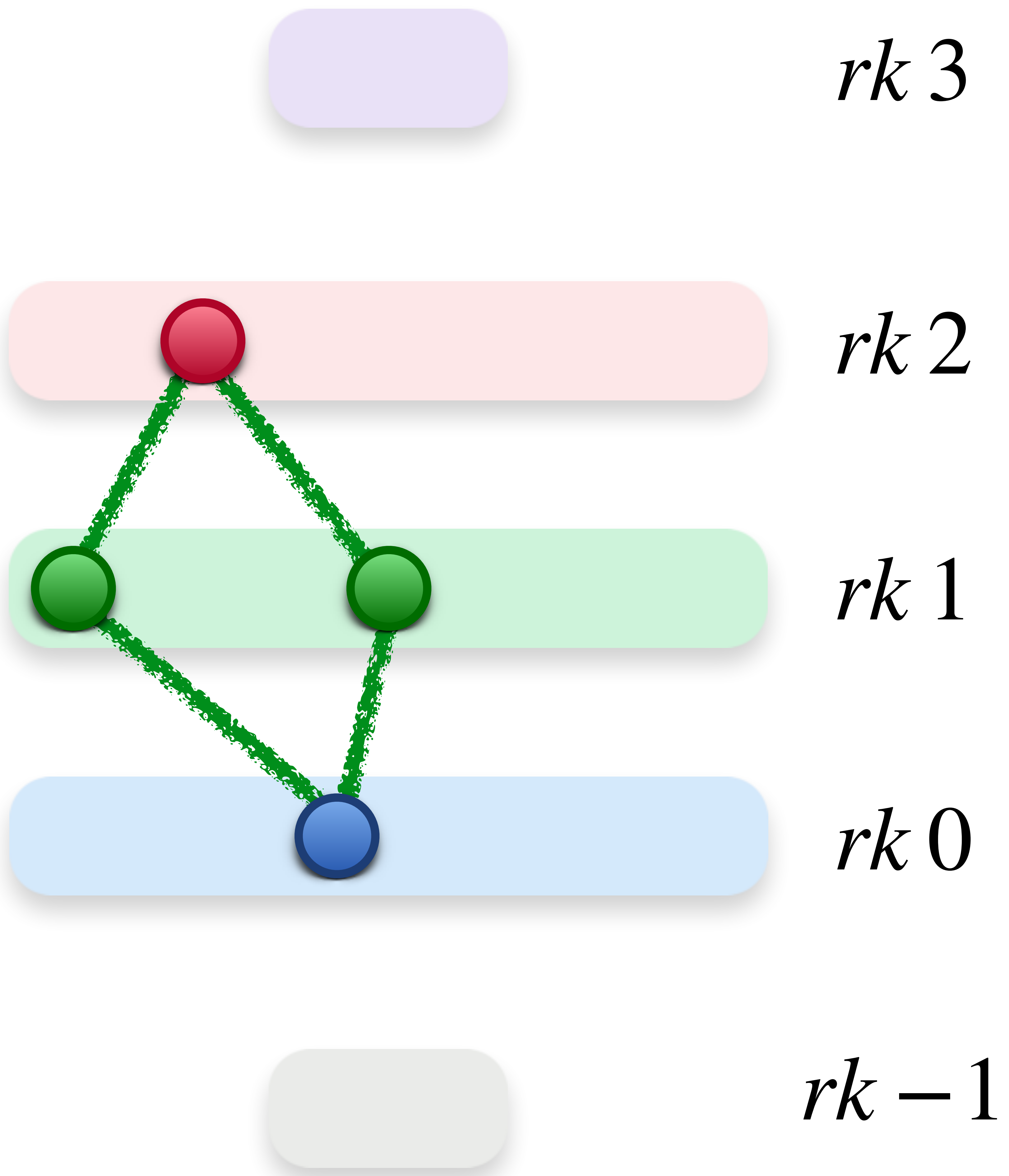

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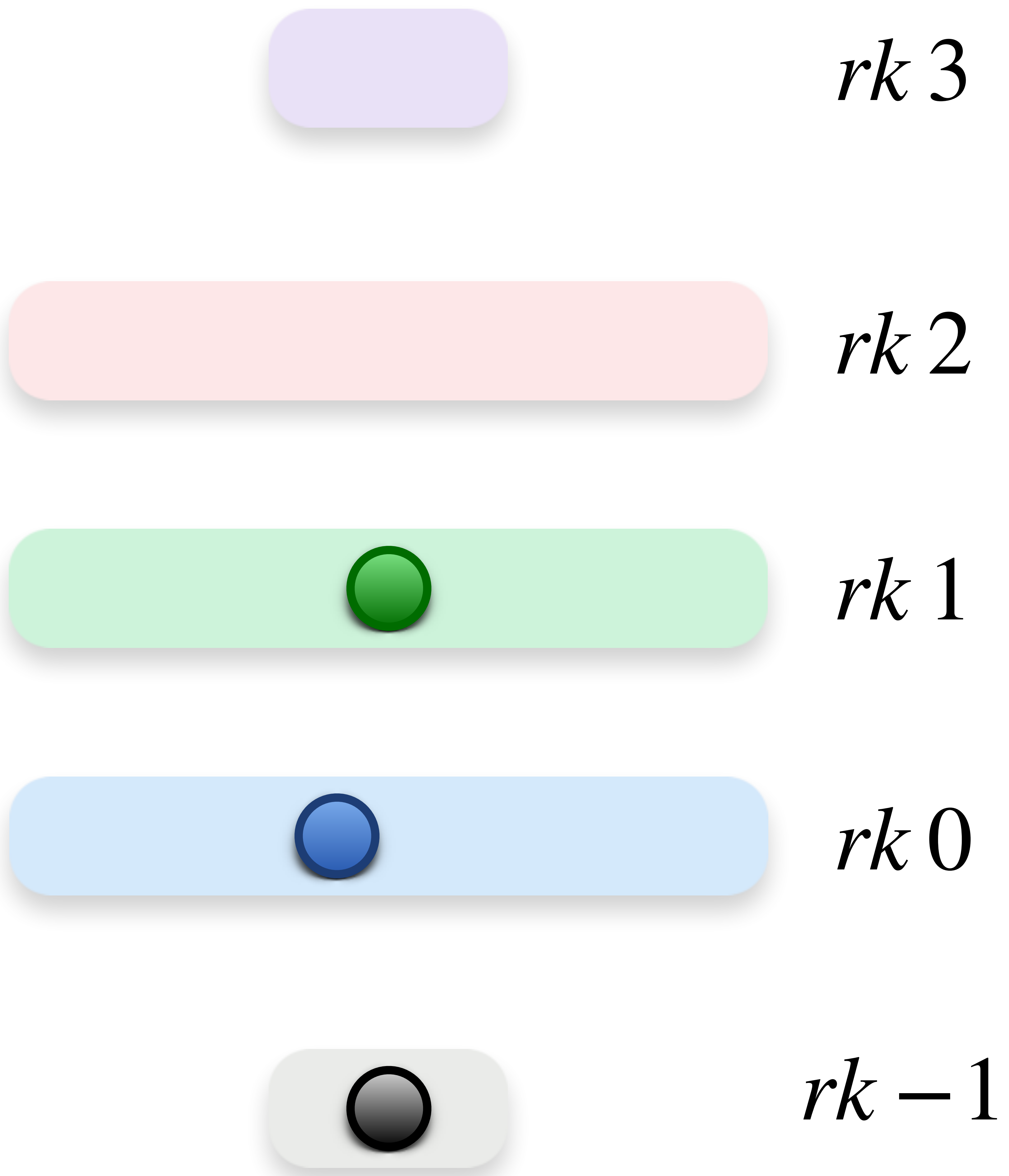

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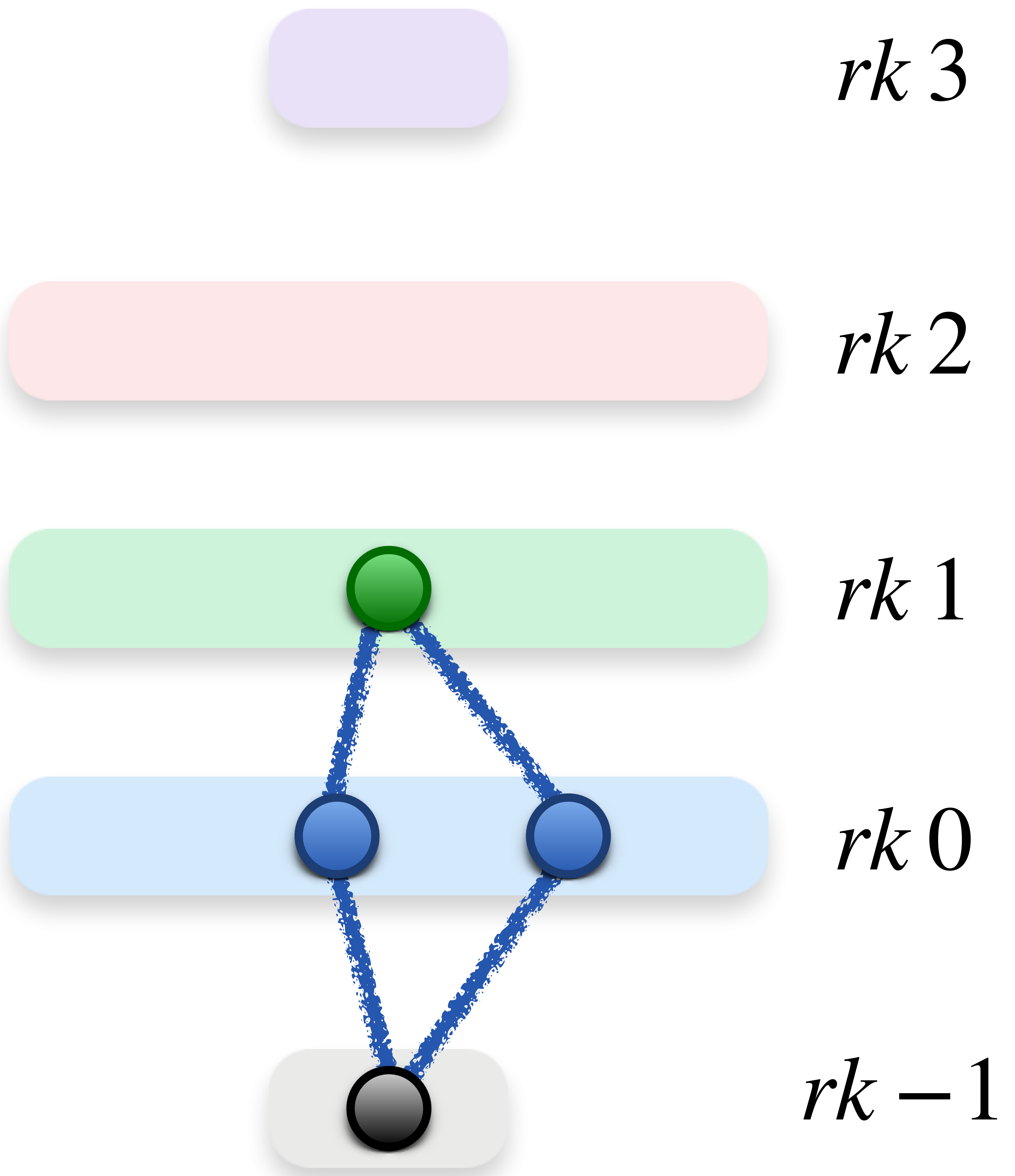
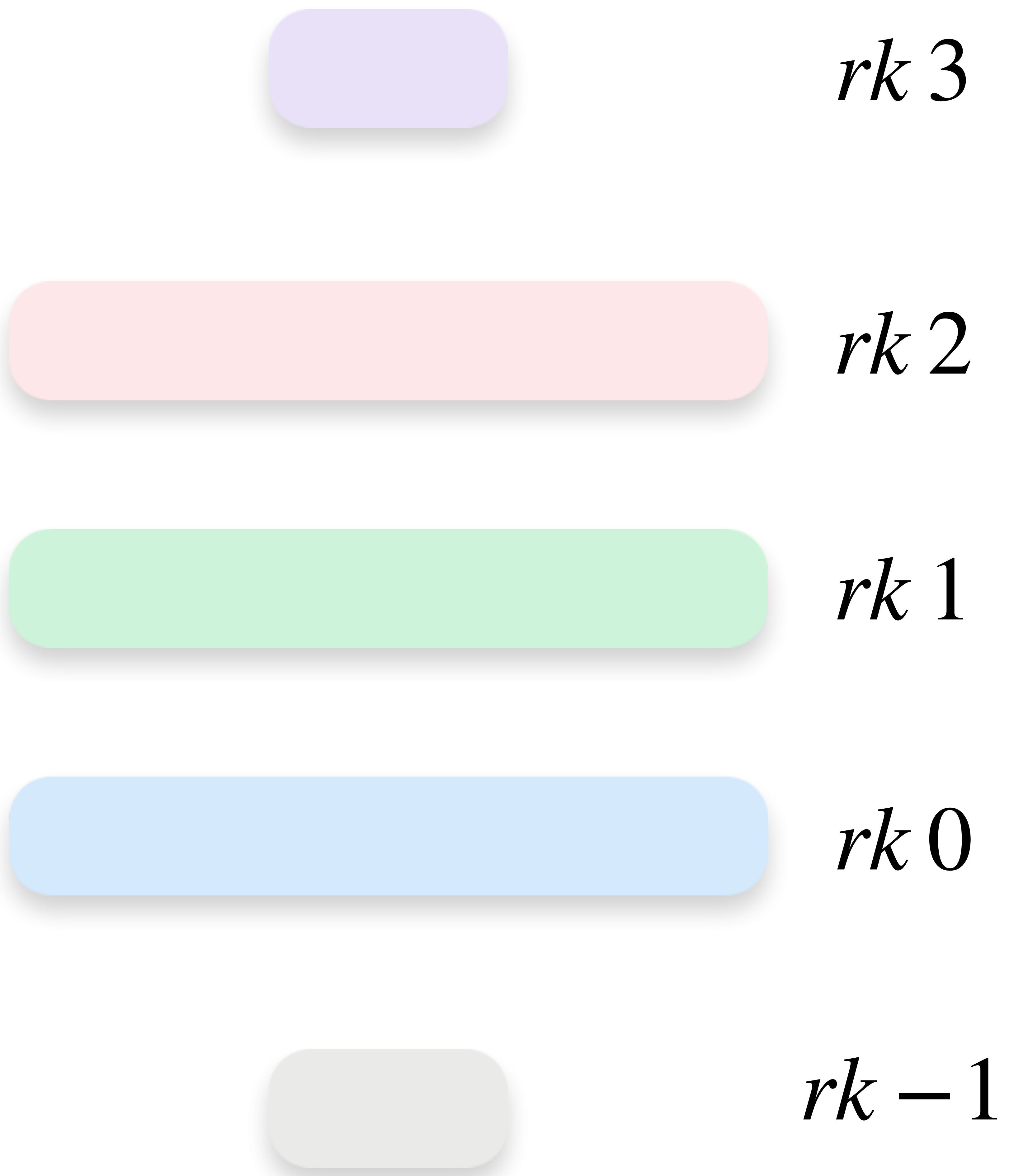


Abstract polytopes

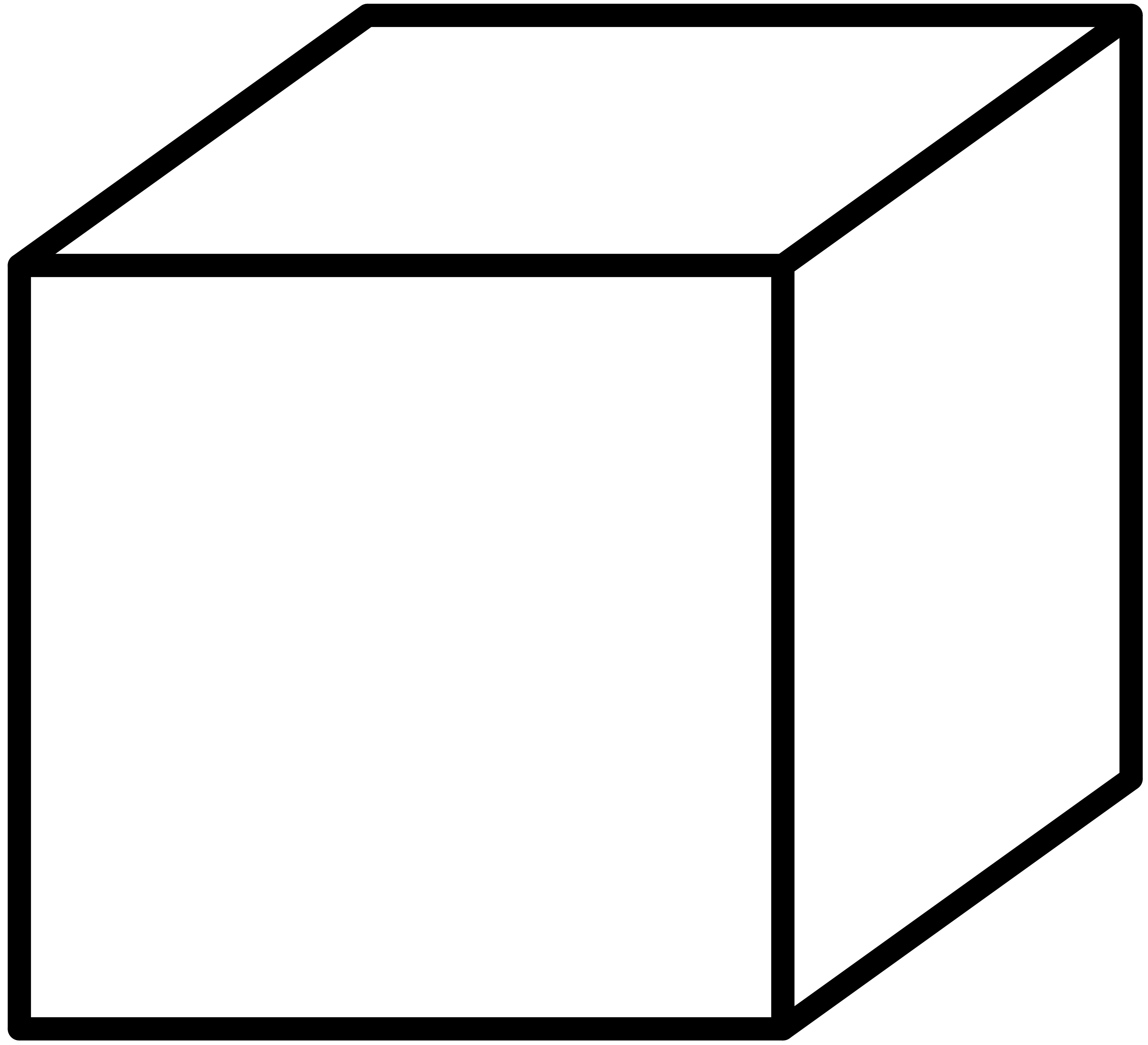
- Ranked poset (each maximal chain contains $(n+2)$ elements);

- Thin (diamond condition);

- Strongly flag connected.



Flag graph



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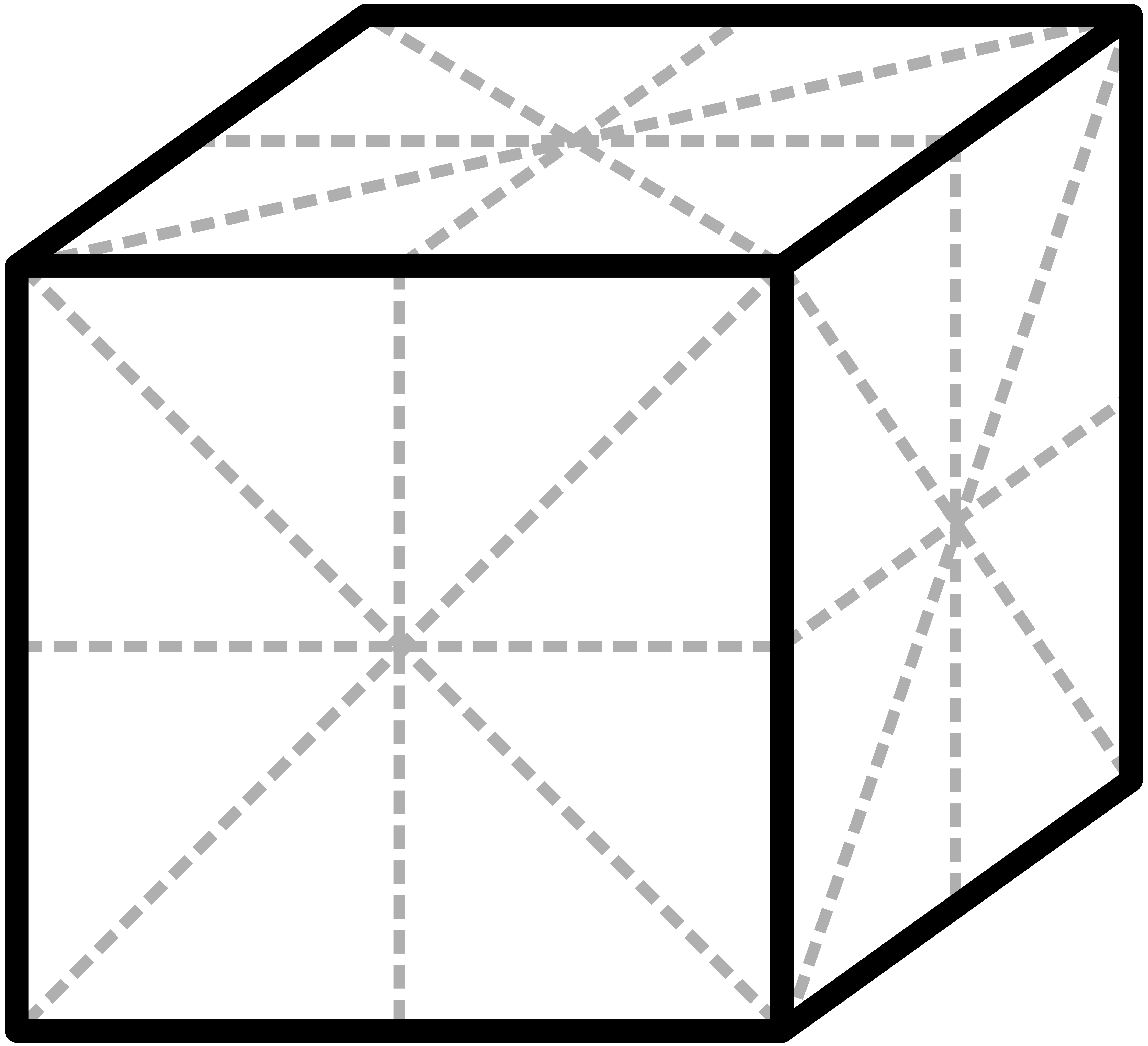

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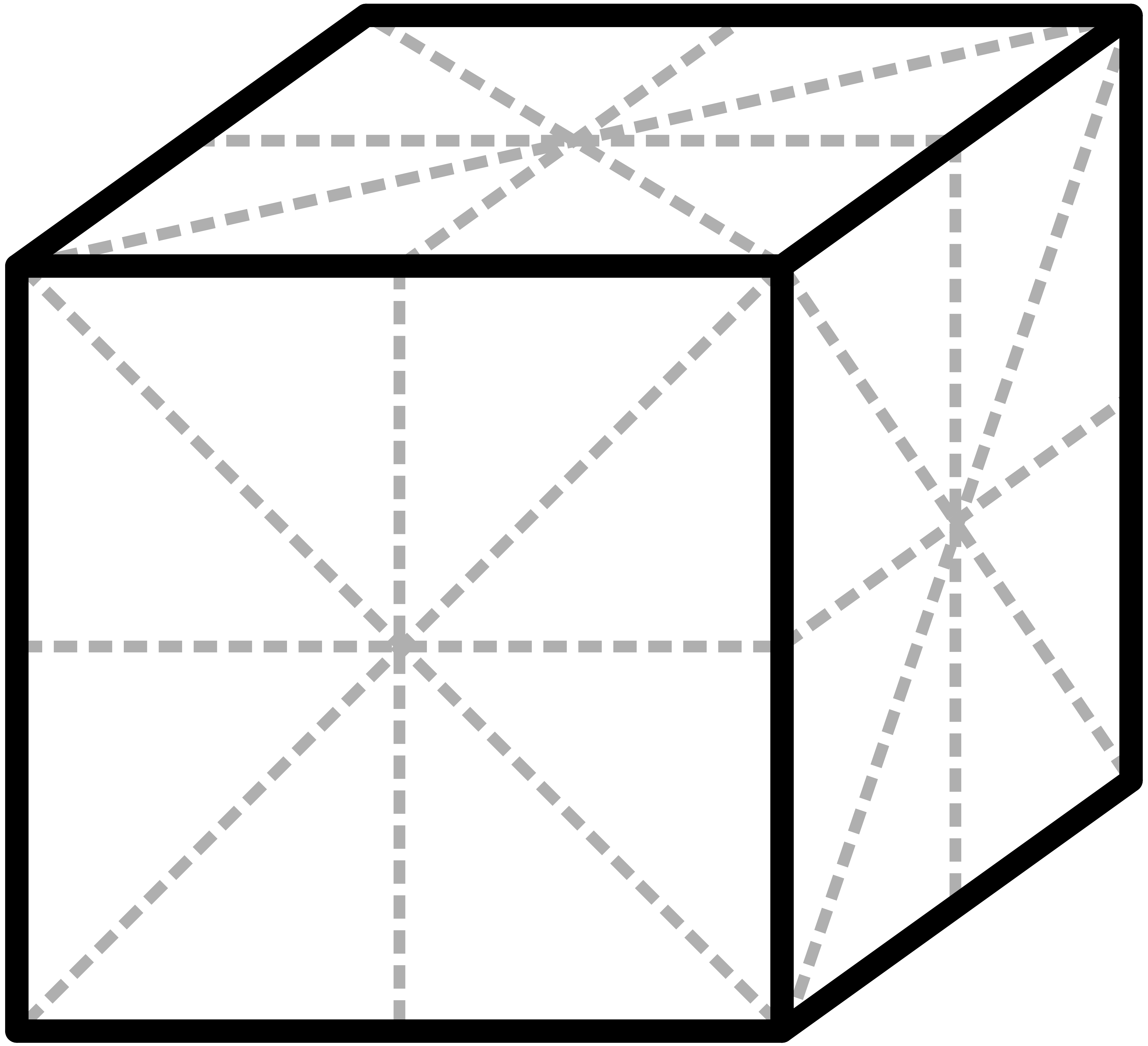

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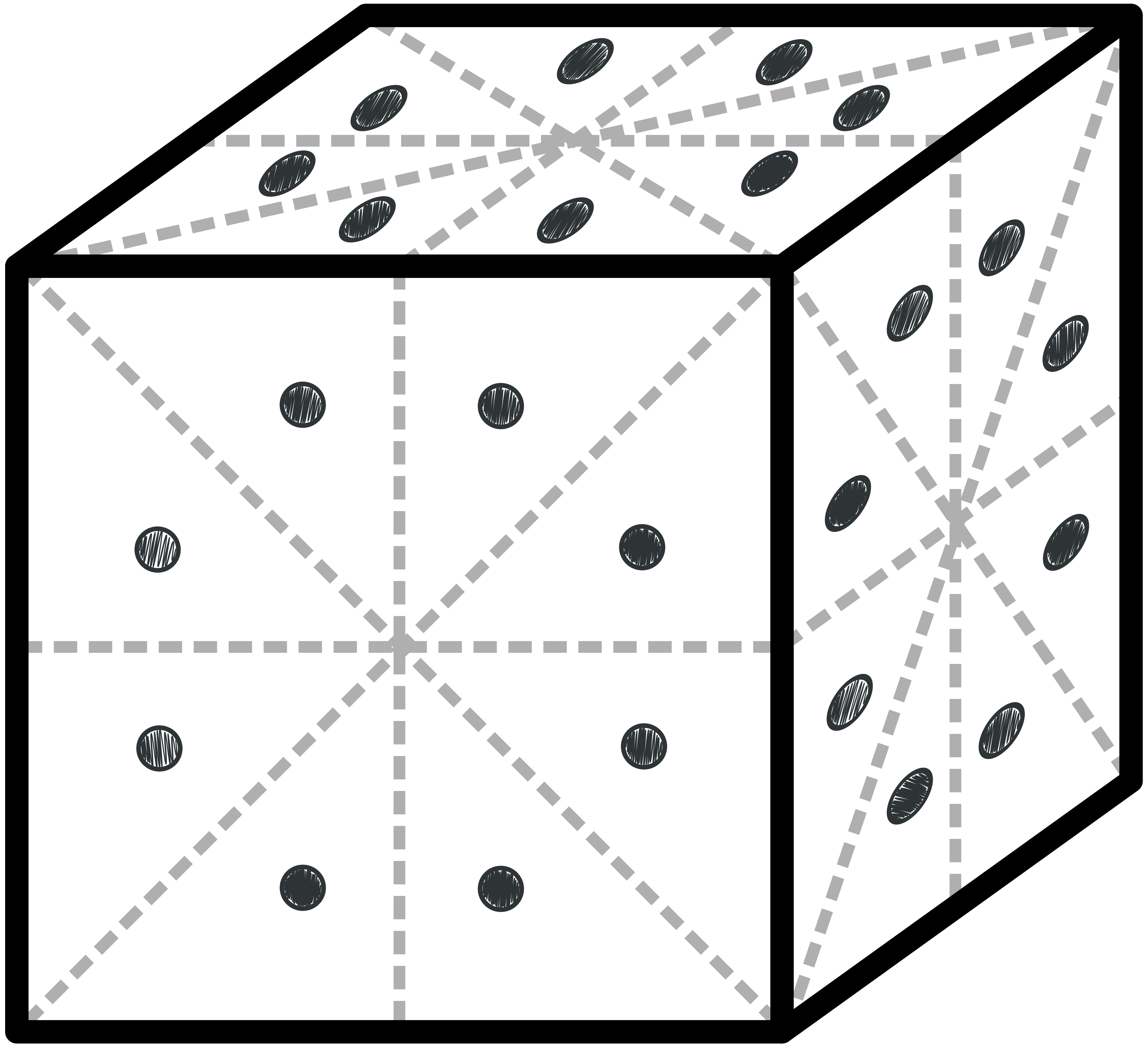

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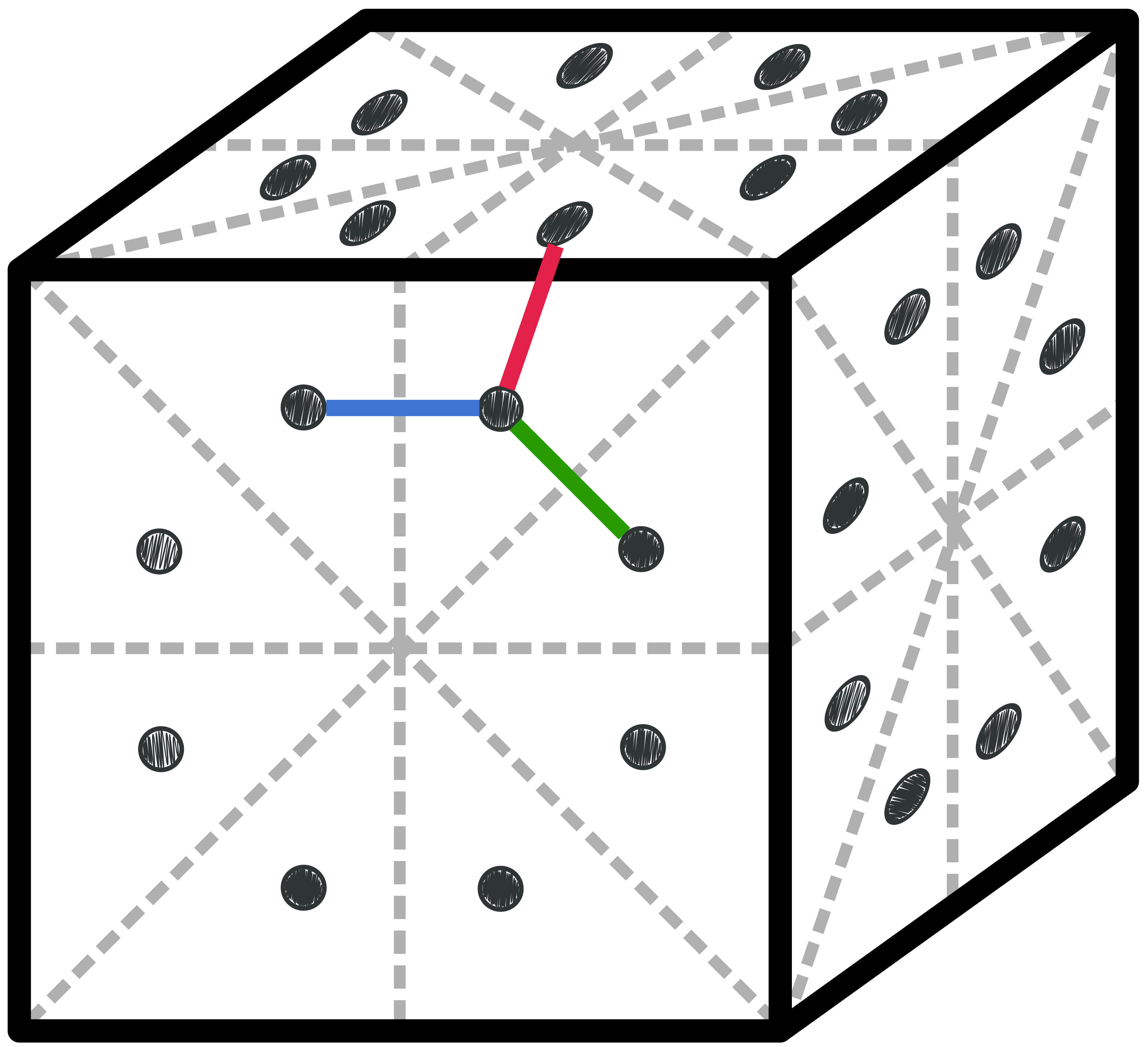
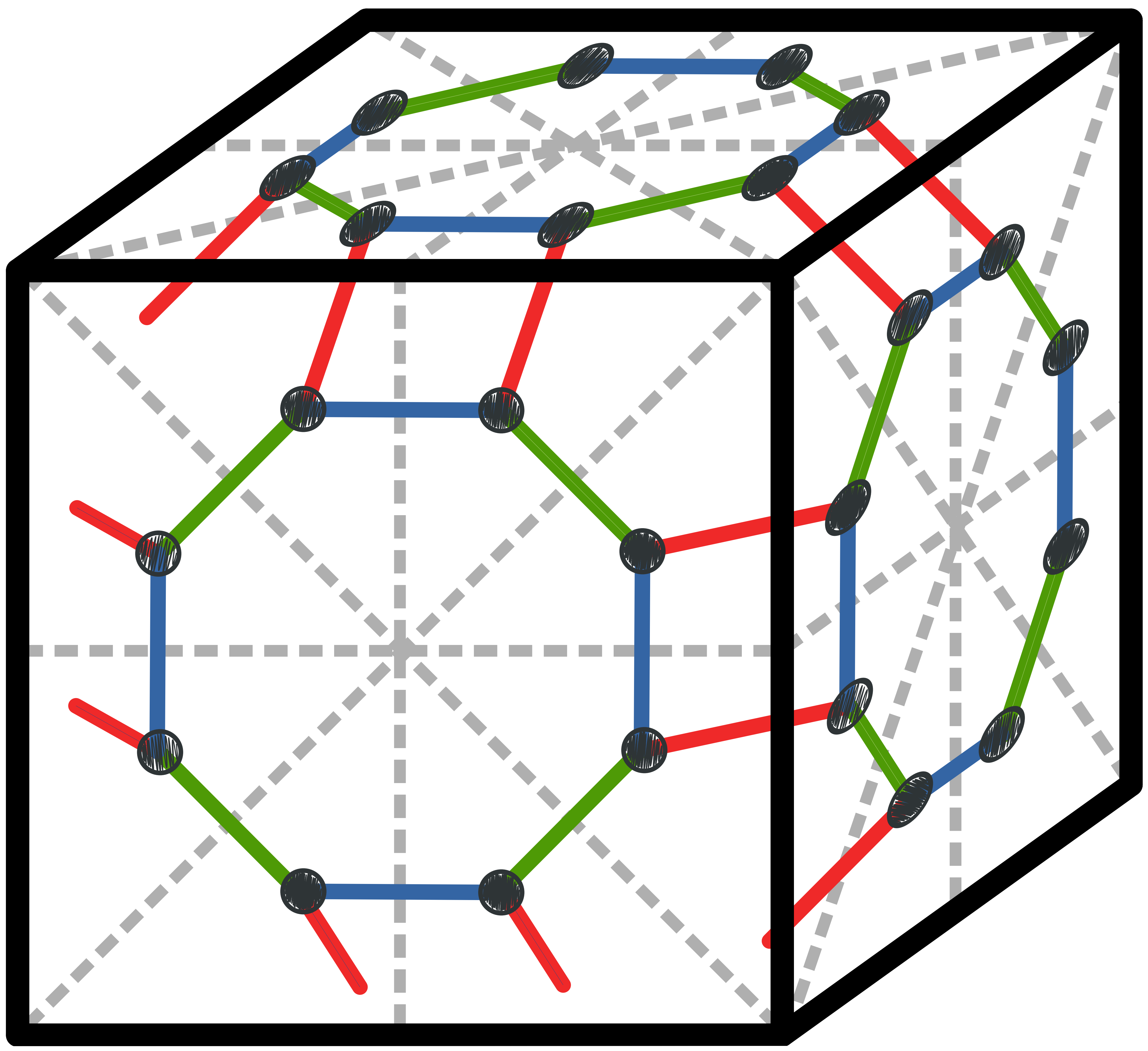


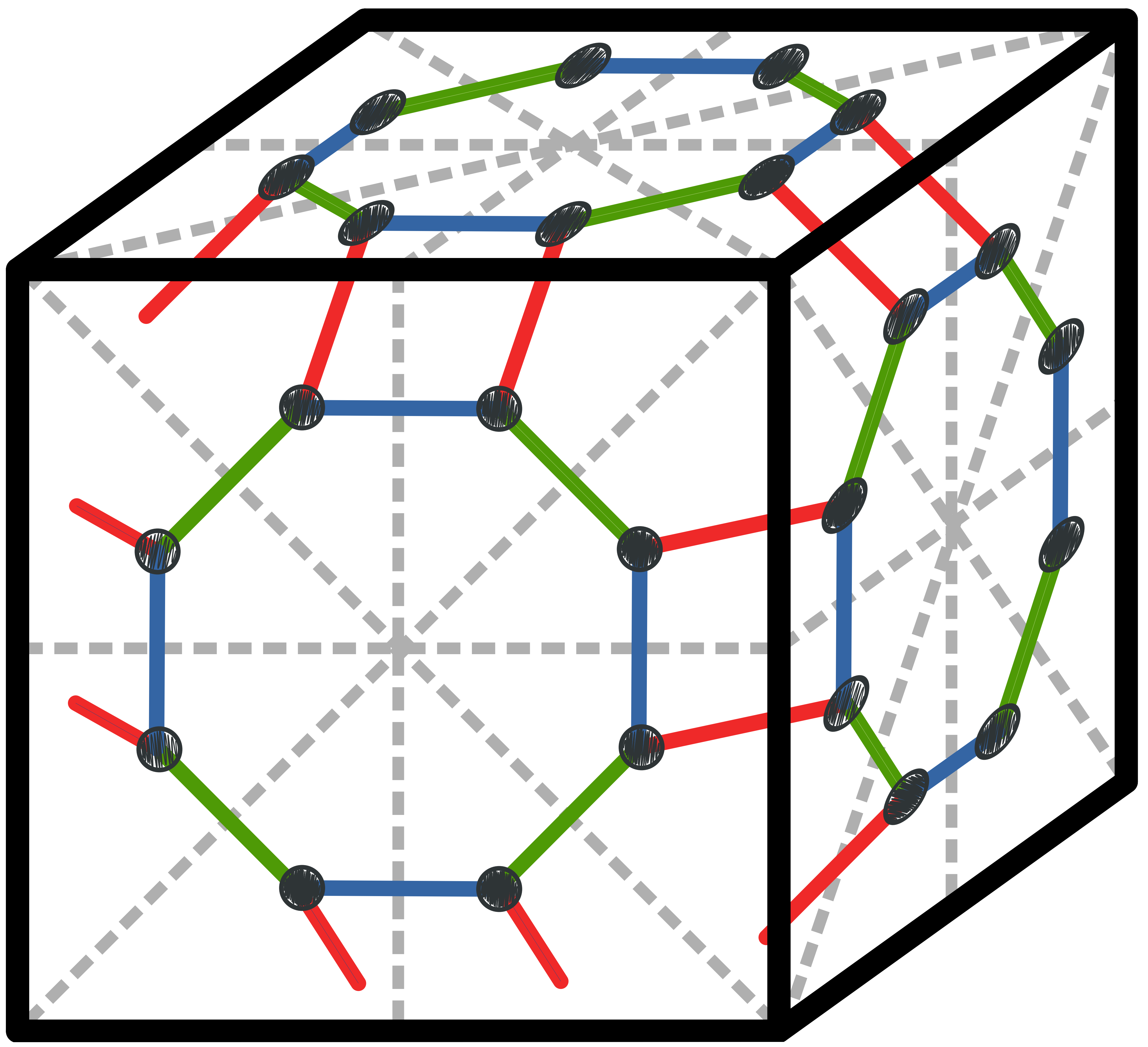
Flag graph



Flag graph

The flag graph $\mathrm{Fl}(\mathscr{P})$ of a polyhedron $\mathscr{P}$:

- Connected and simple,
- Valency 3,
- Properly-edge 3-coloured,
- The (0,2,0,2)-paths are alternating squares.



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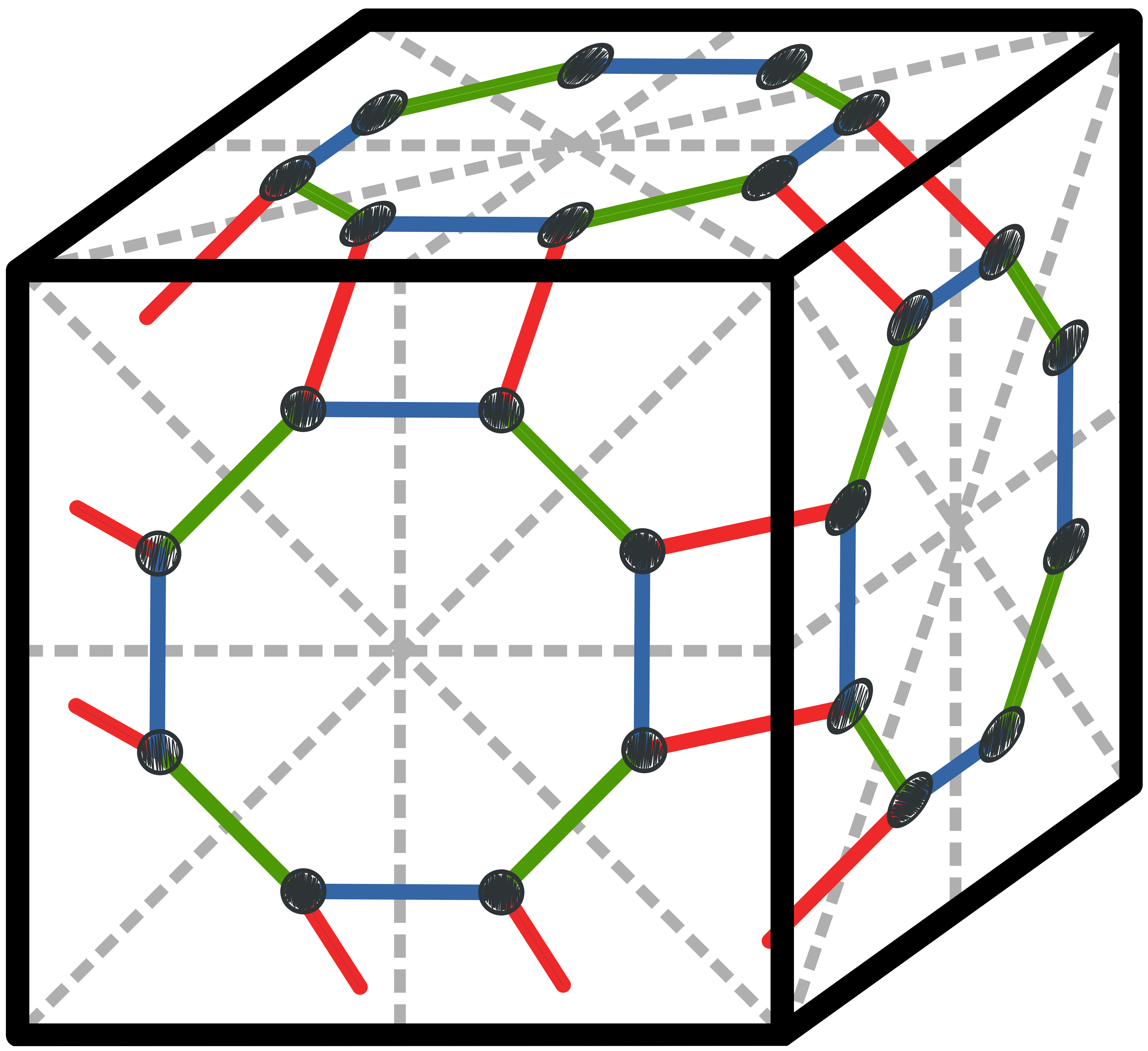

Maniplexes

A 3-maniplex is a graph:

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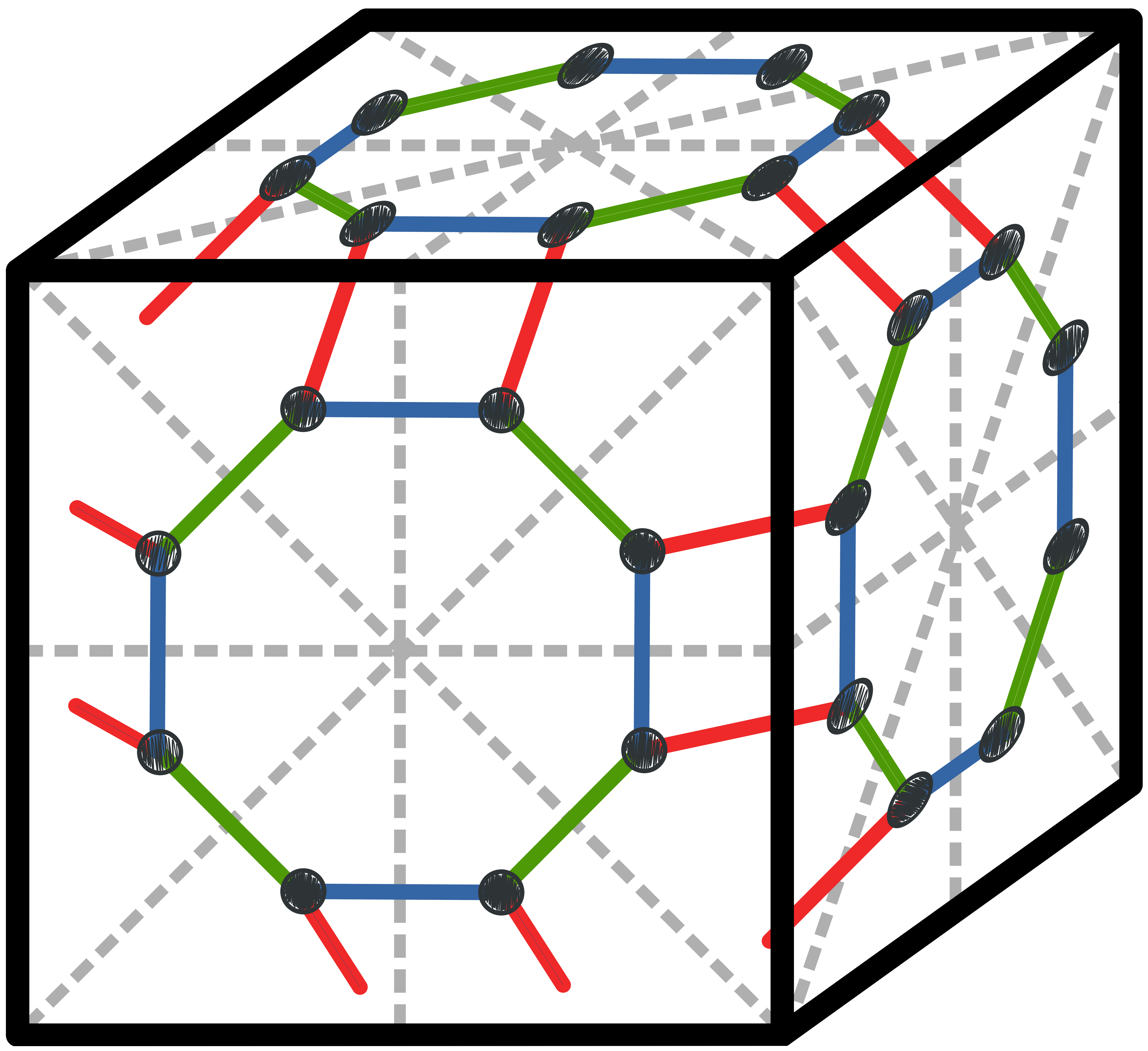
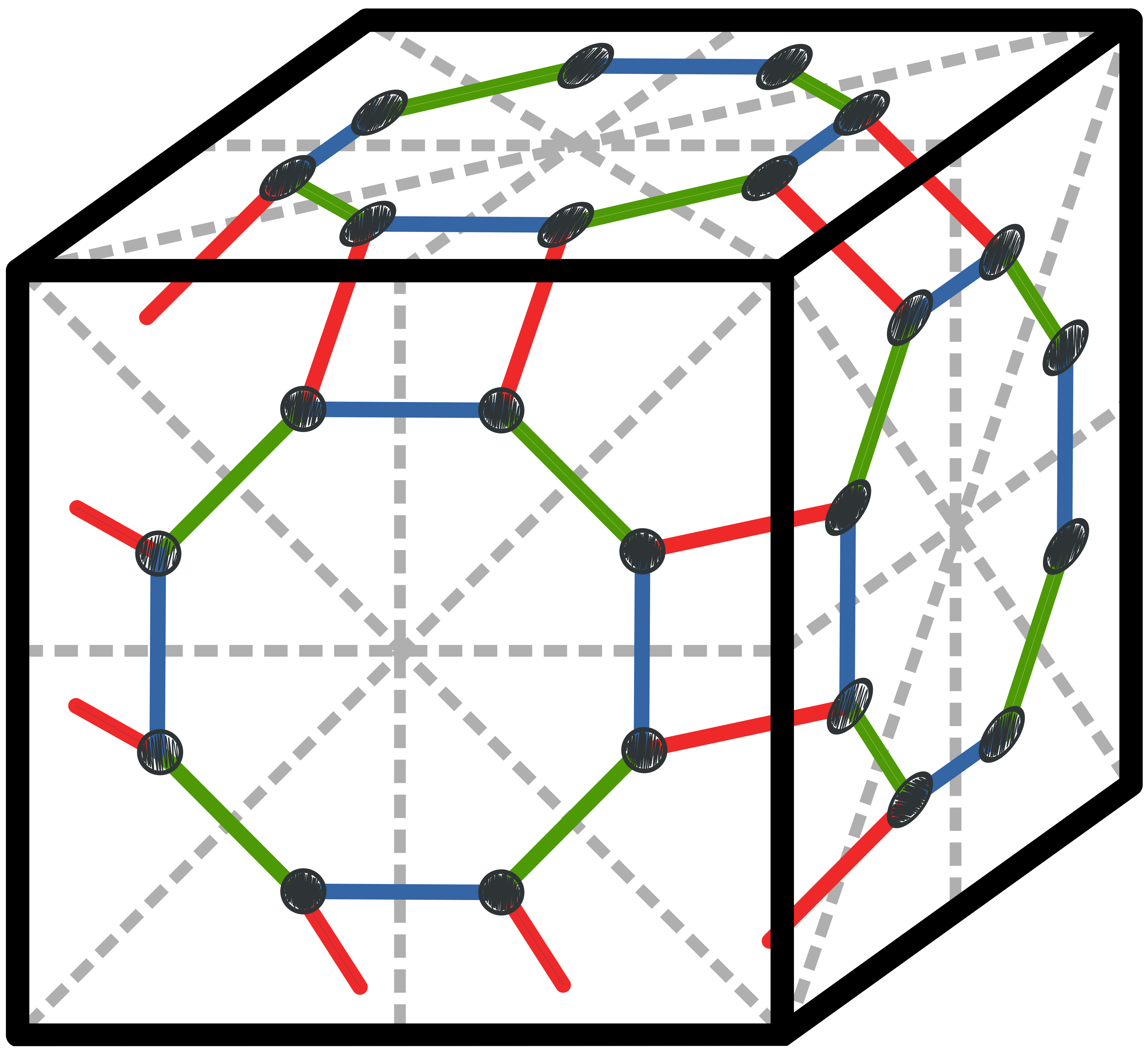


Maniplexes

A n-maniplex is a graph:

- Connected and simple,
- Valency n,
- Properly-edge n-coloured,
- The (i, j, i, j)-paths are alternating squares, whenever $|i - j| > 1$



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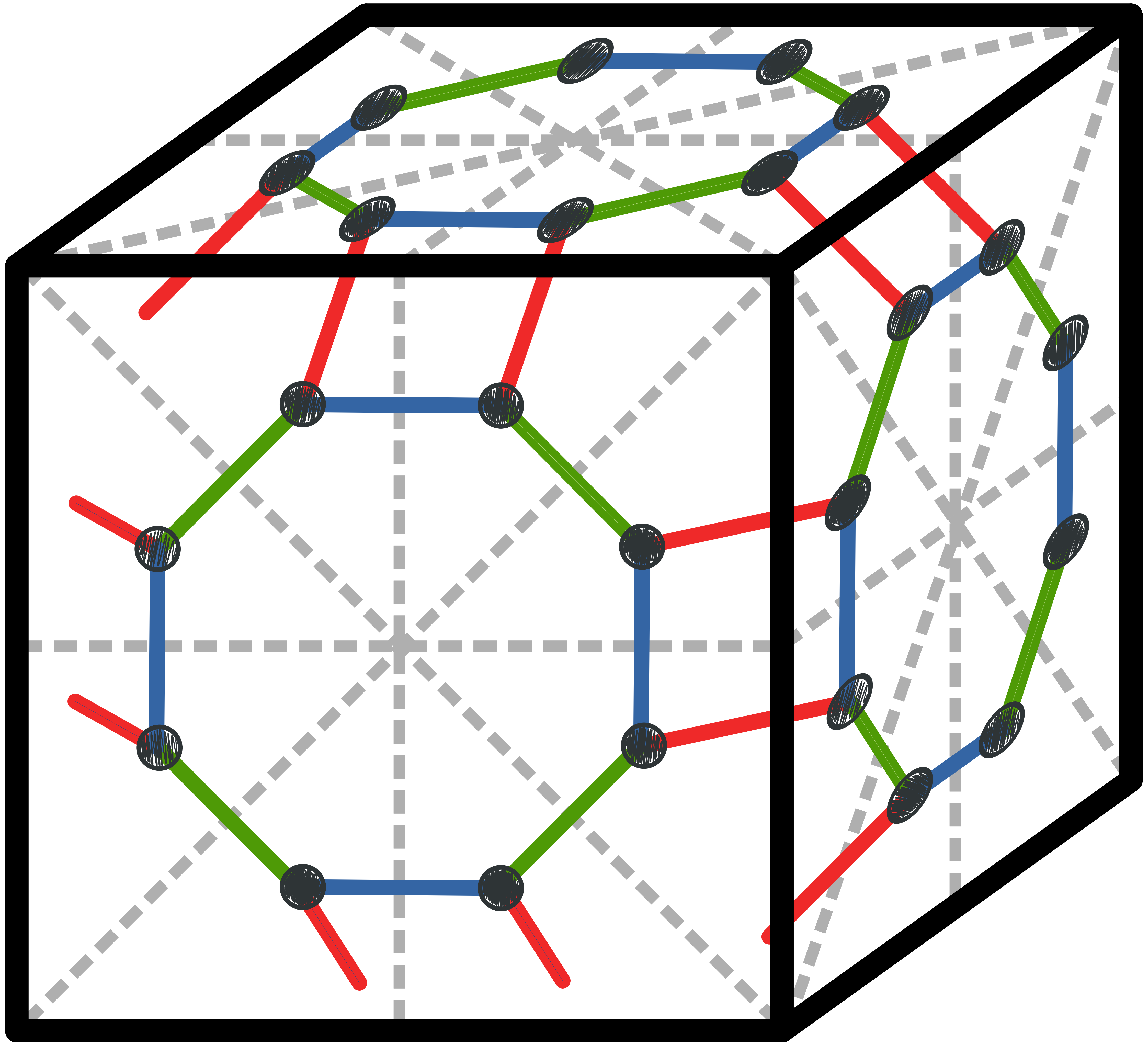
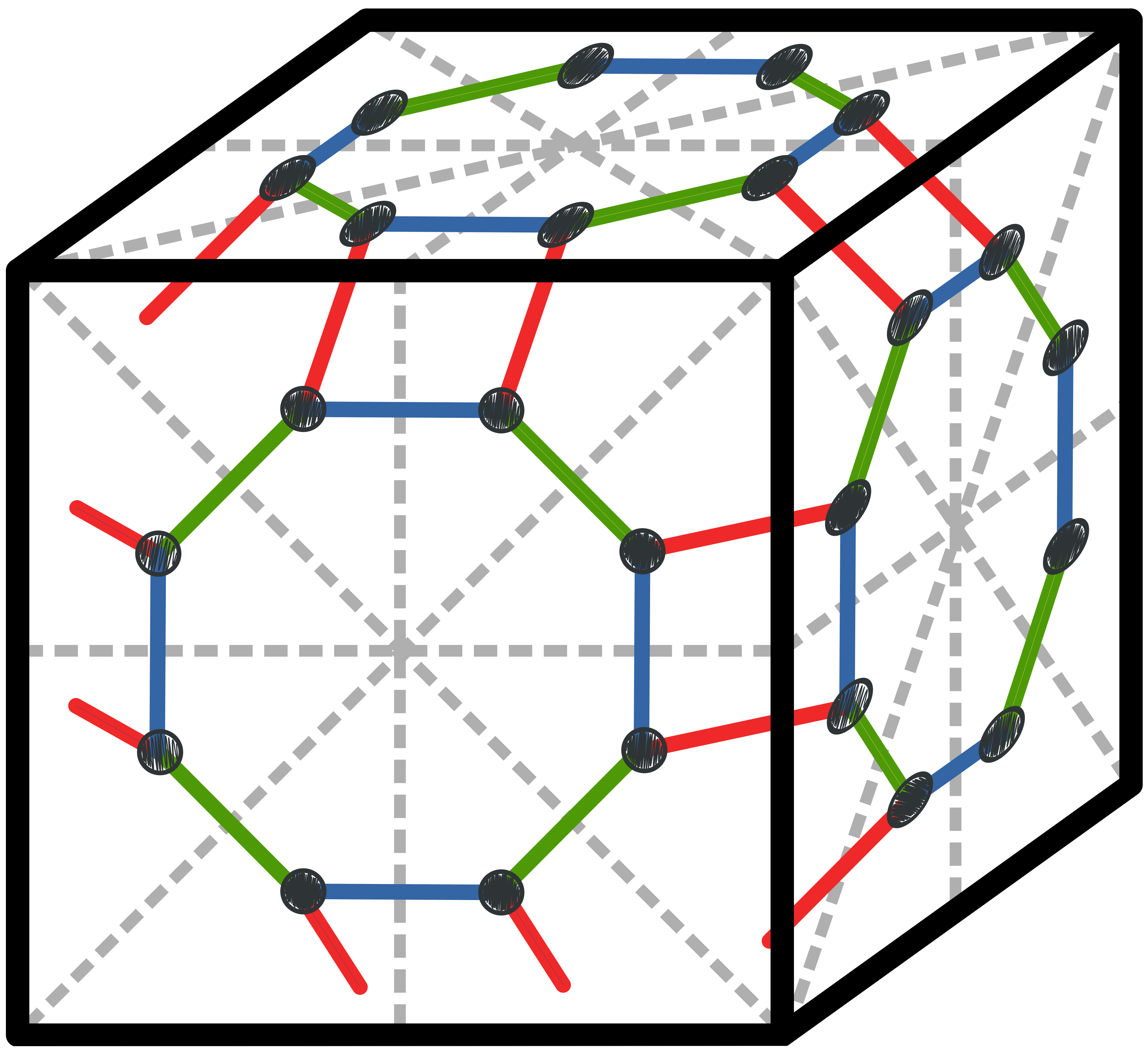


Gabe might or might not have talked about this.

Polytopes vs maniplexes



Polytopes vs maniplexes

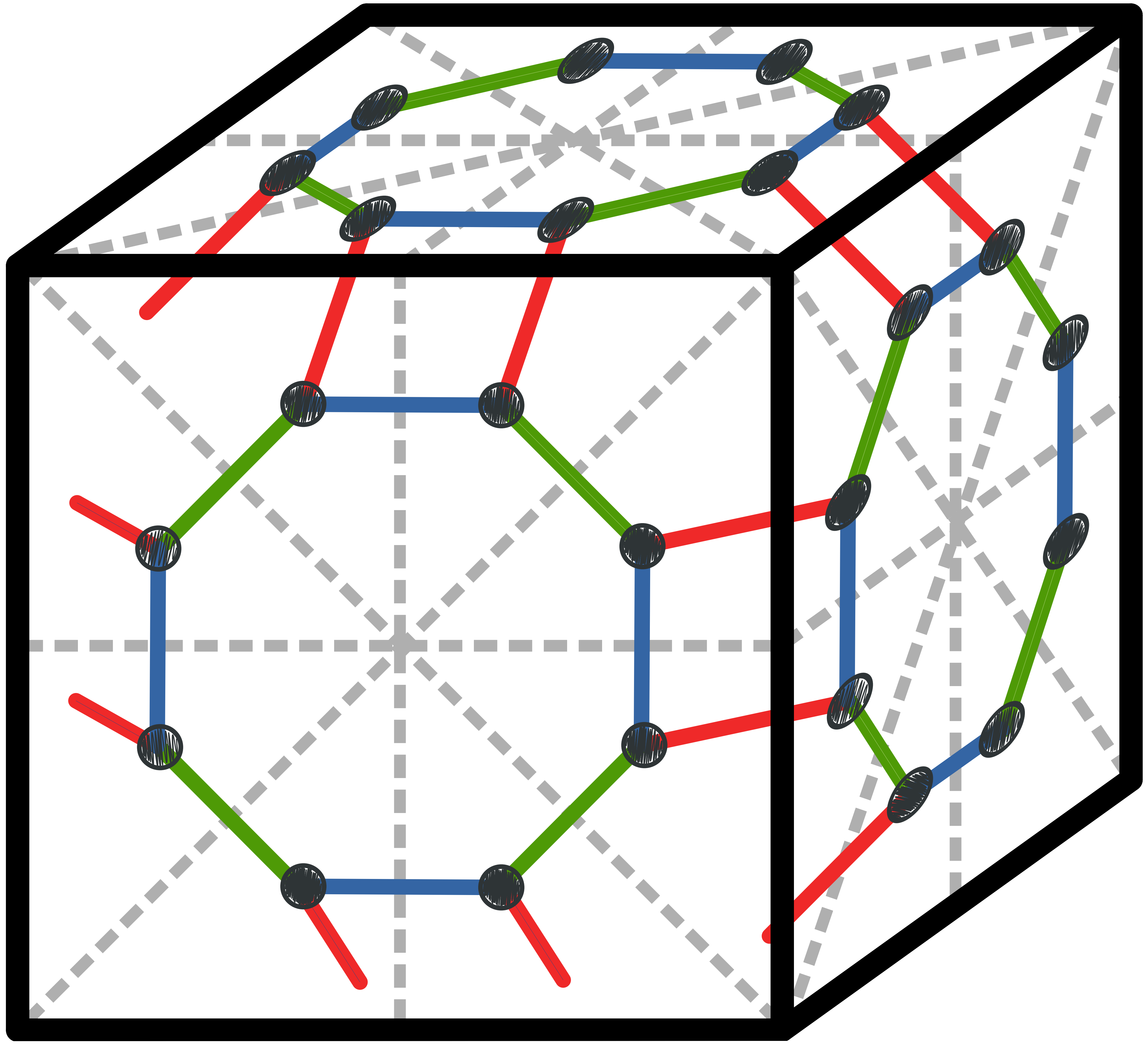
rk 3

rk 2

rk 1

rk 0

rk −1



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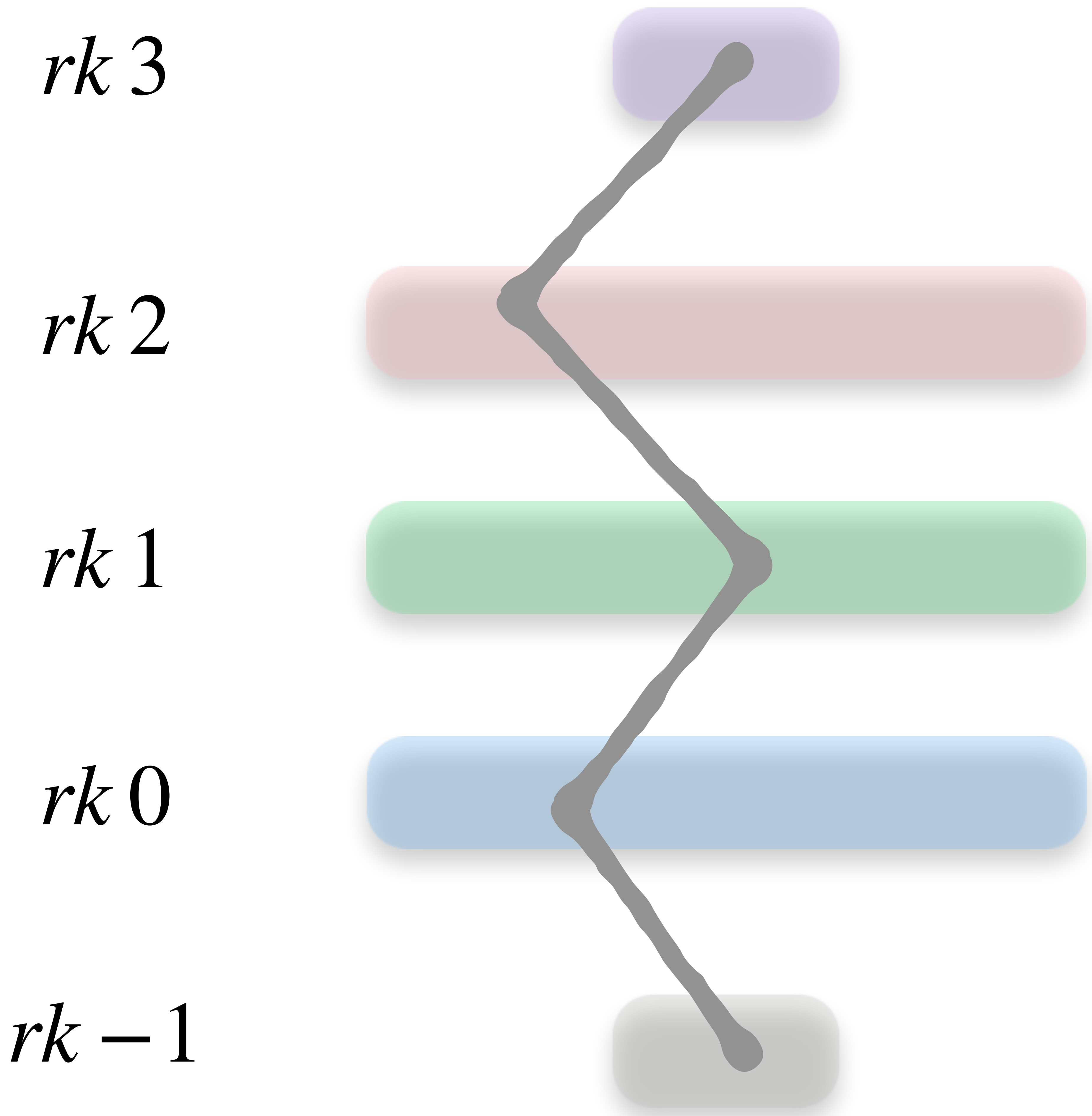

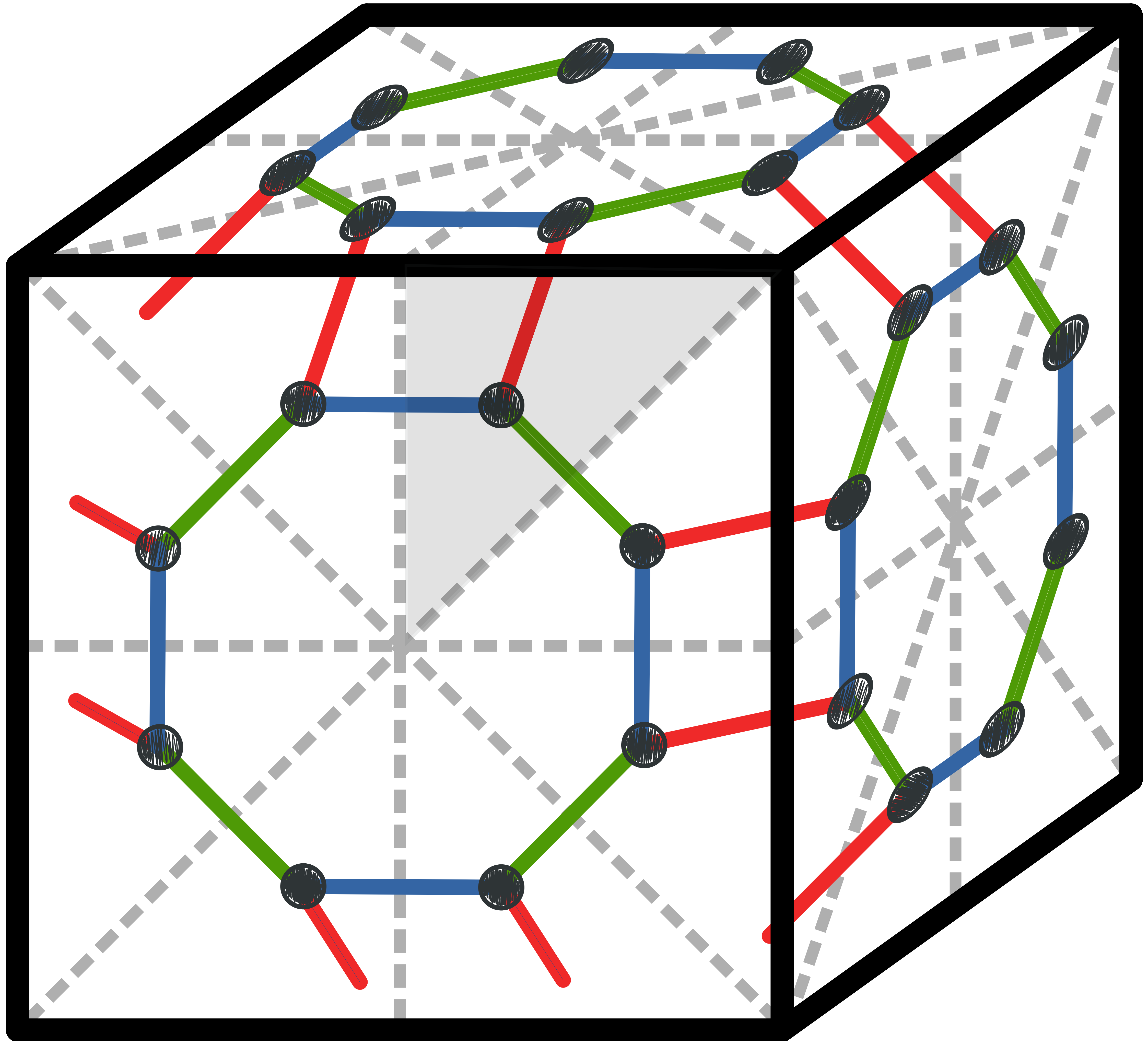

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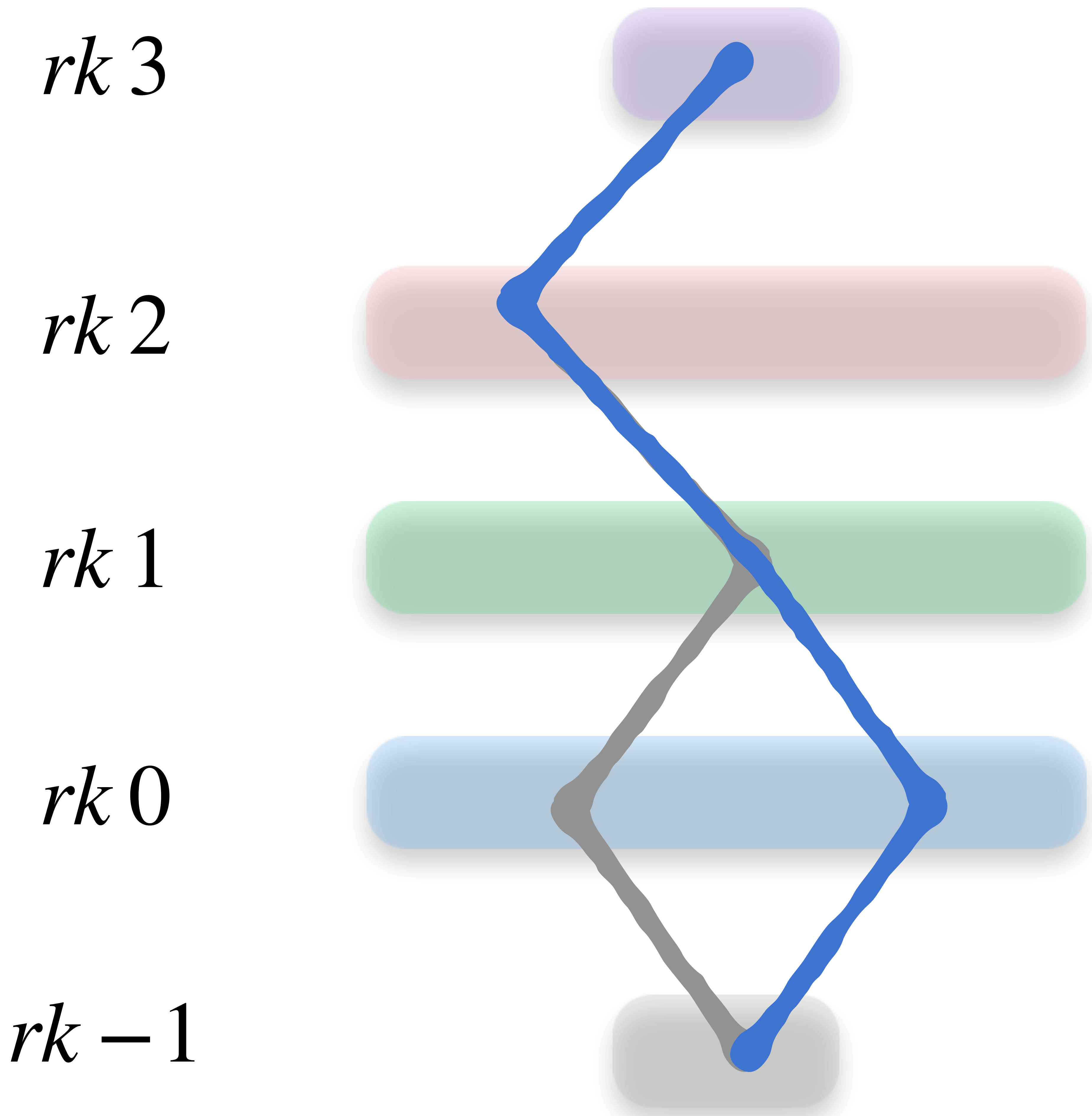

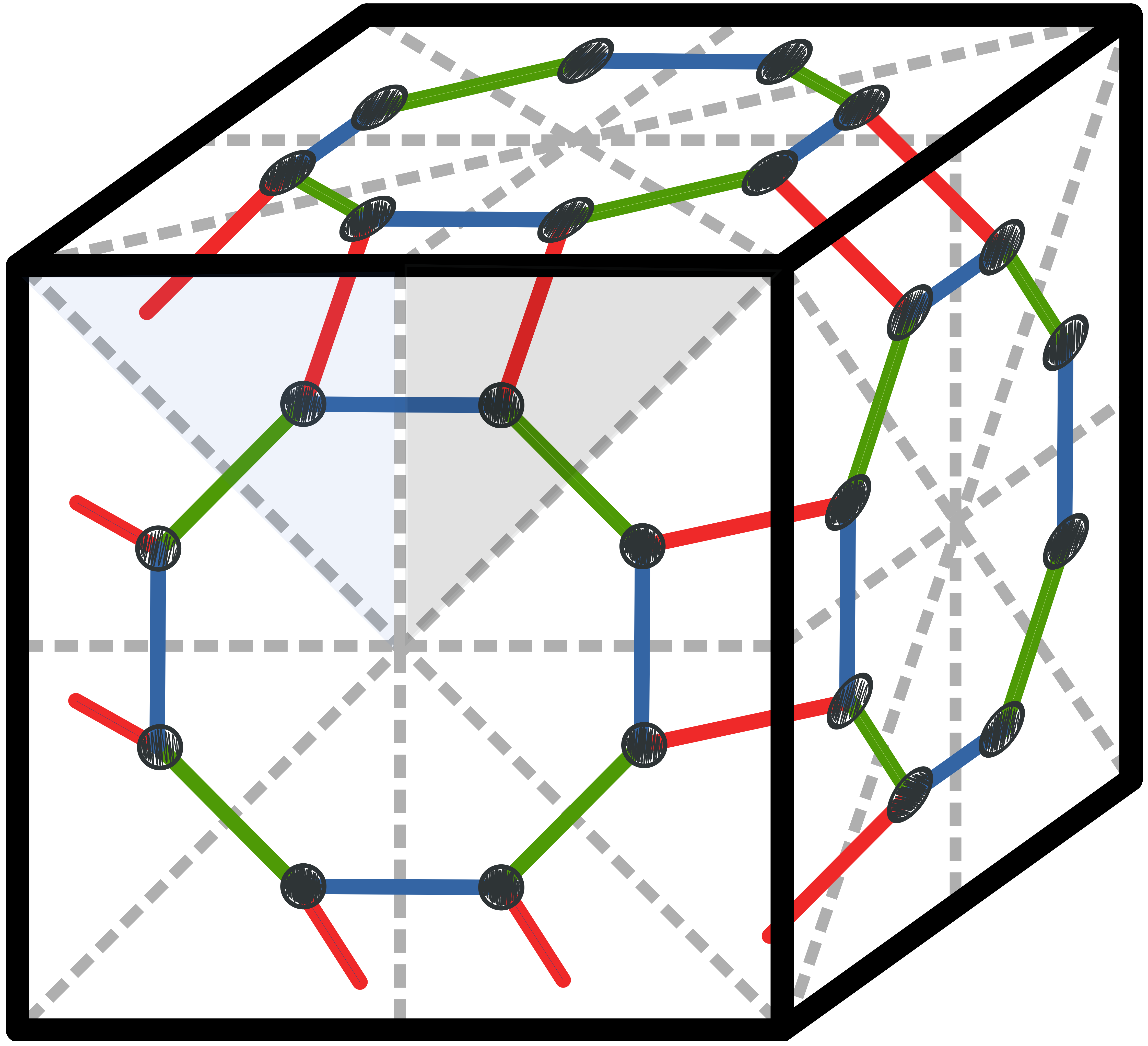

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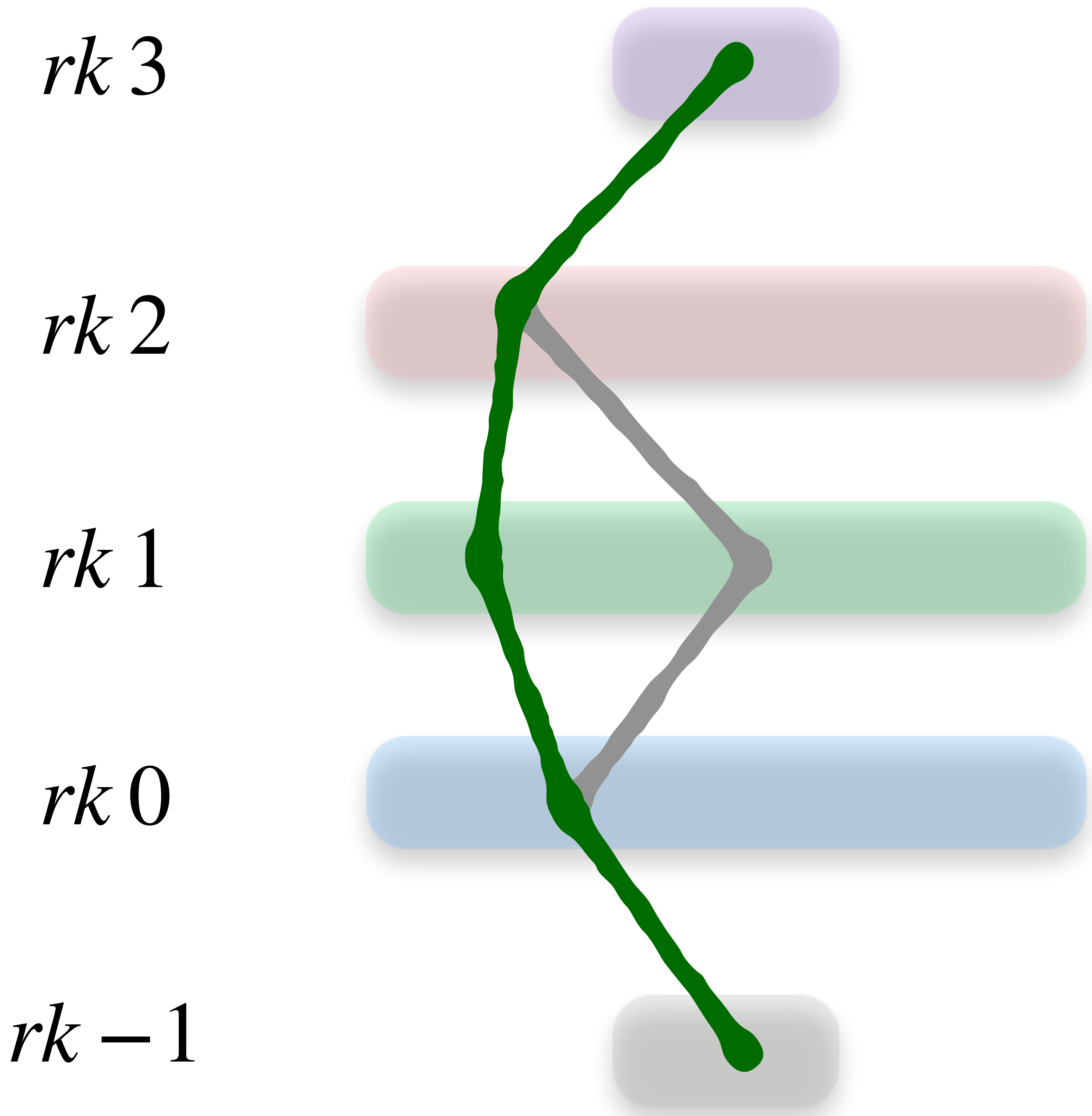

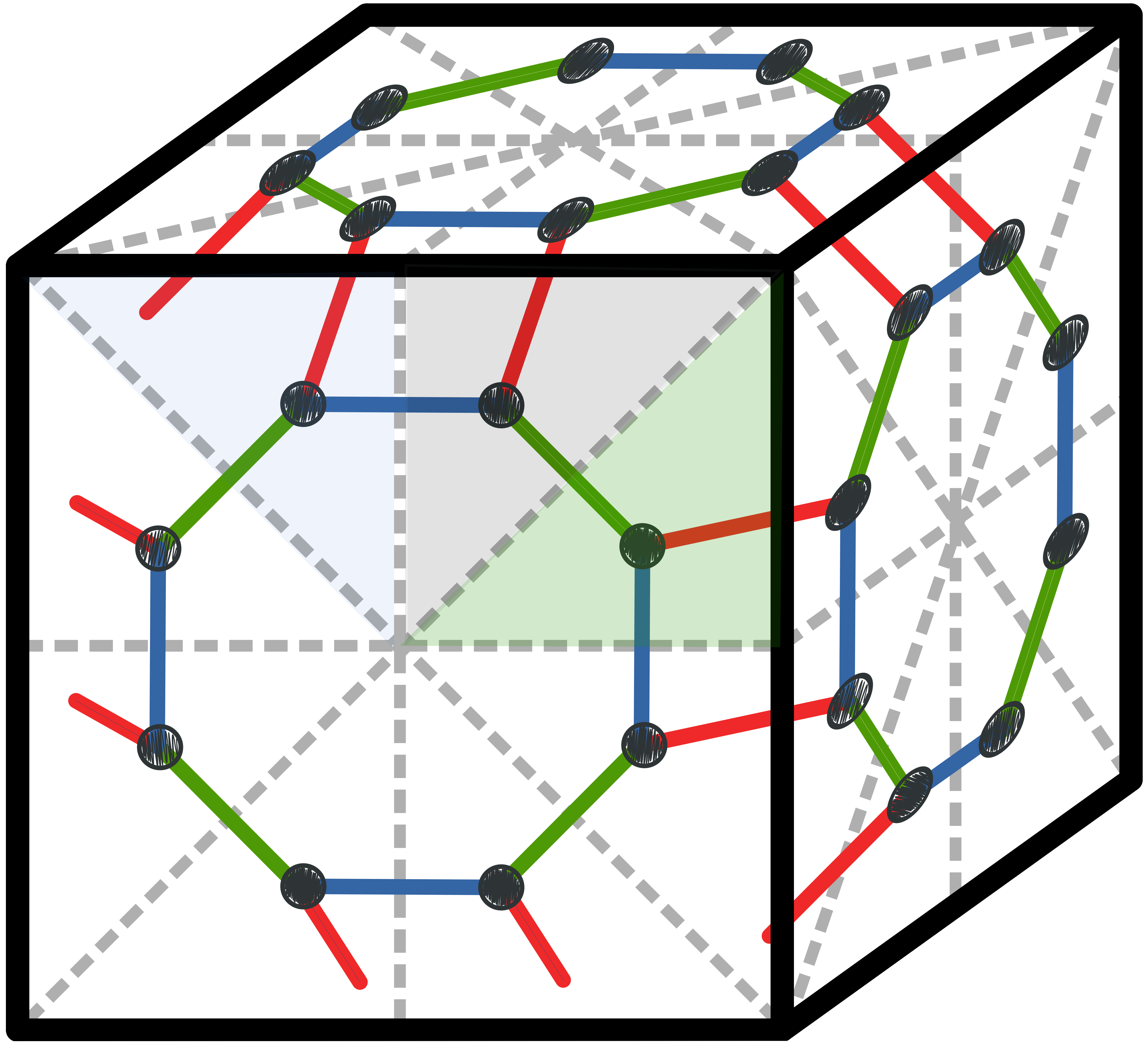

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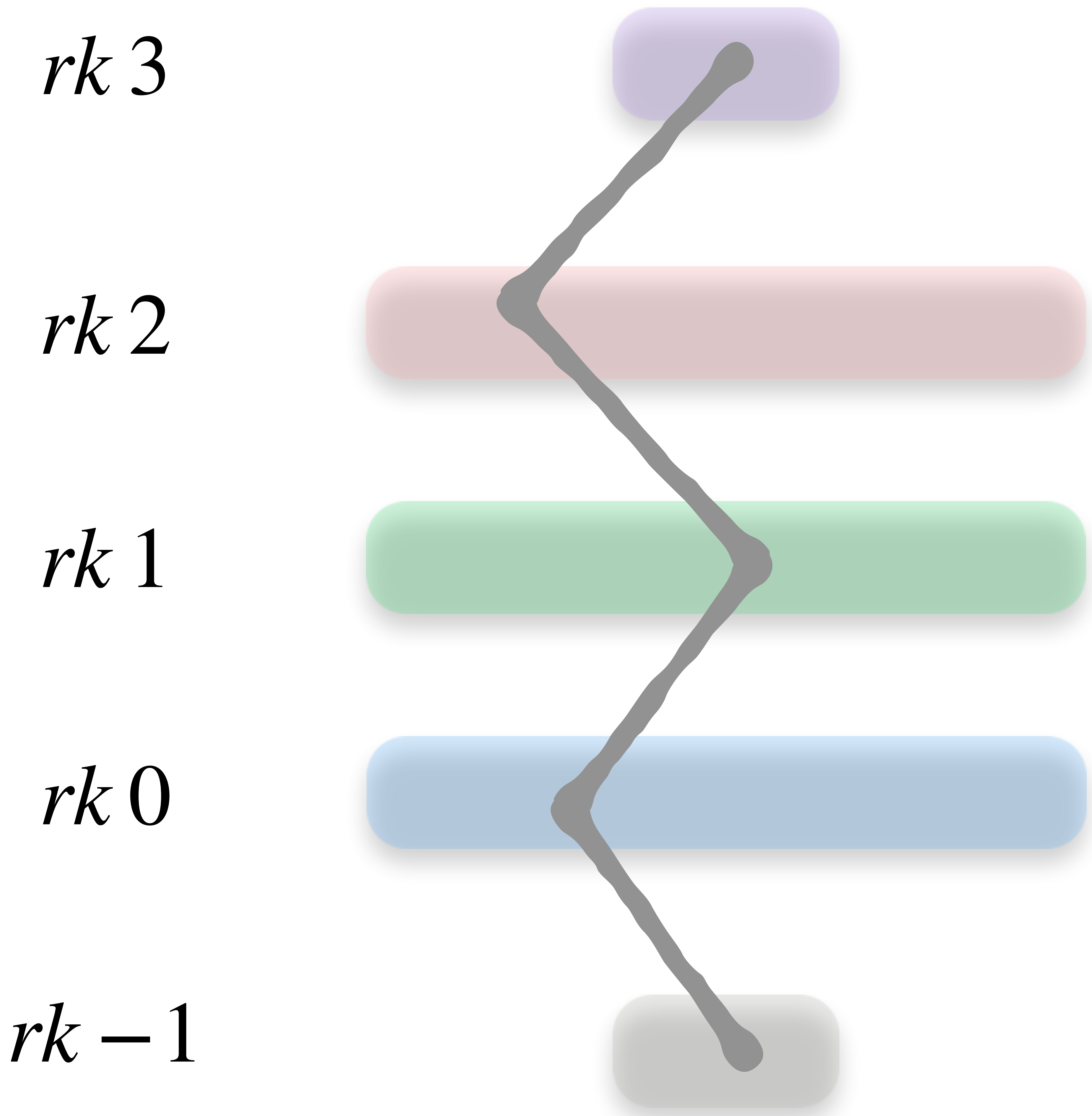

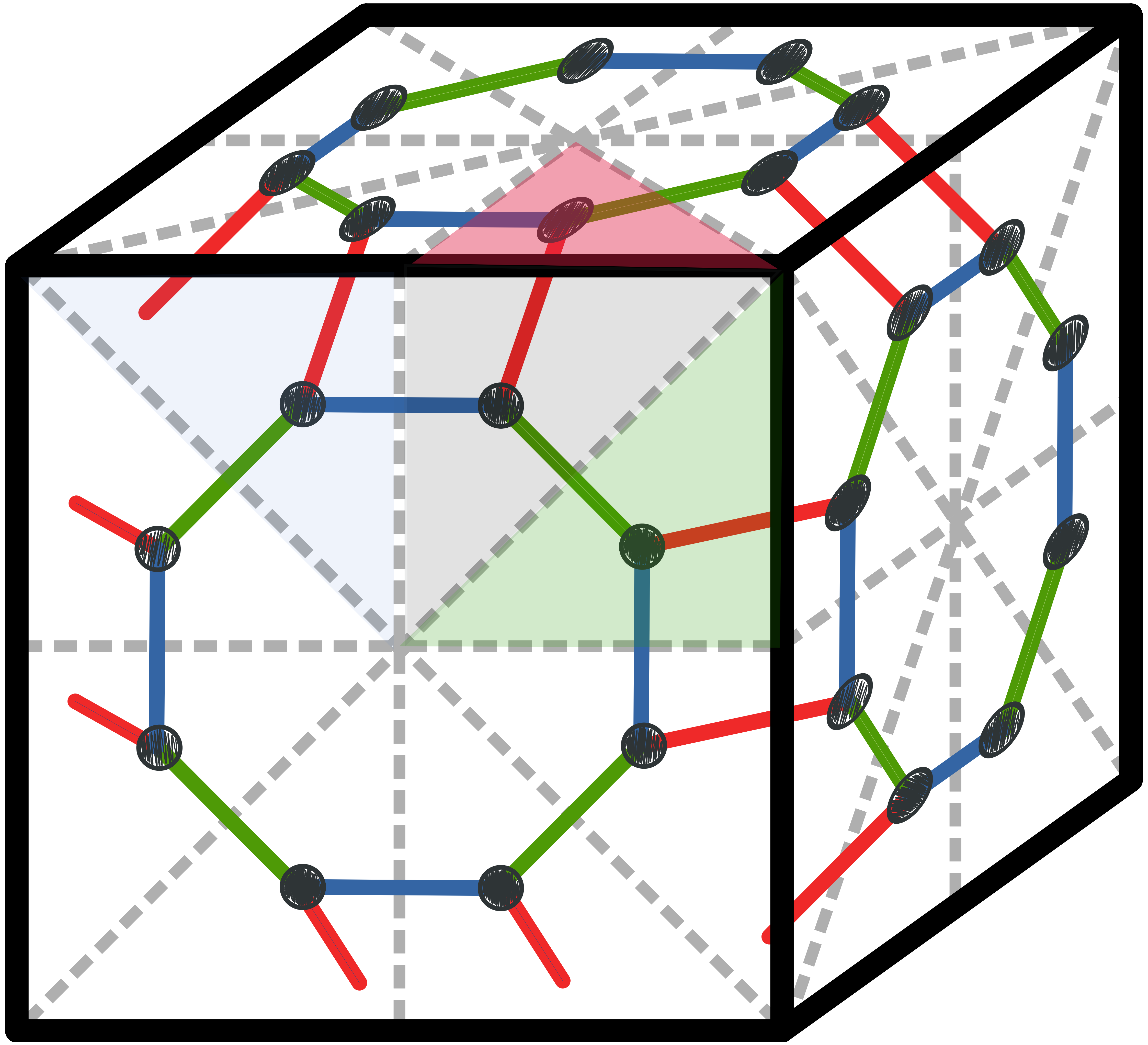

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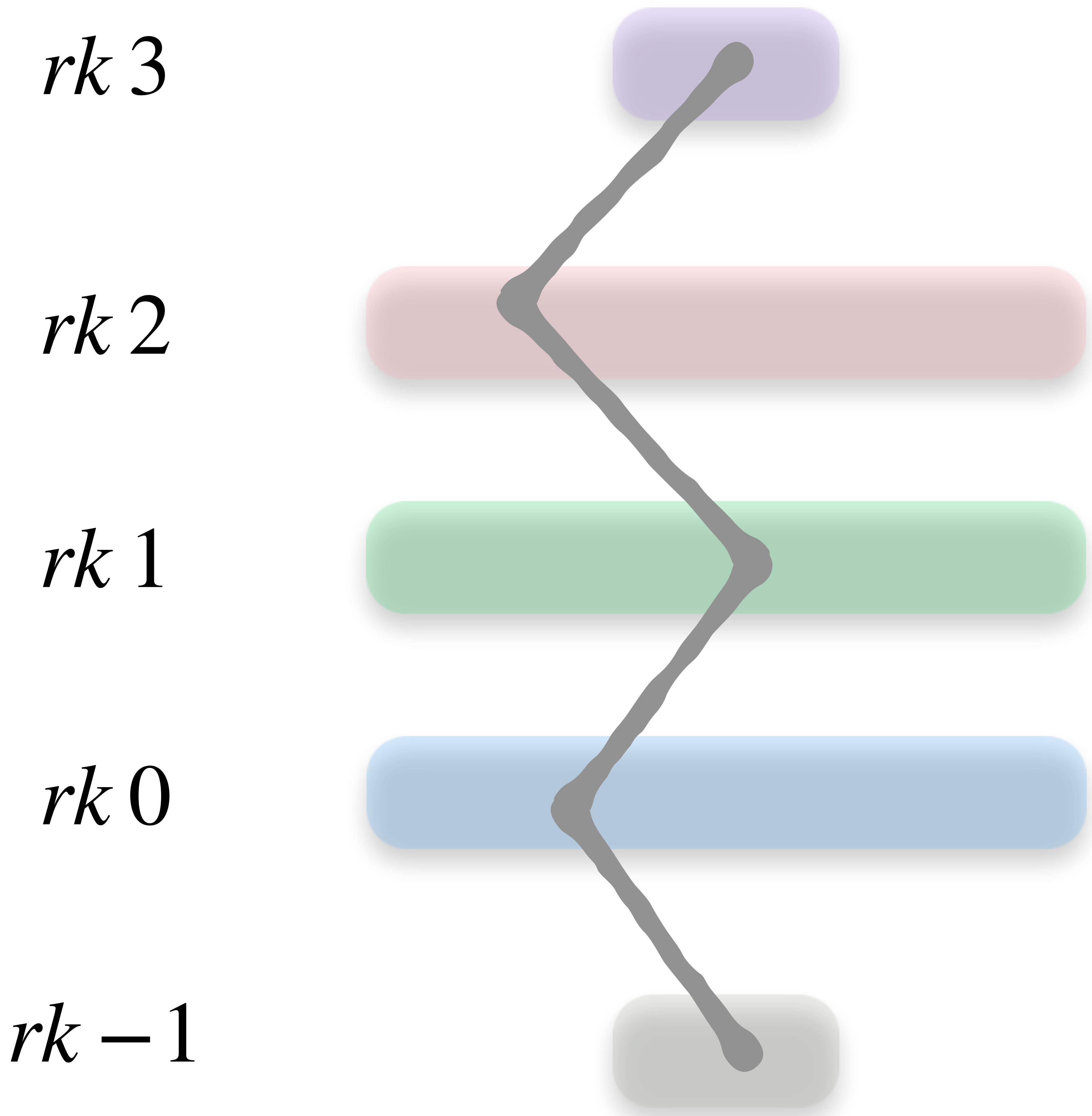
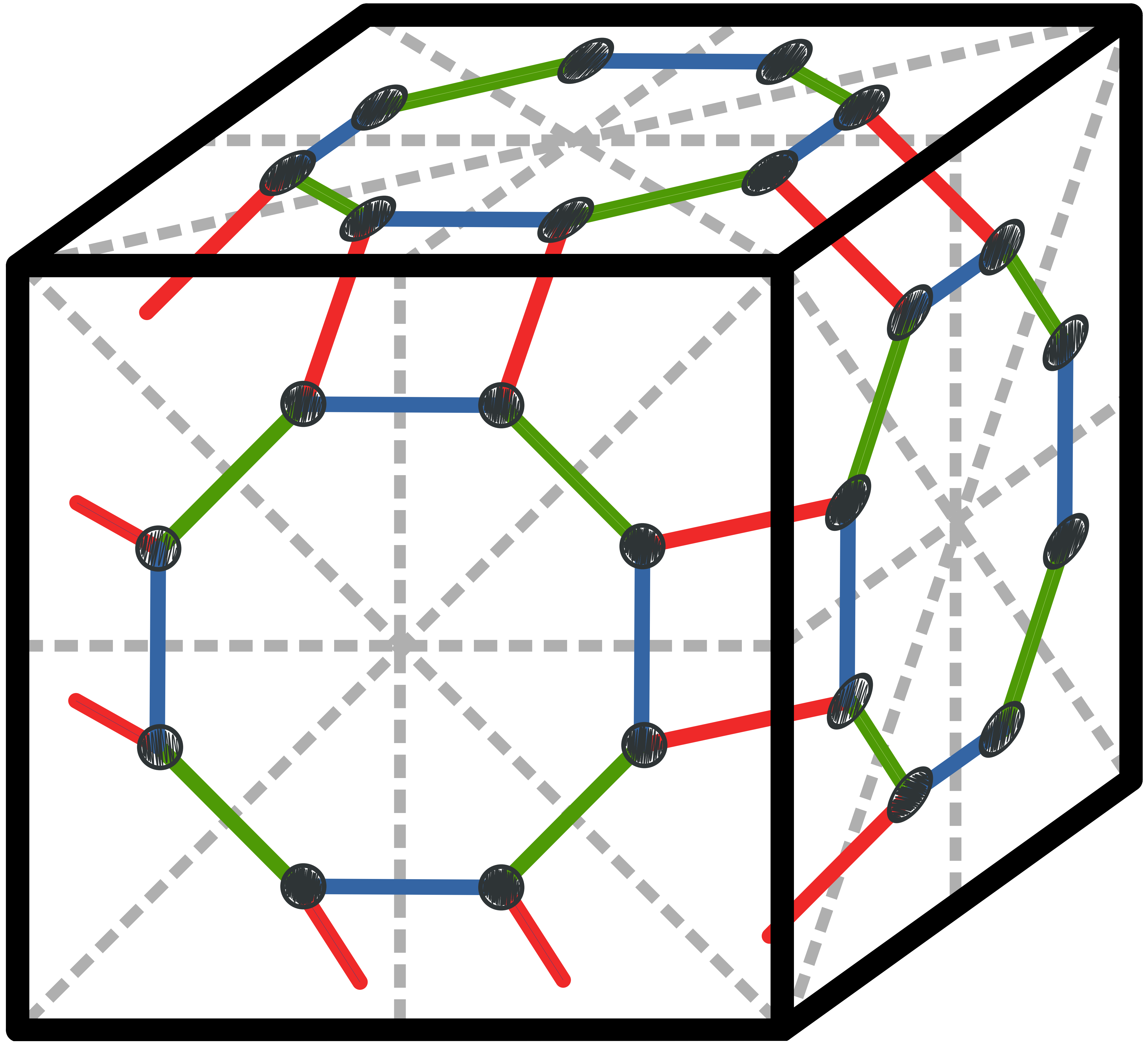




Polytopes vs maniplexes

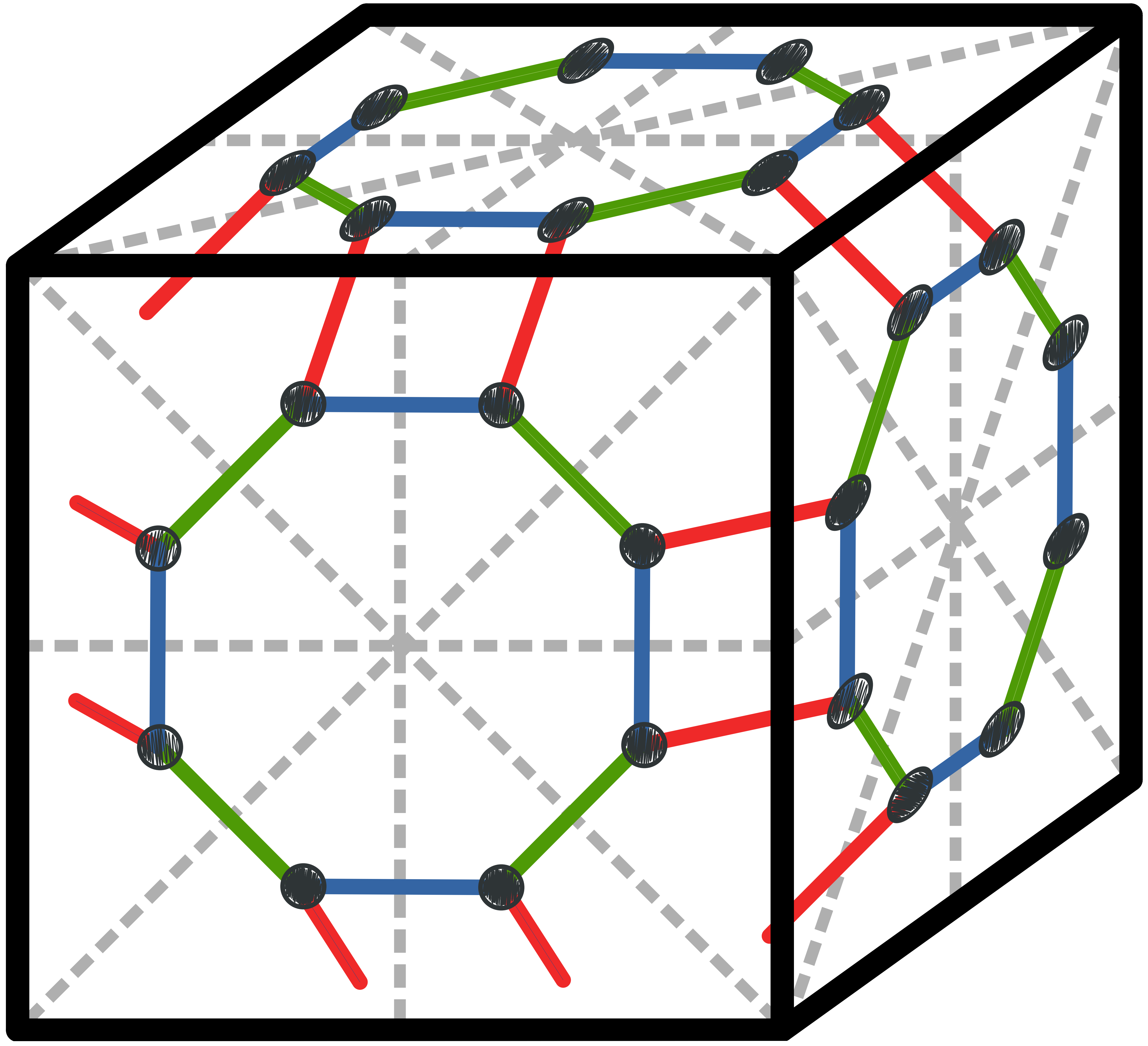
$rk\,3$

$rk\,2$

$rk\,1$

$rk\,0$

$rk\,-1$



Polytopes vs maniplexes

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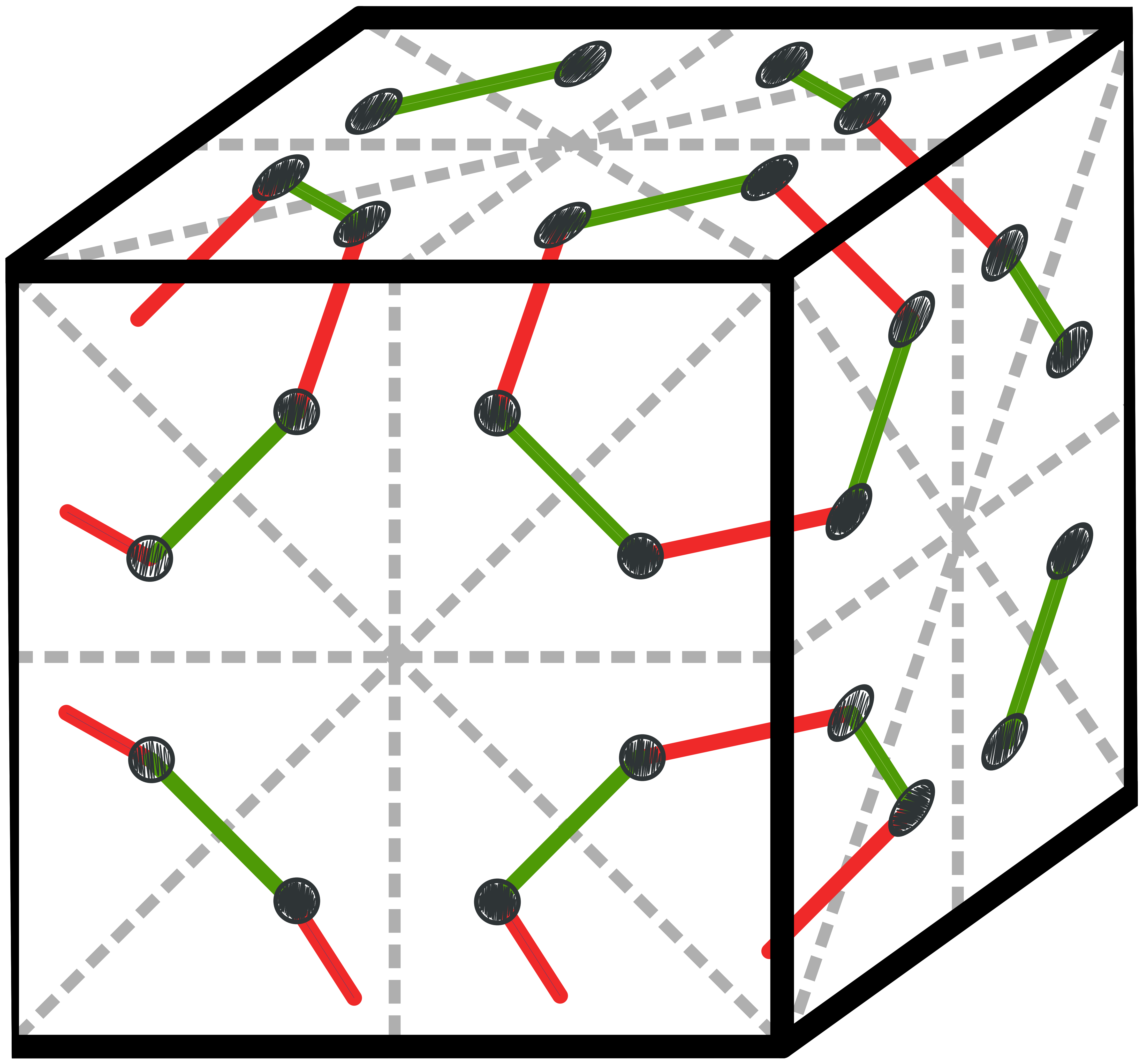
rk 2

rk 1

rk 0

rk −1



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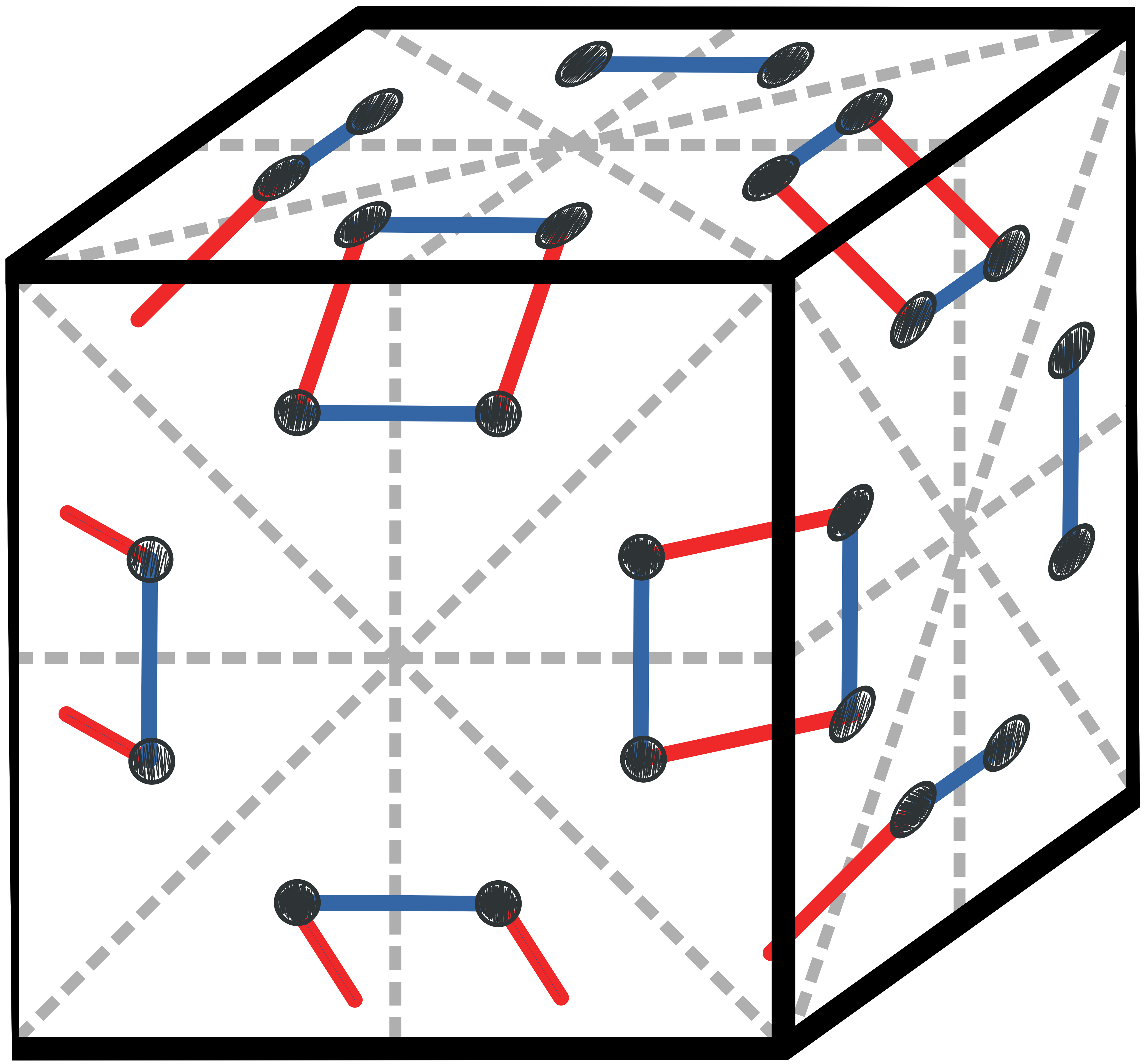

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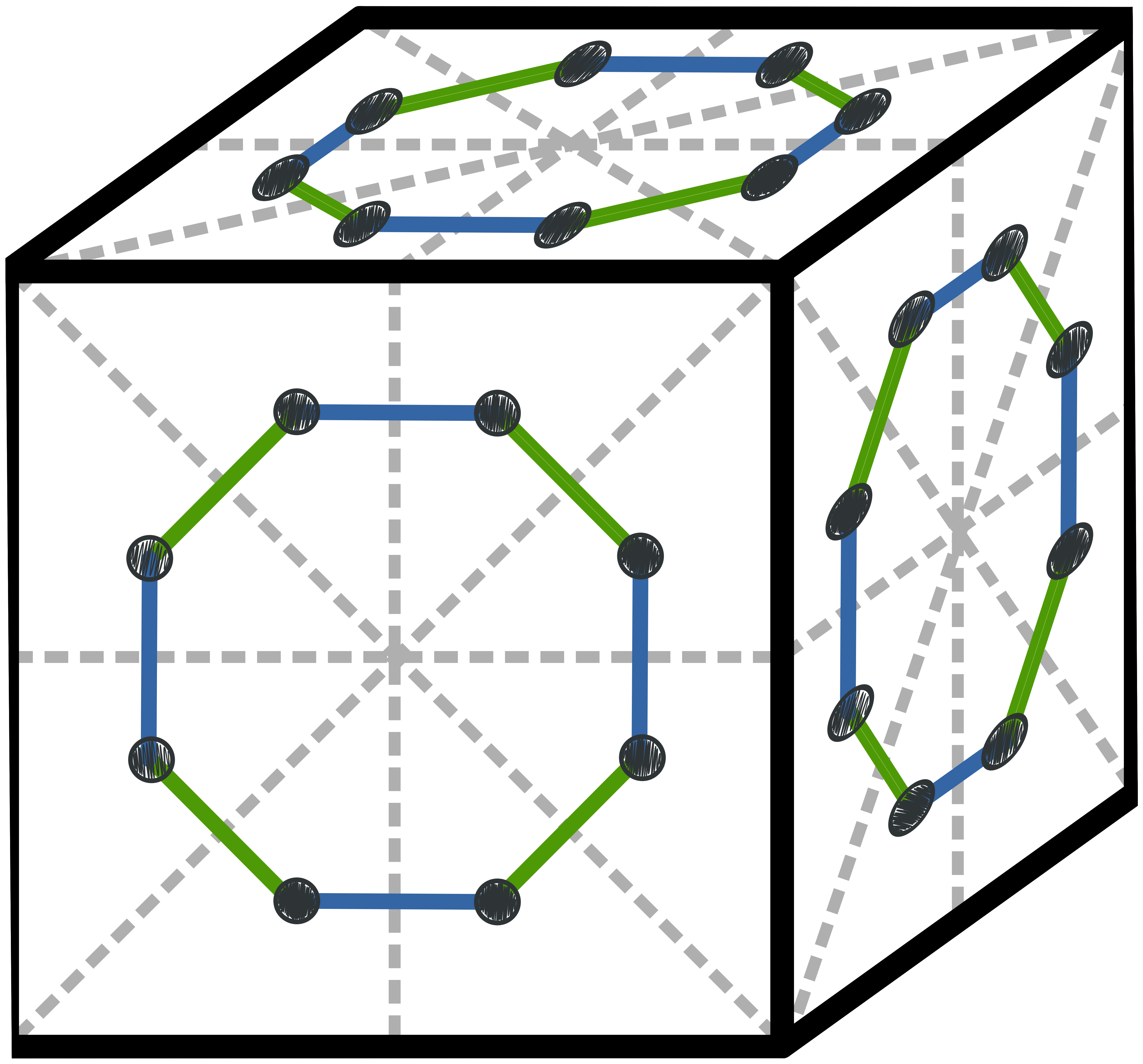


Polytopes vs maniplexes

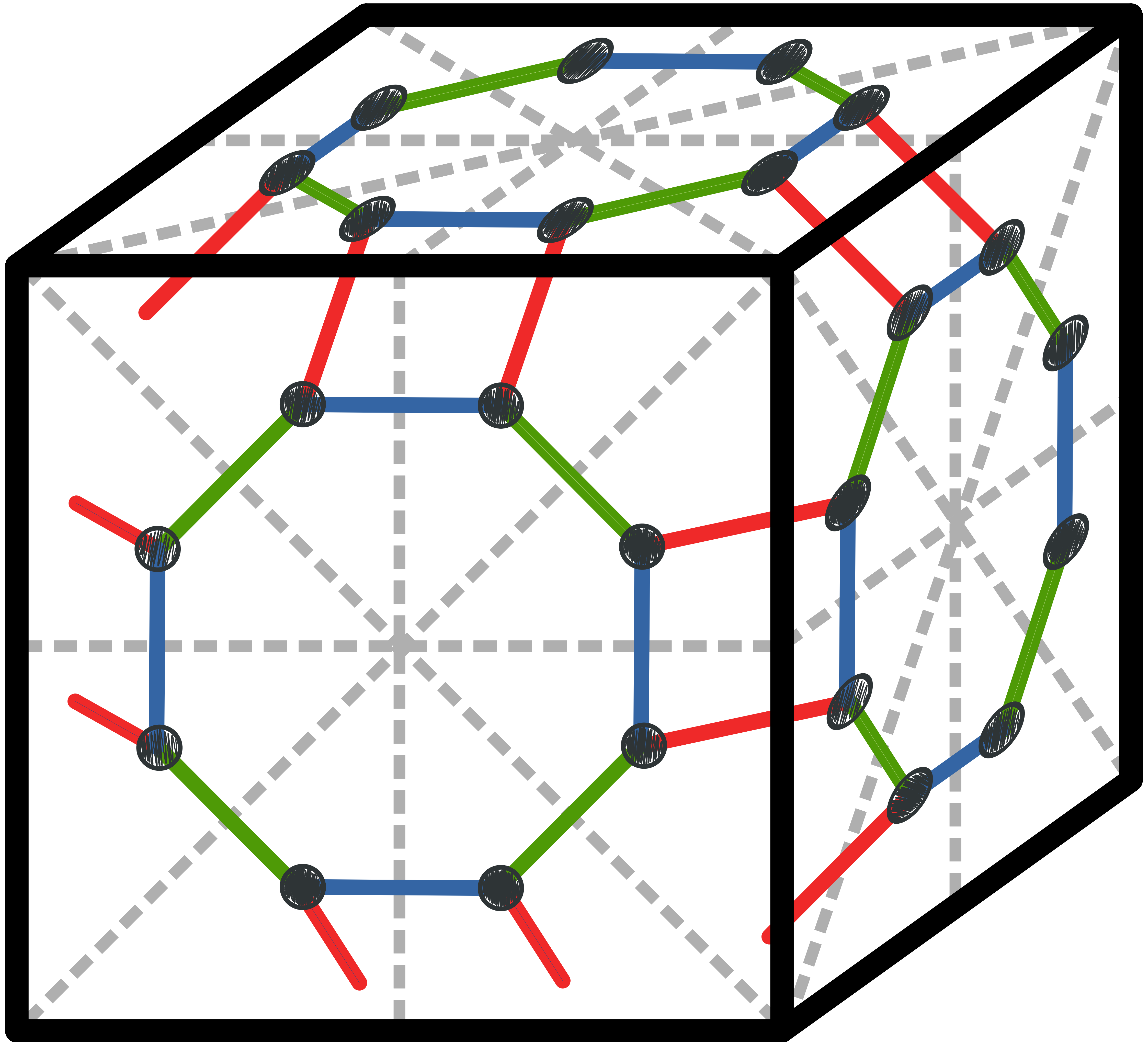
$rk\ 3$

$rk\ 2$

$rk\ 1$

$rk\ 0$

$rk\ {-1}$



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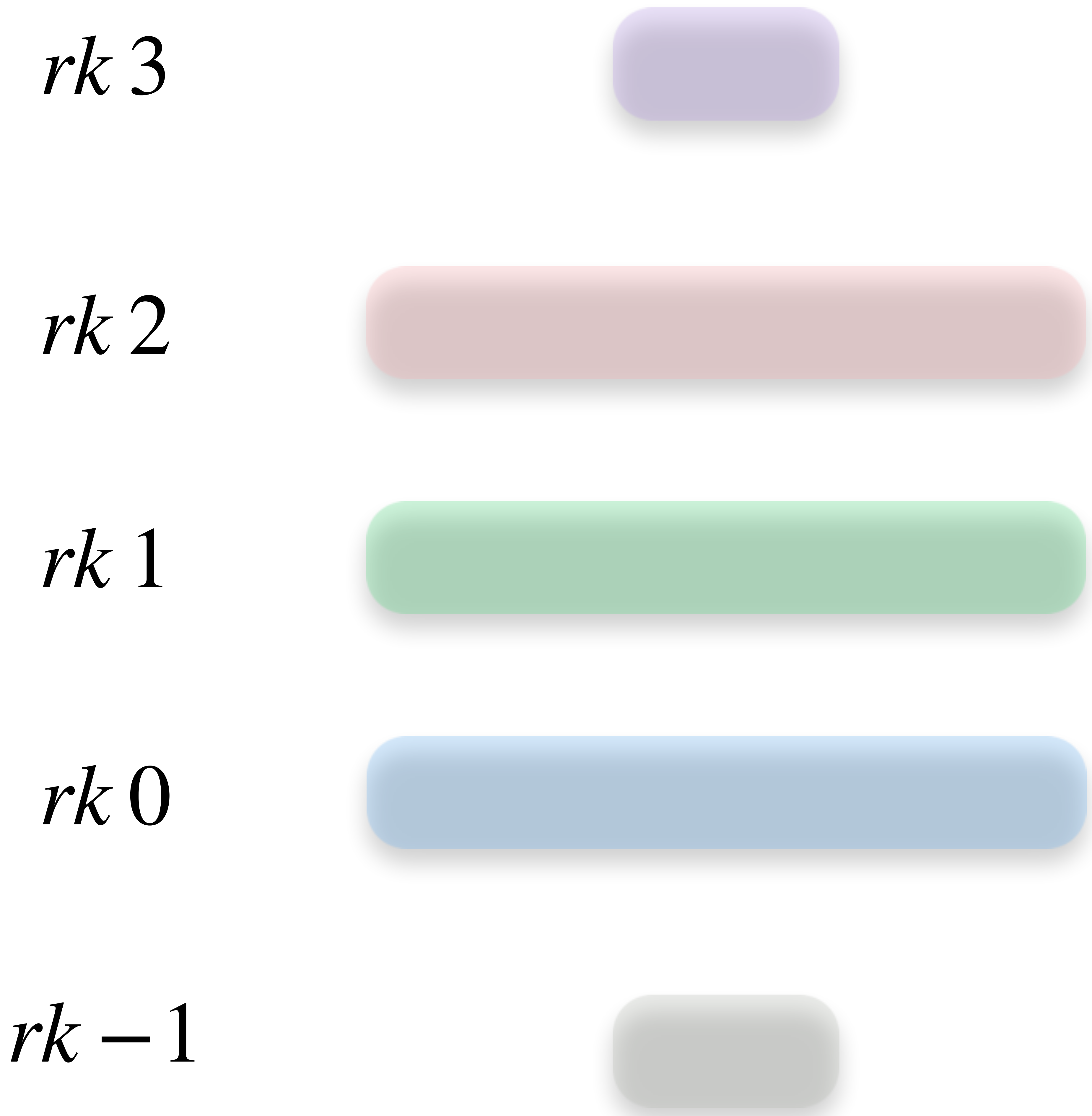

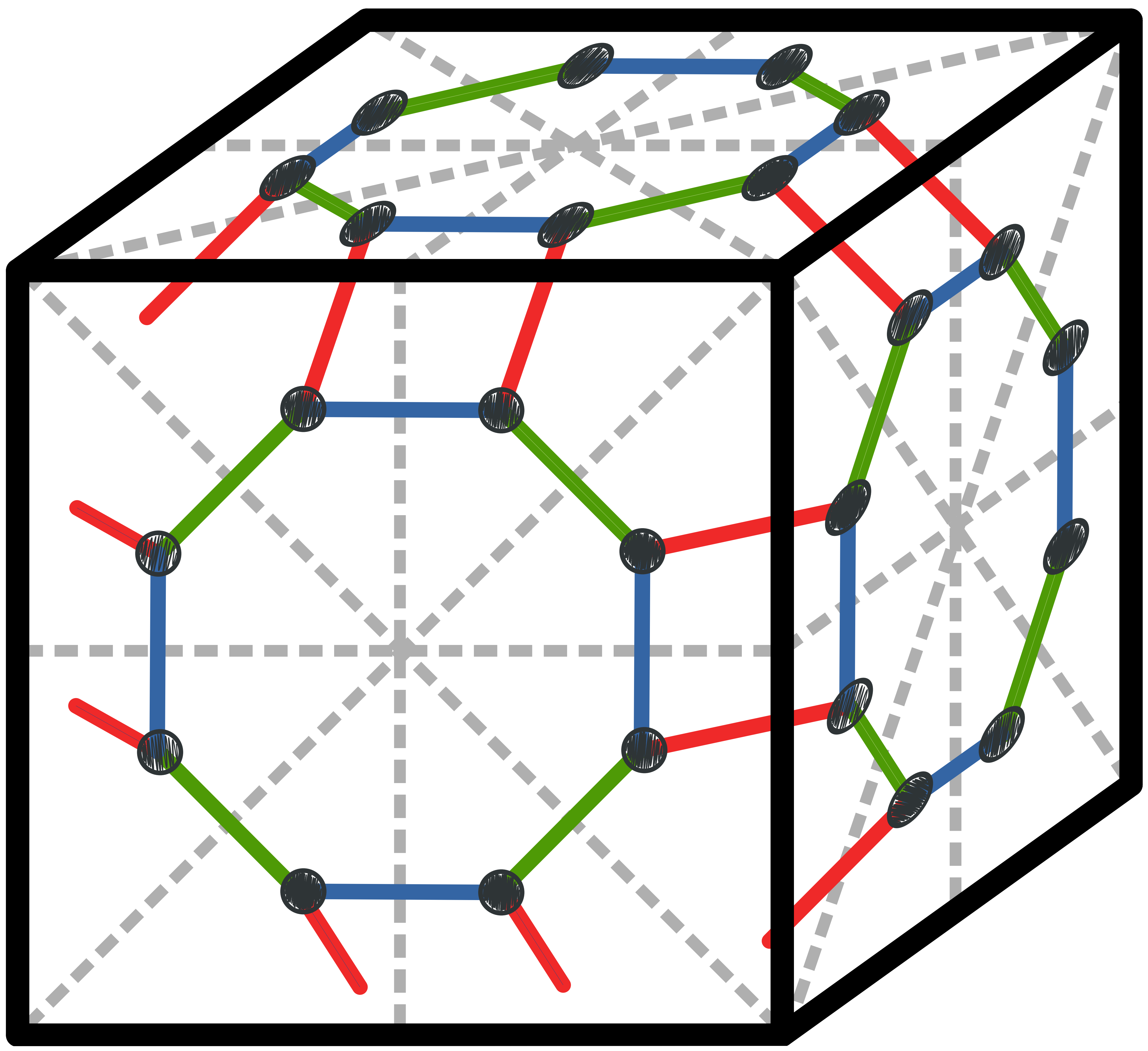

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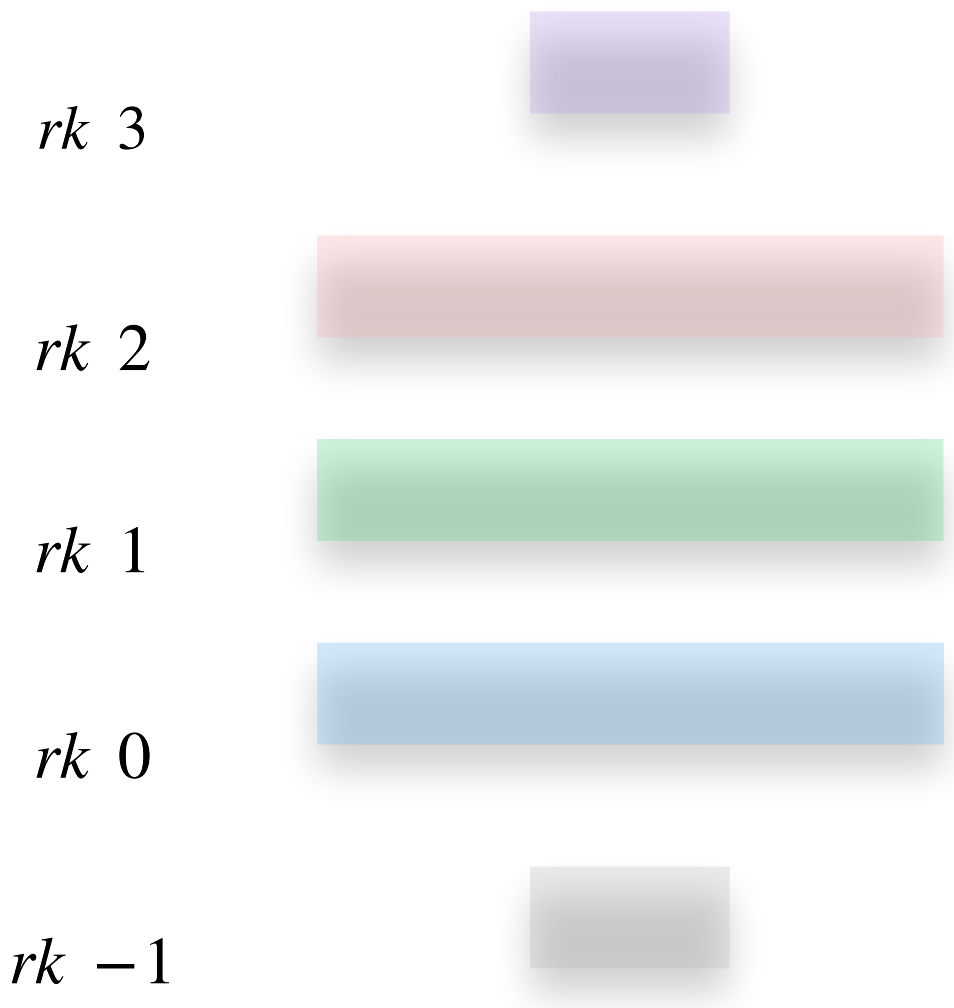
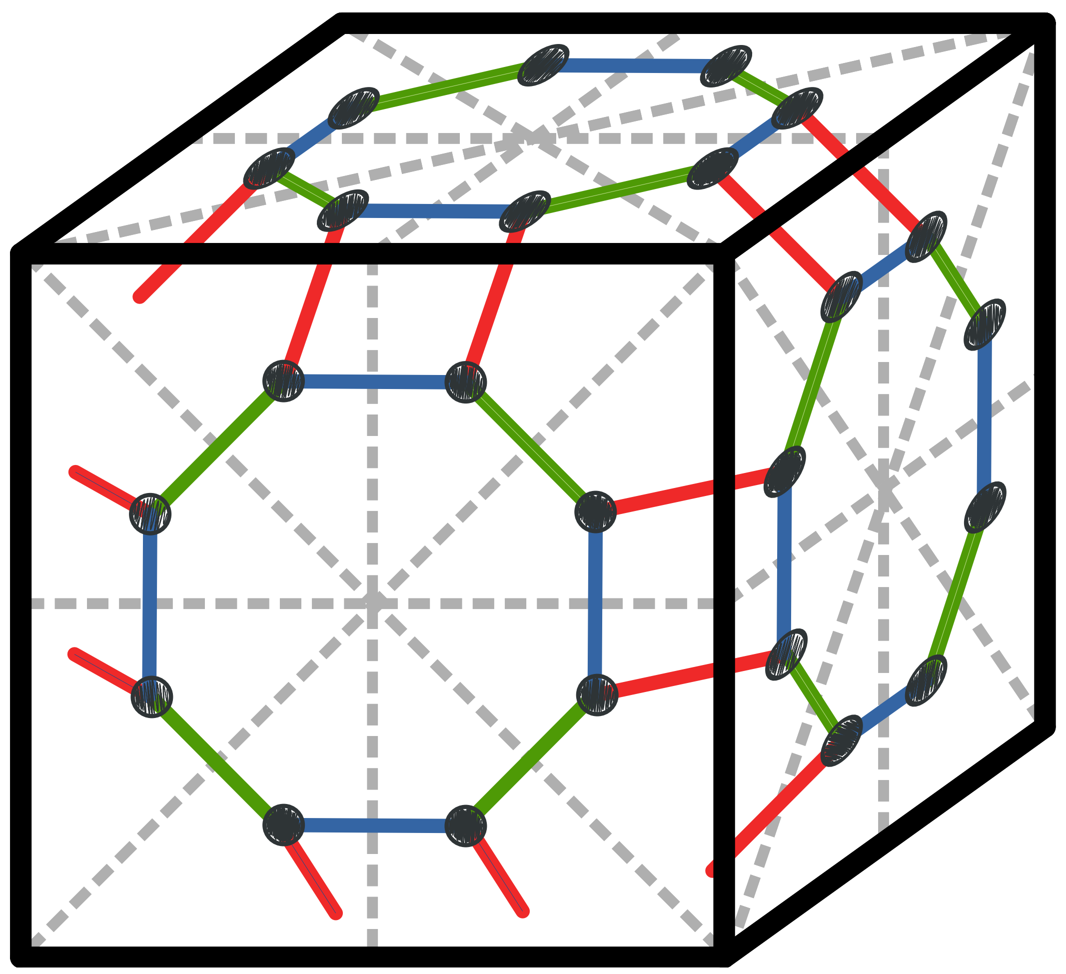
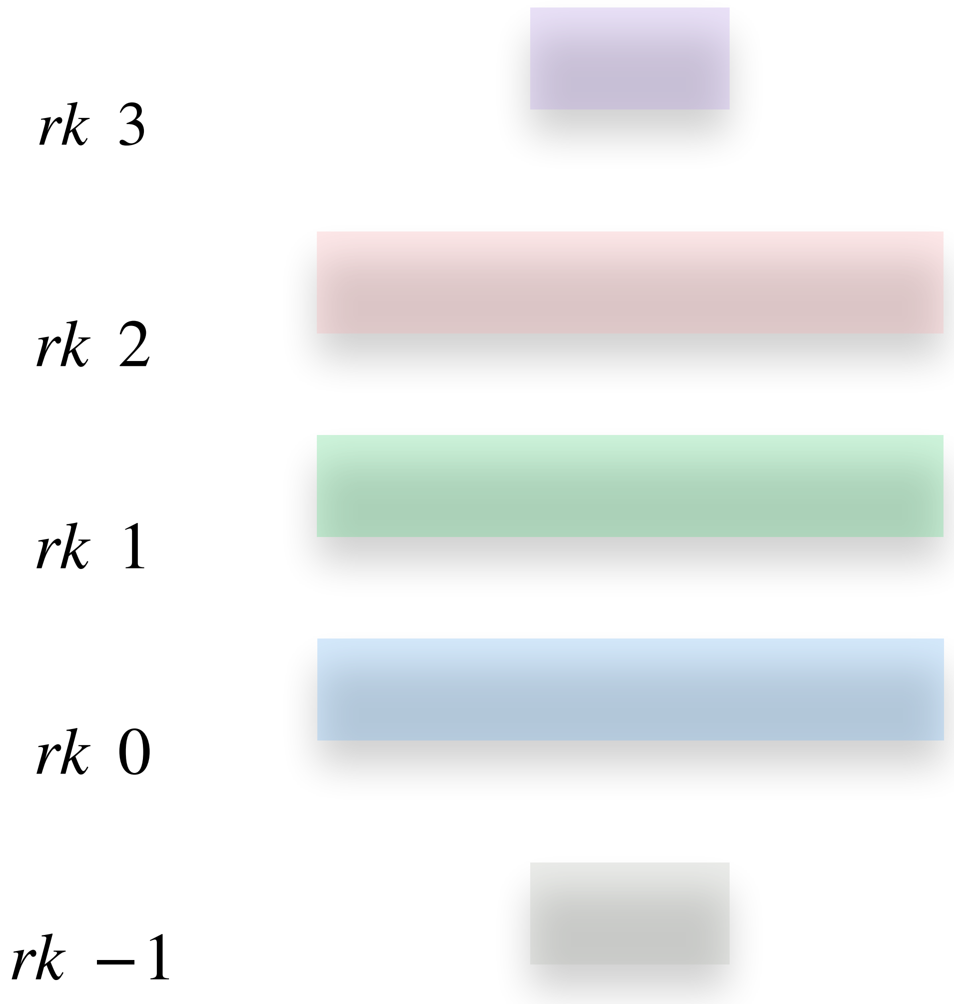
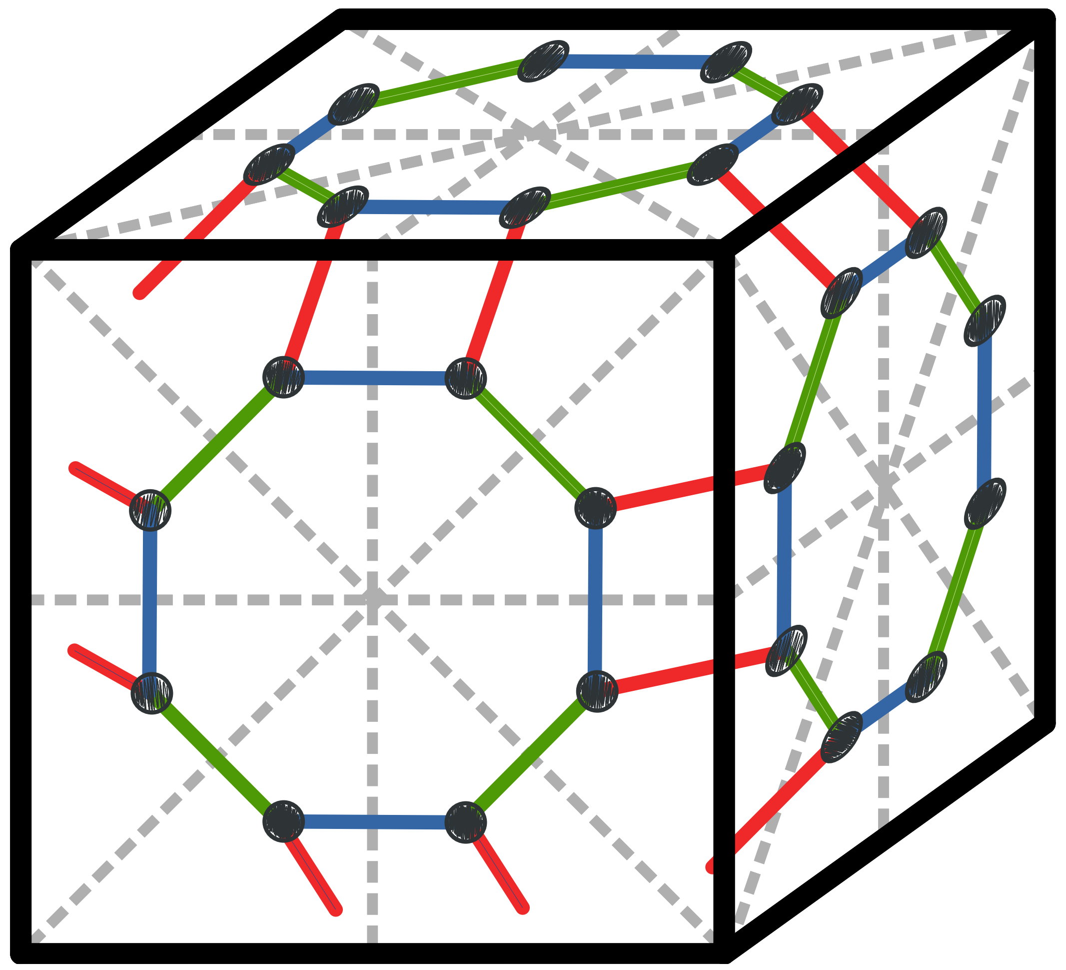




Polytopes vs maniplexes

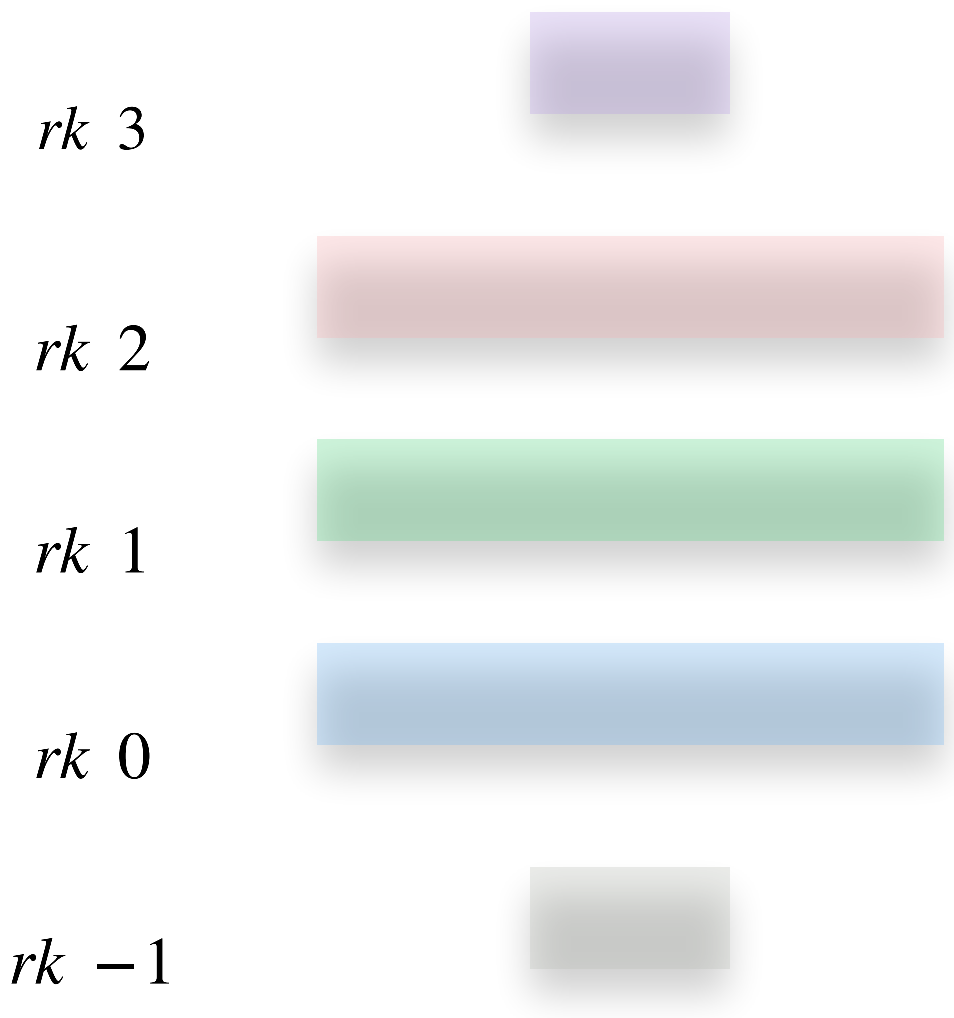
$\mathcal{P} \longrightarrow \mathrm{Fl}(\mathcal{P})$

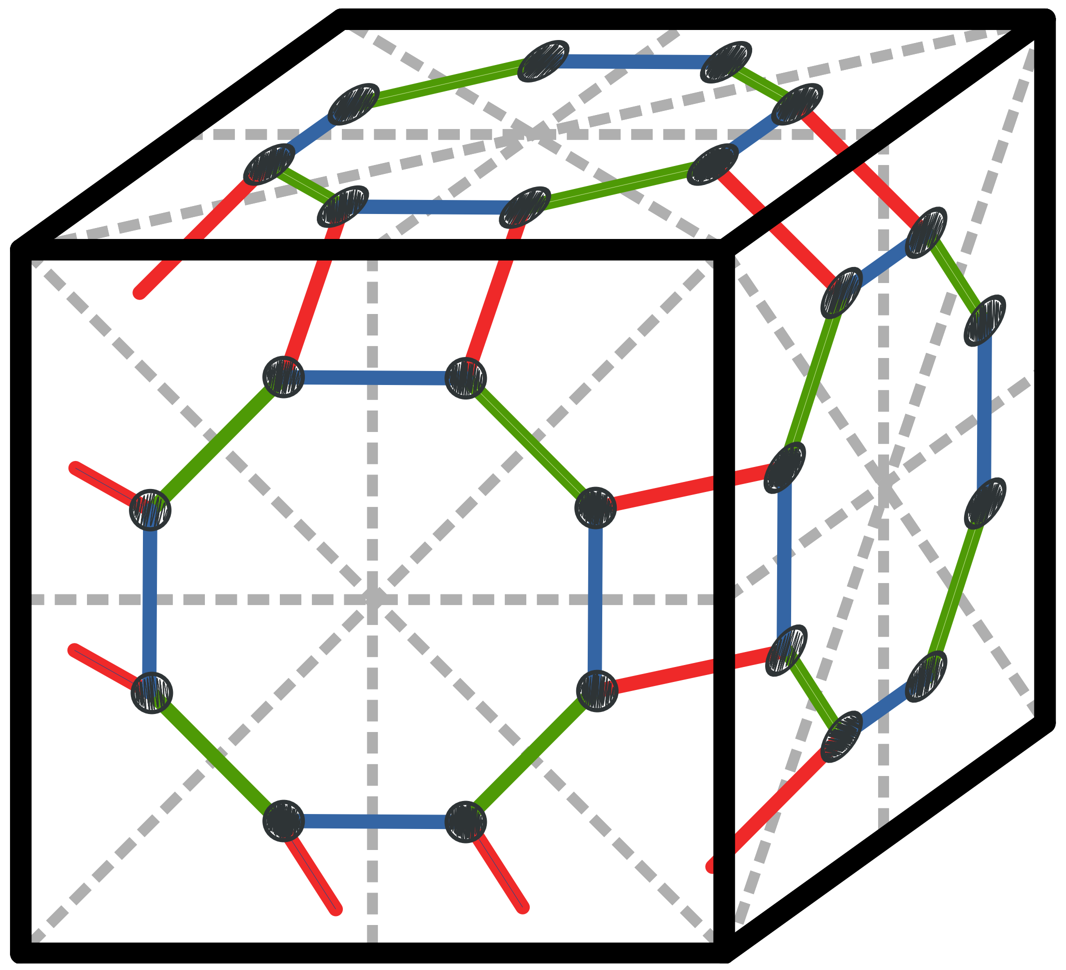




Polytopes vs maniplexes

$\mathcal{P}$ $\longrightarrow$ $\mathrm{Fl}(\mathcal{P})$



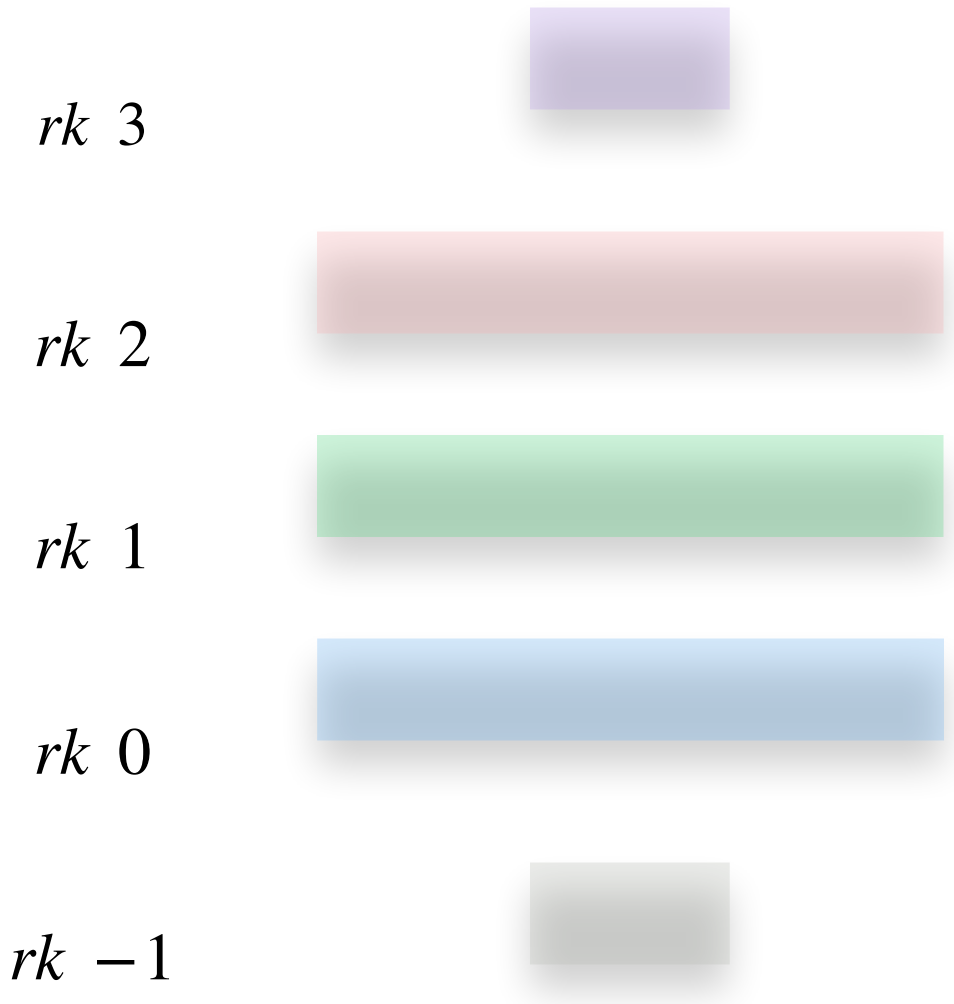


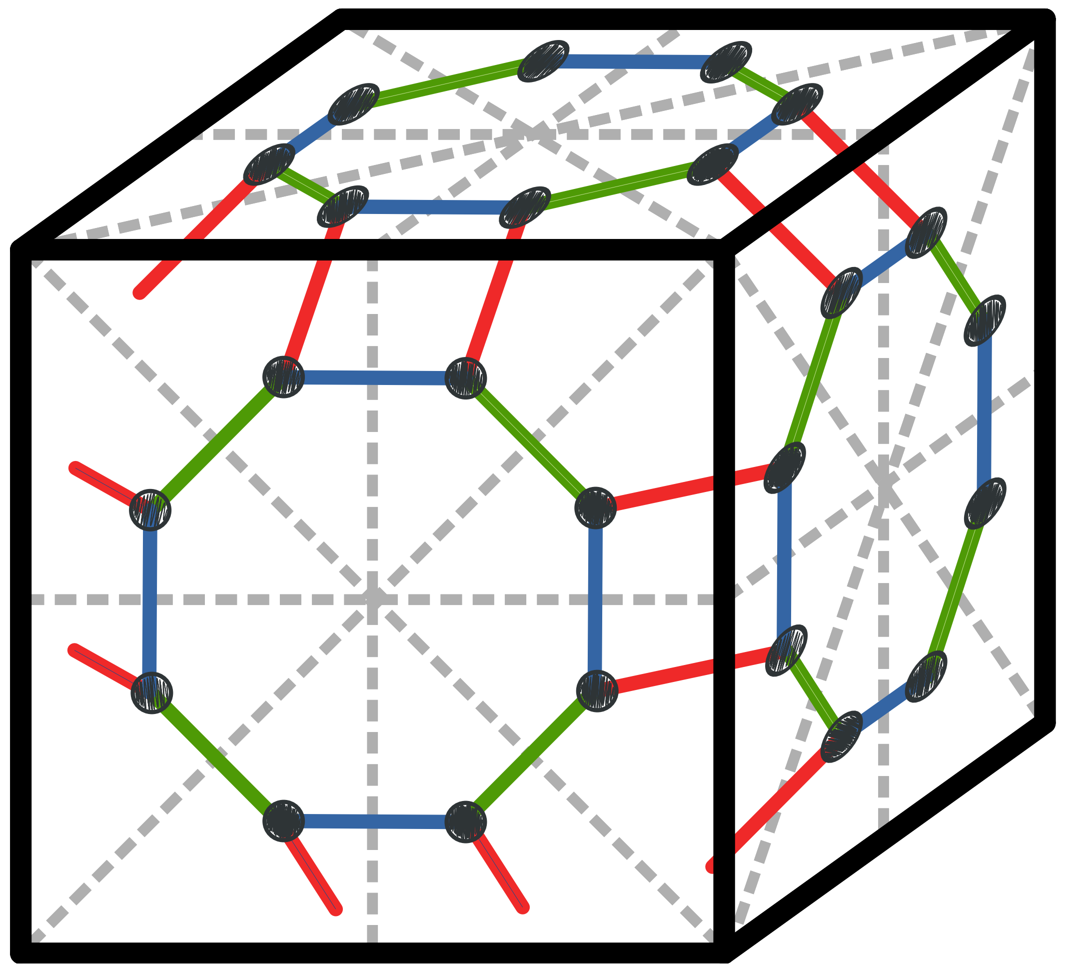
$\mathrm{Pos}(\mathcal{M})$ $\longleftarrow$ $\mathcal{M}$

Polytopes vs maniplexes

$\mathscr{P}$ → $\mathrm{Fl}(\mathscr{P})$

Always a Maniplex





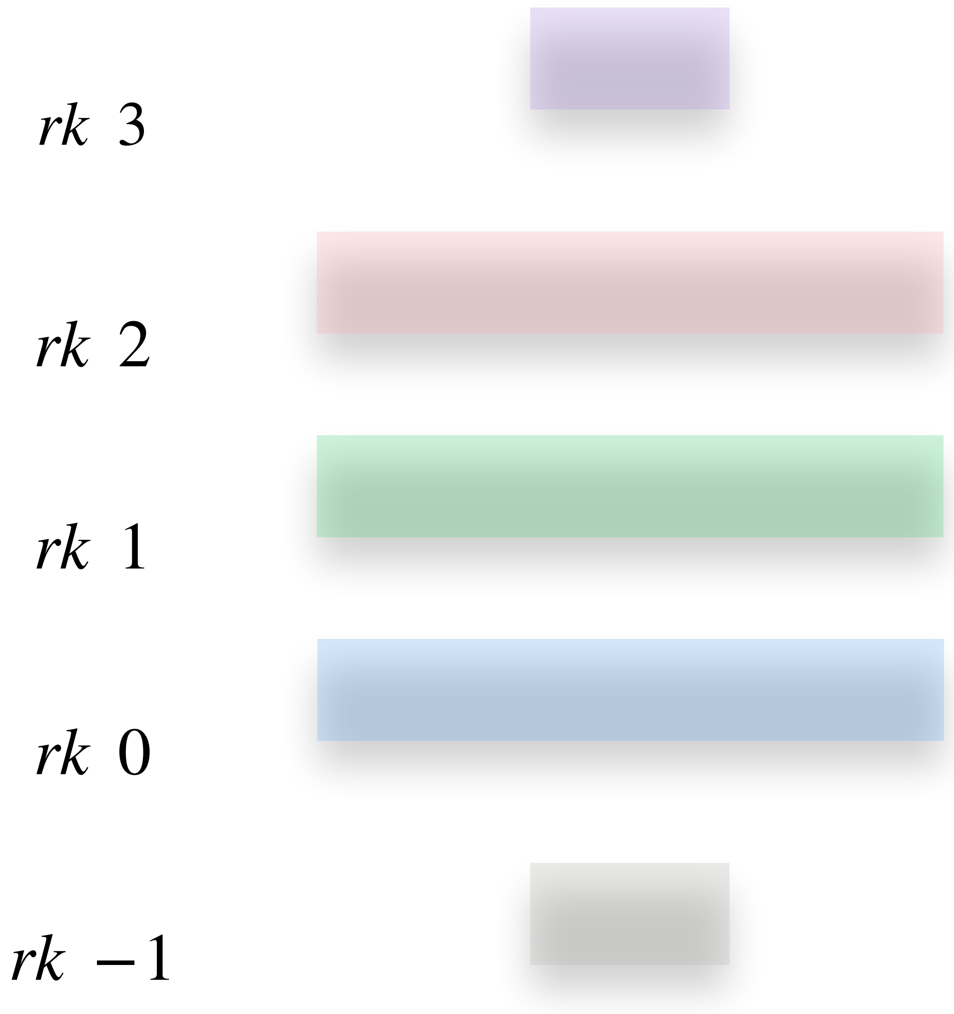
$\mathrm{Pos}(\mathscr{M})$ ← $\mathscr{M}$

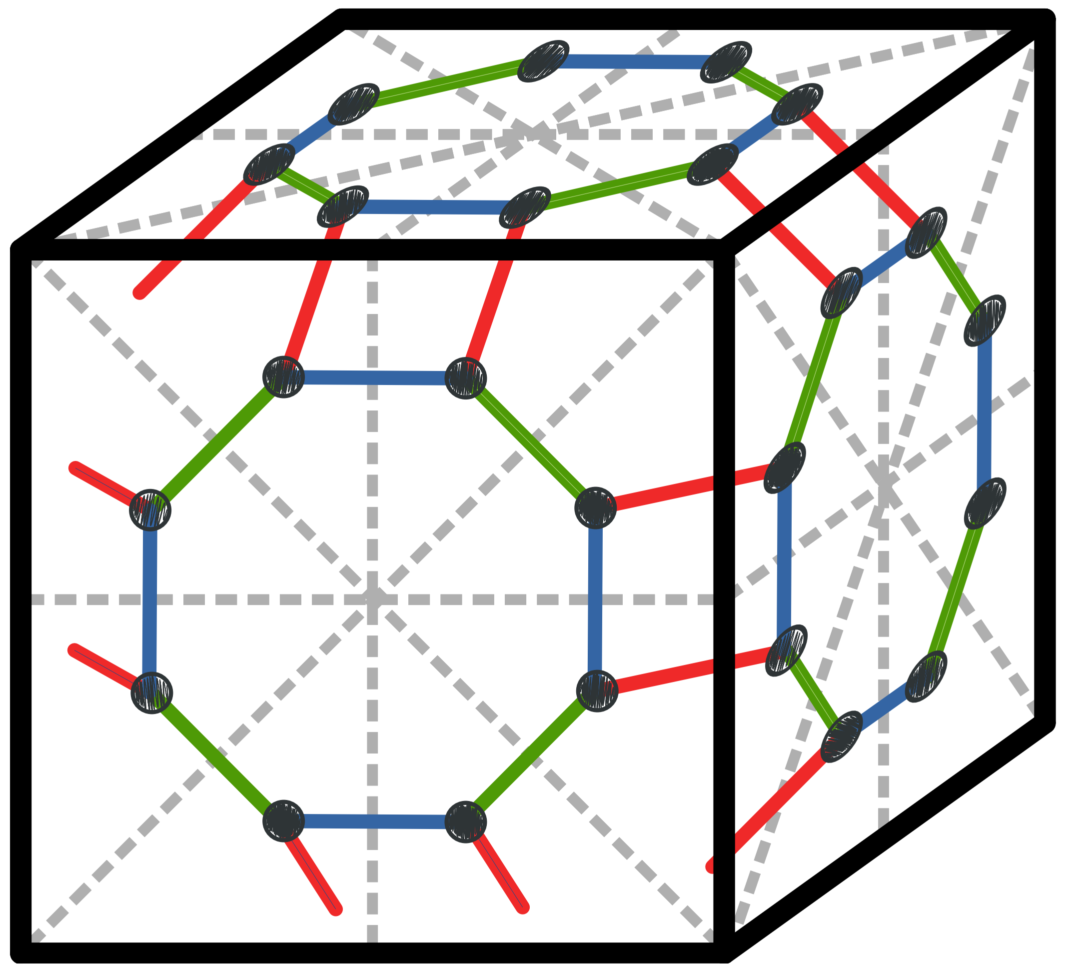
Polytopes vs maniplexes

Always a Maniplex

$\mathscr{P}$

$\mathrm{Fl}(\mathscr{P})$





$\mathrm{Pos}(\mathscr{M})$ ← $\mathscr{M}$

Polytopes vs maniplexes

$\mathcal{M}$ $\mathrm{Pos}(\mathcal{M})$

Polytopes vs maniplexes

$$\mathscr{M} \longrightarrow \mathrm{Pos}(\mathscr{M})$$

Polytopes vs maniplexes

$$\mathscr{M} \longrightarrow \mathrm{Pos}(\mathscr{M})$$

Why?

Polytopes vs maniplexes

$$\mathcal{M} \longrightarrow \mathrm{Pos}(\mathcal{M})$$

Why?

- Being a polytope is very nice, but very restrictive.

Polytopes vs maniplexes

$$\mathcal{M} \longrightarrow \mathrm{Pos}(\mathcal{M})$$

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- Classical operations on polytopes often do no yield polytopes.

Polytopes vs maniplexes

$$\mathscr{M} \longrightarrow \mathrm{Pos}(\mathscr{M})$$

Why?

- Being a polytope is very nice, but very restrictive.
- Classical operations on polytopes often do no yield polytopes.
  - $\delta(\mathscr{P})$ (dual) ✅

Polytopes vs maniplexes

$\mathcal{M} \longrightarrow \mathrm{Pos}(\mathcal{M})$

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Polytopes vs maniplexes

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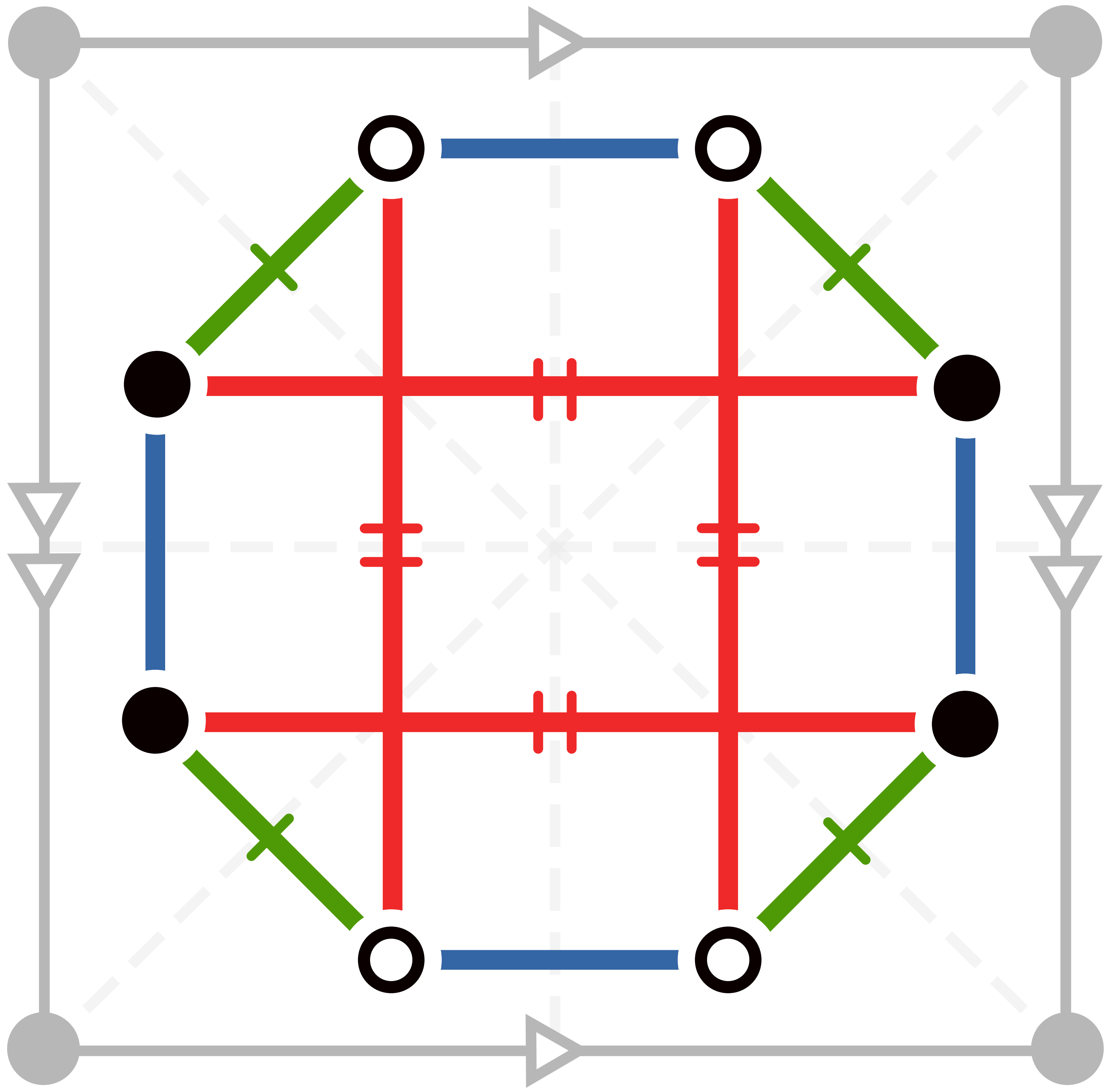

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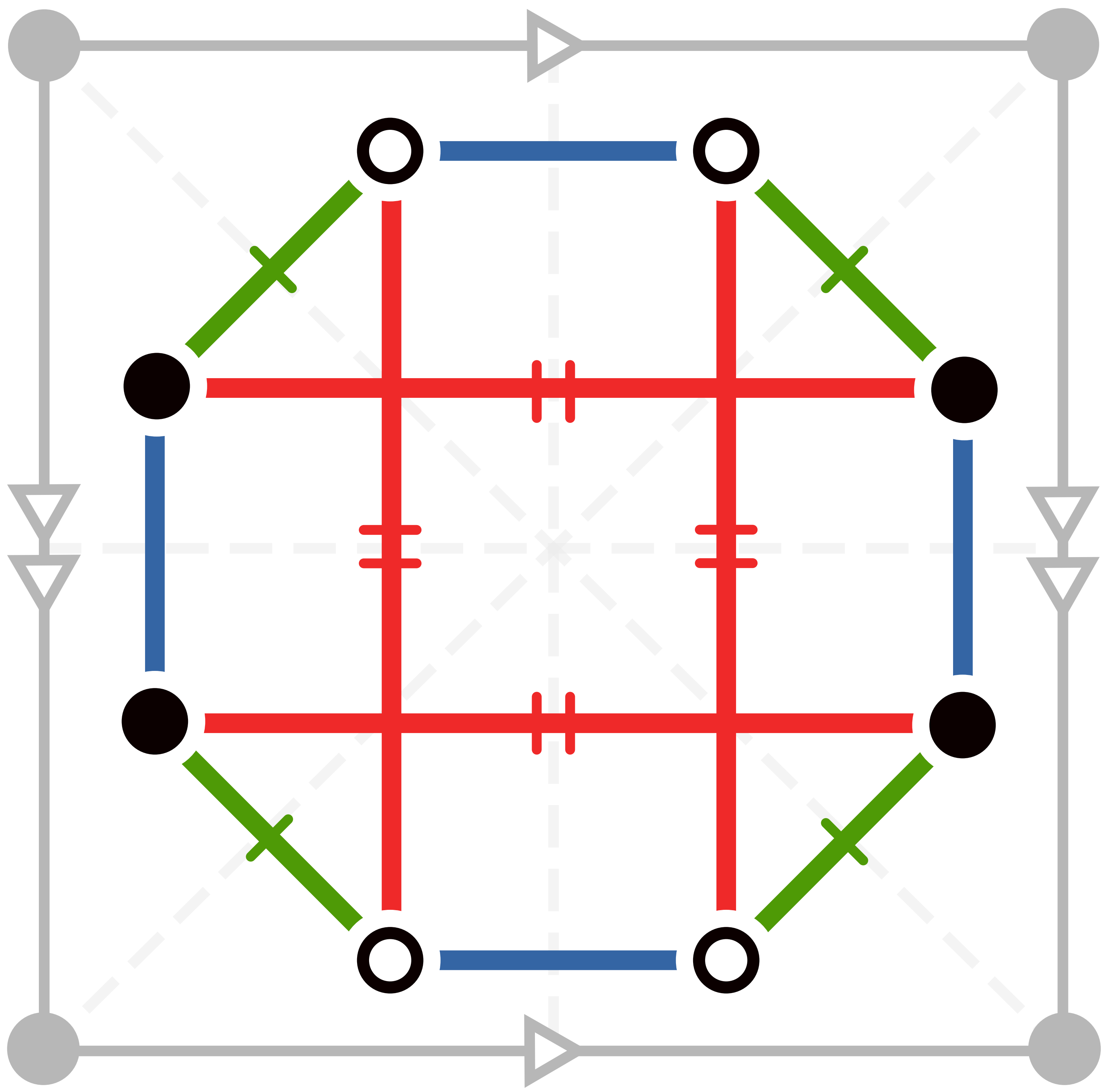

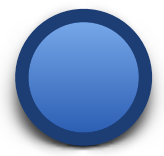

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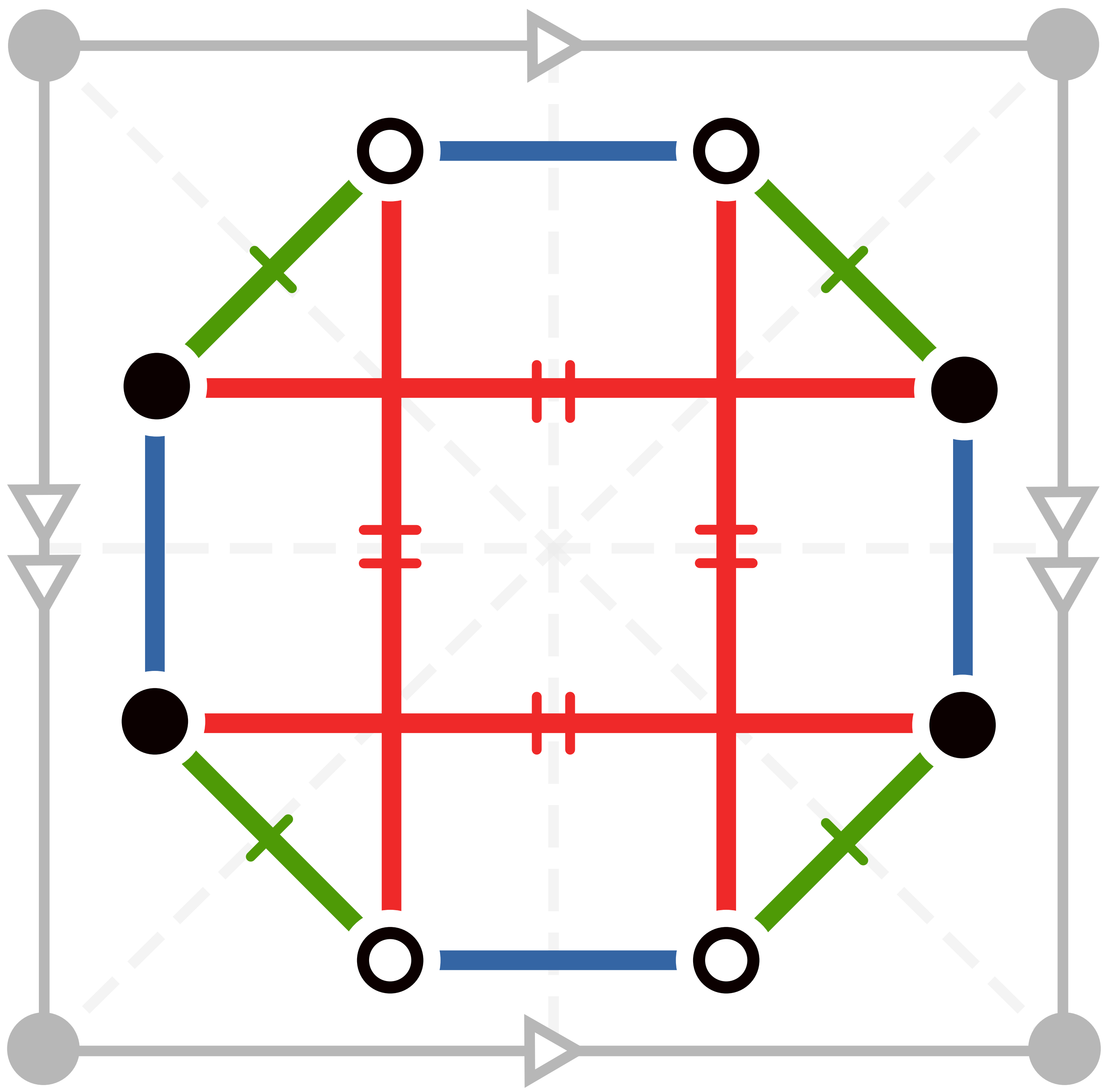

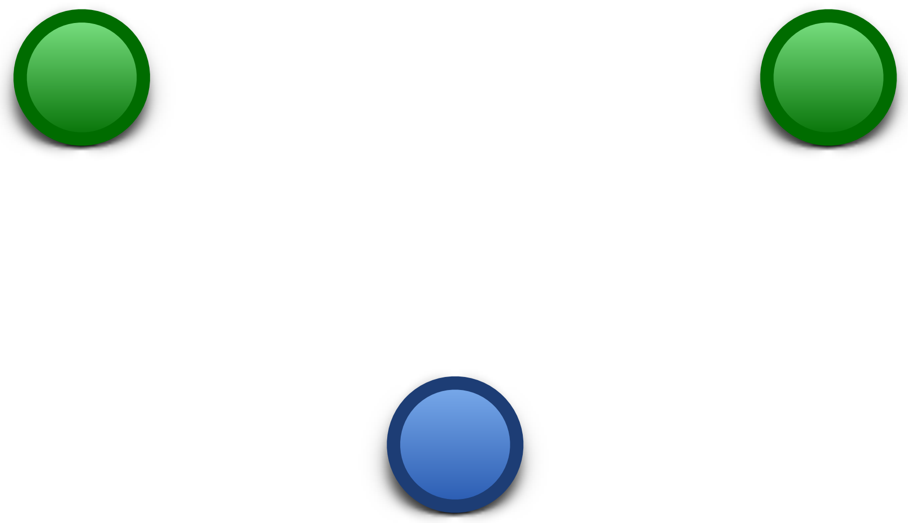

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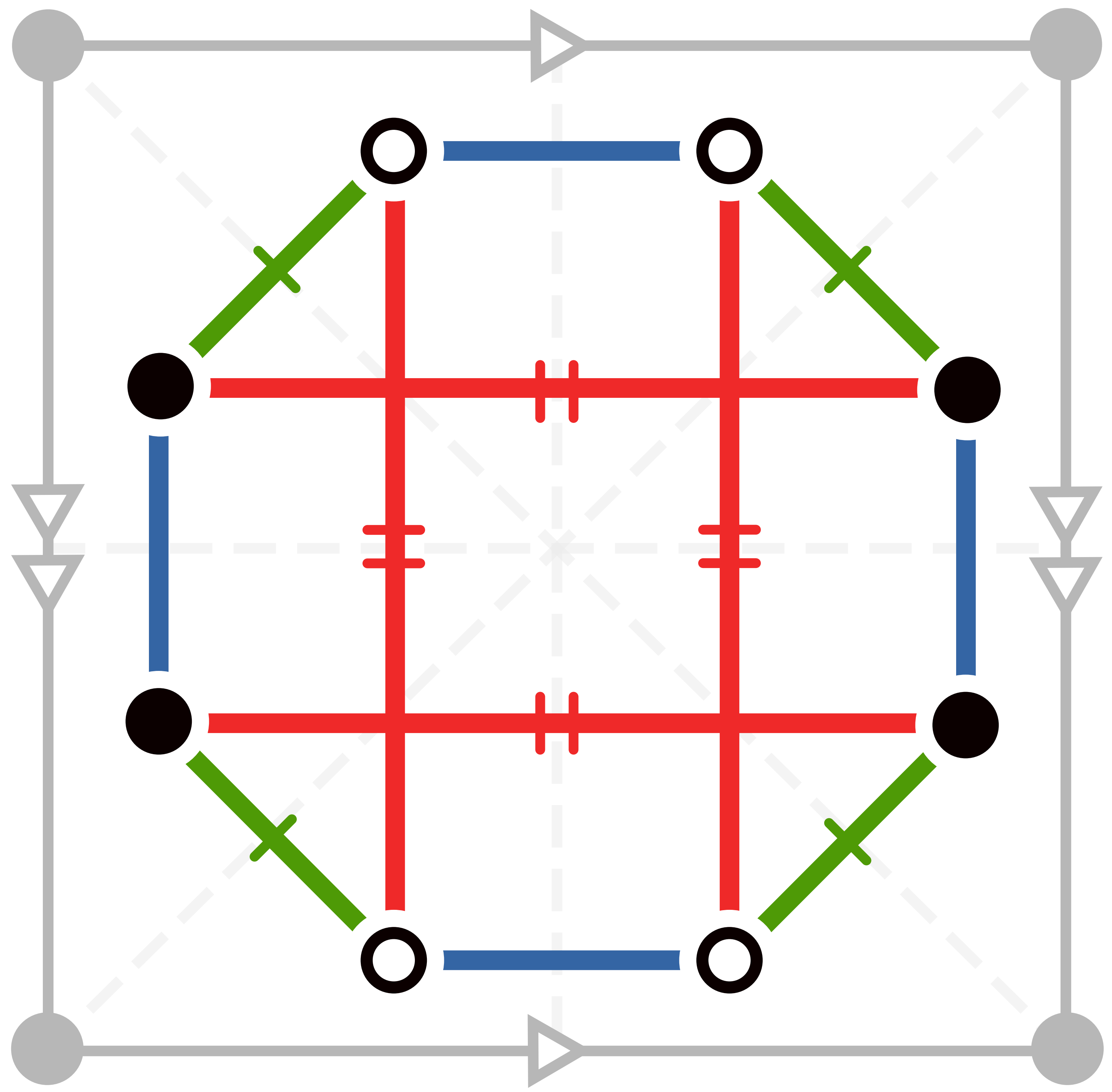

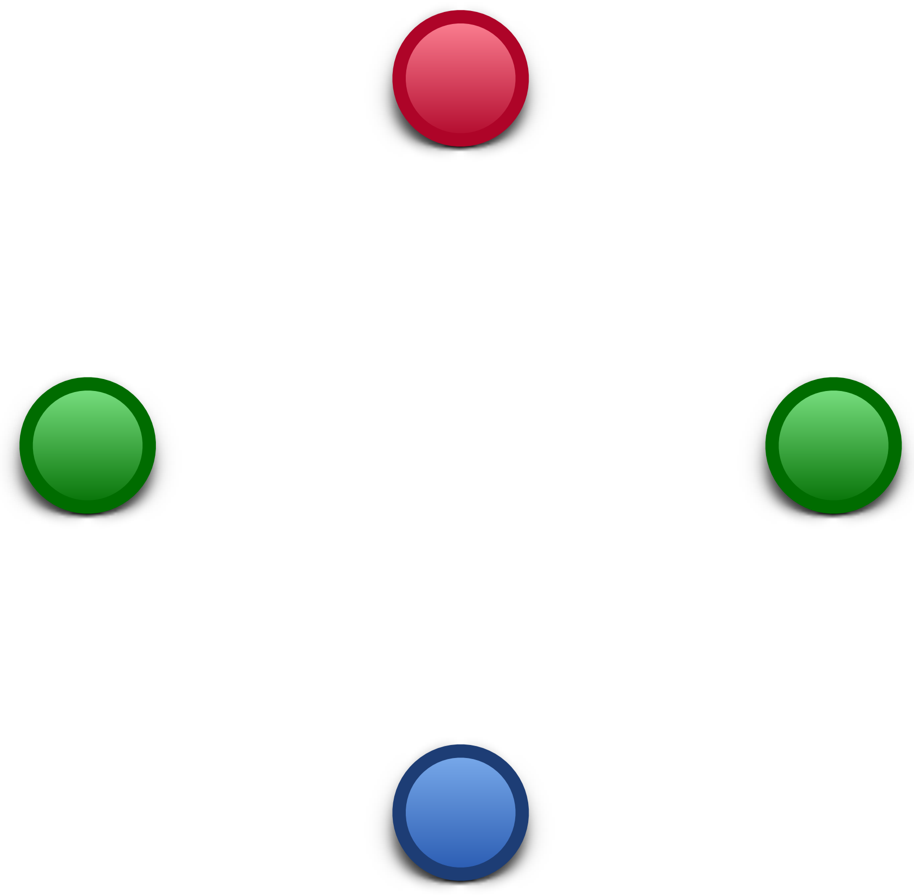

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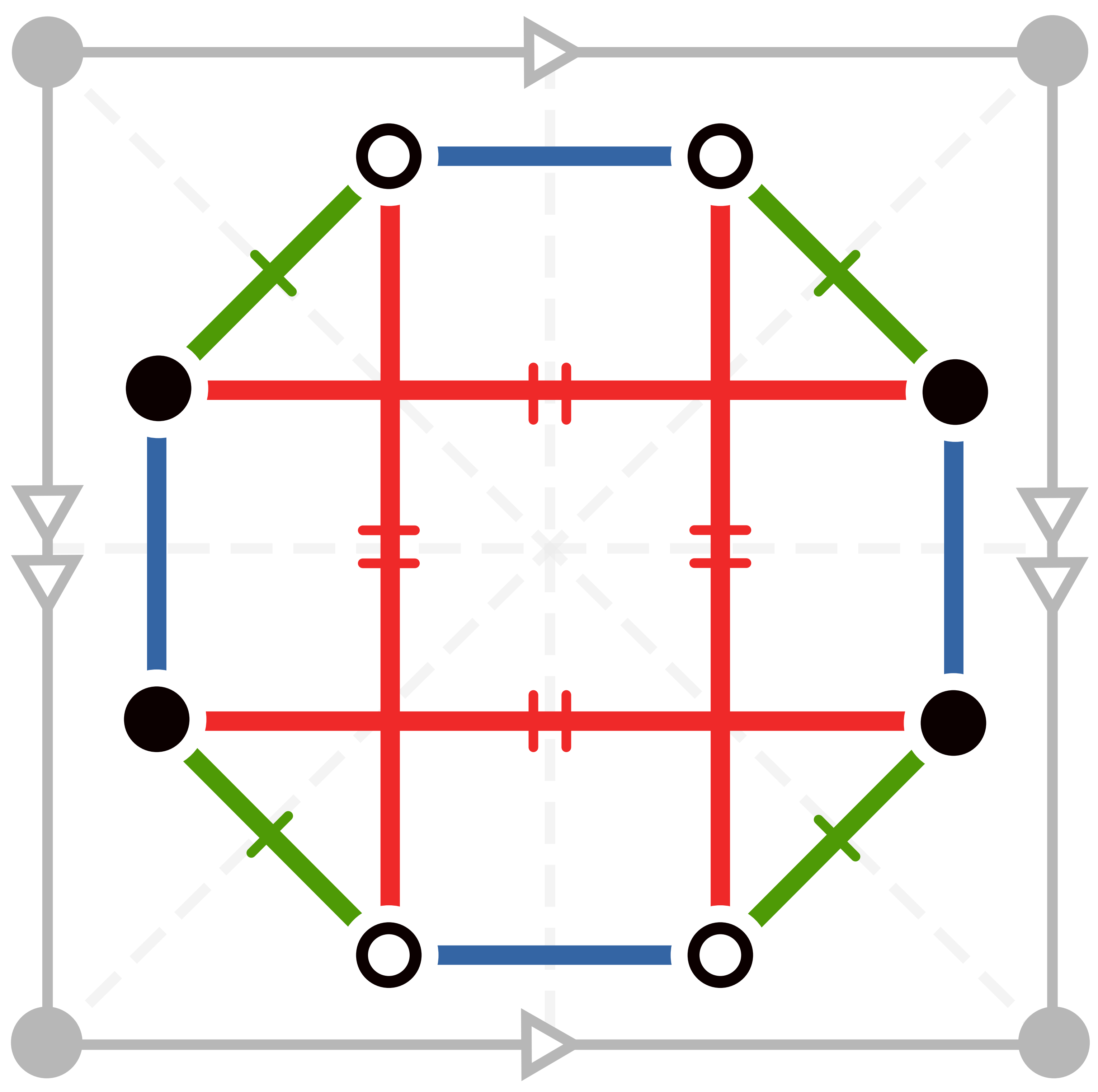

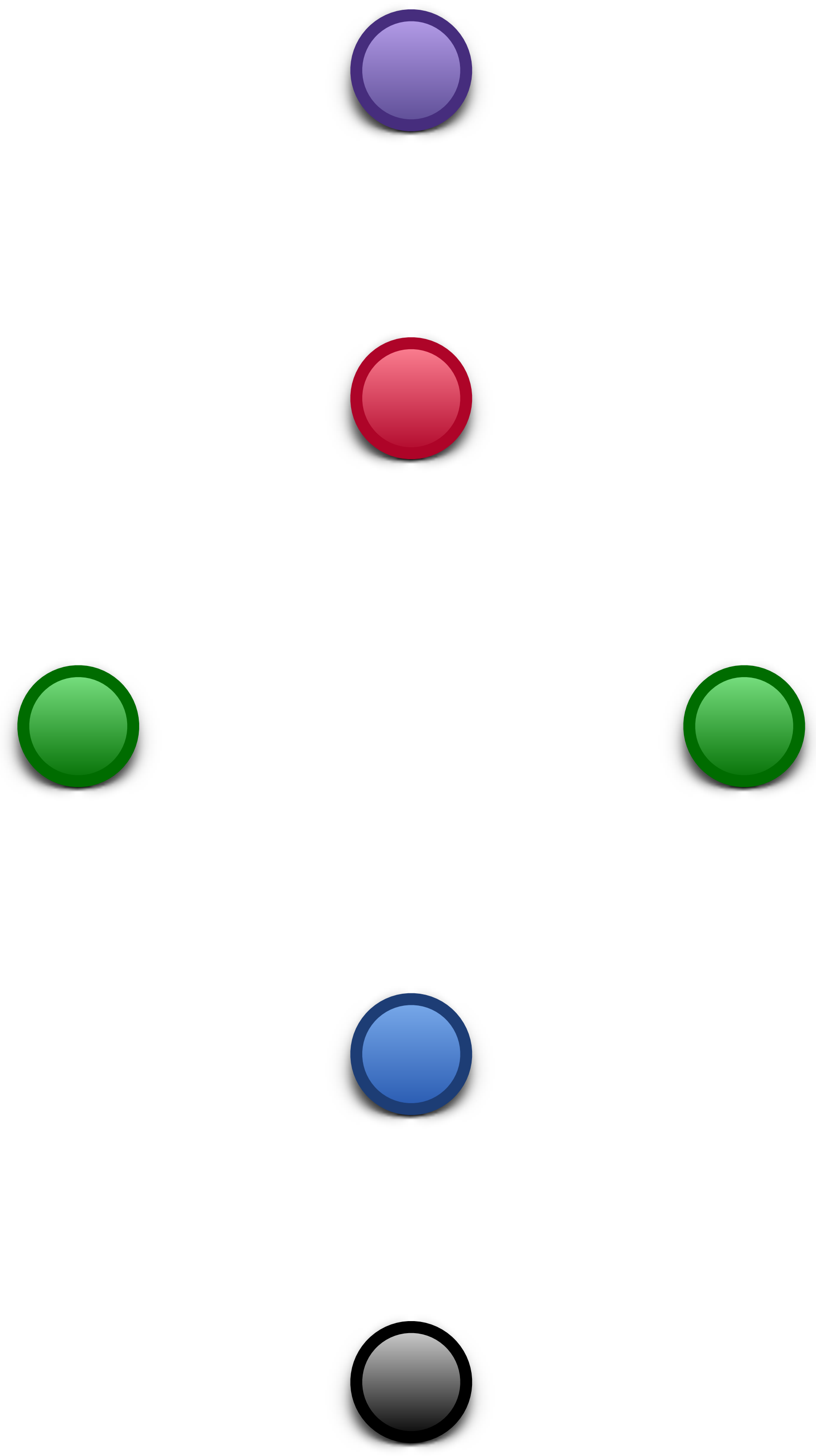

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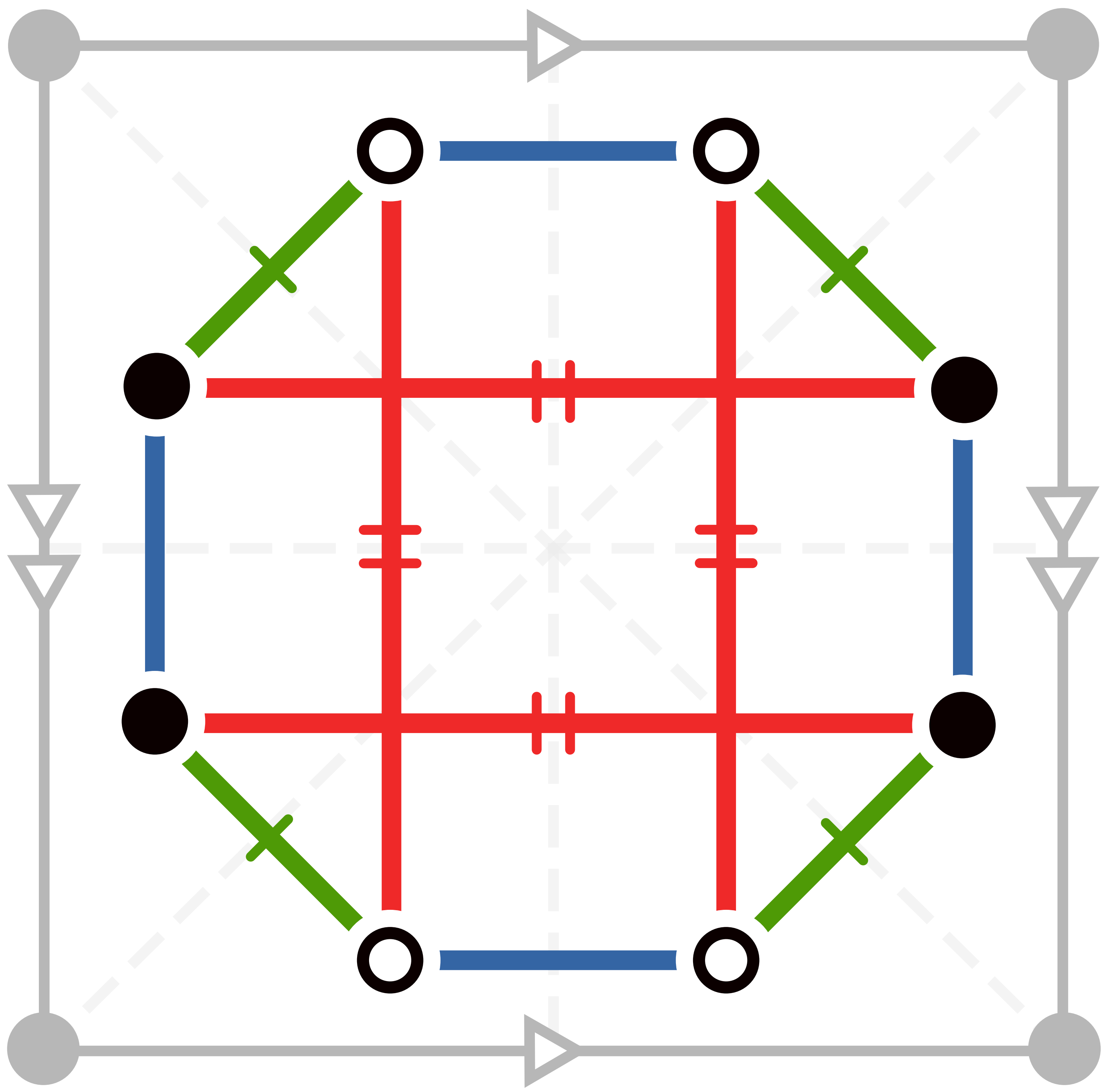

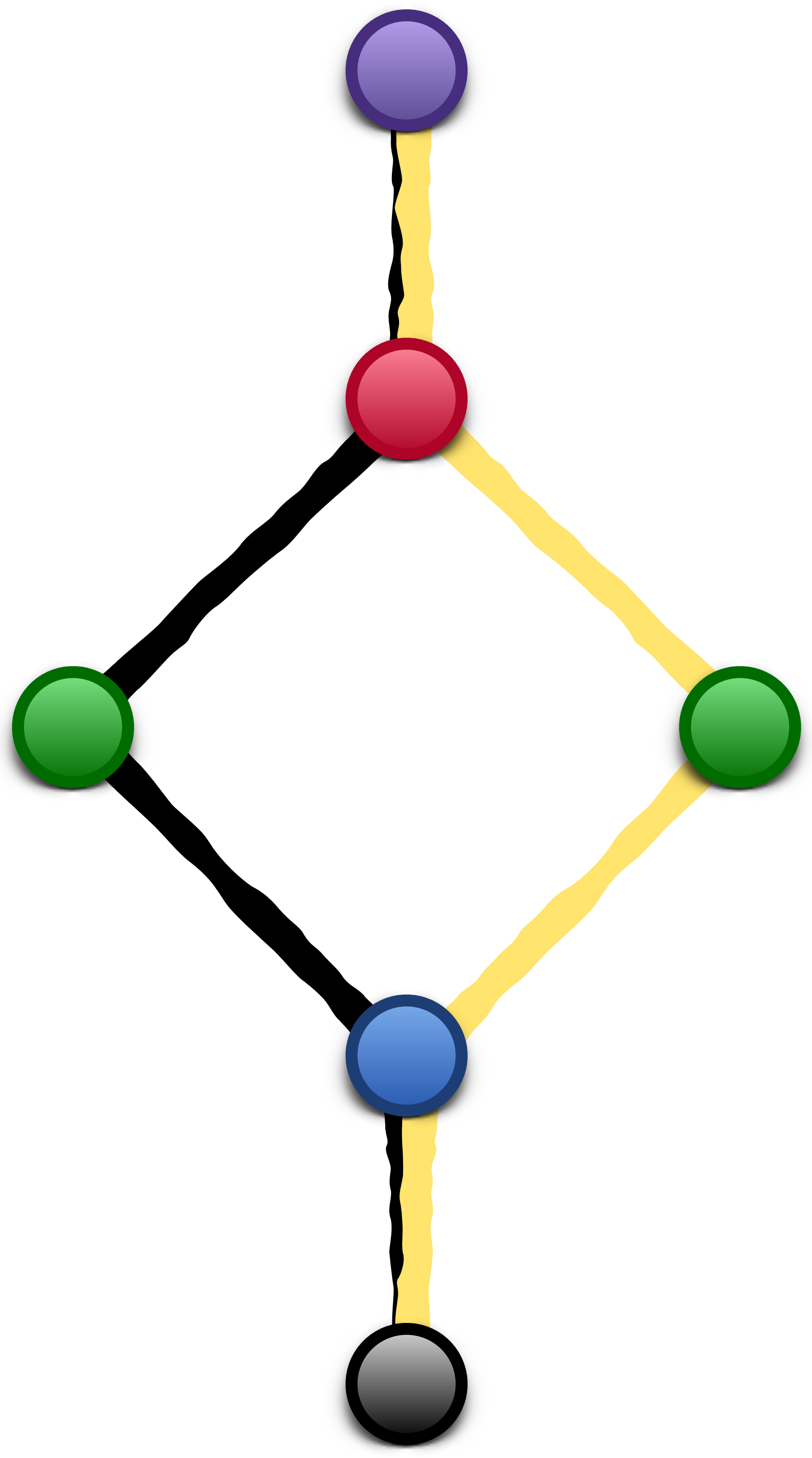

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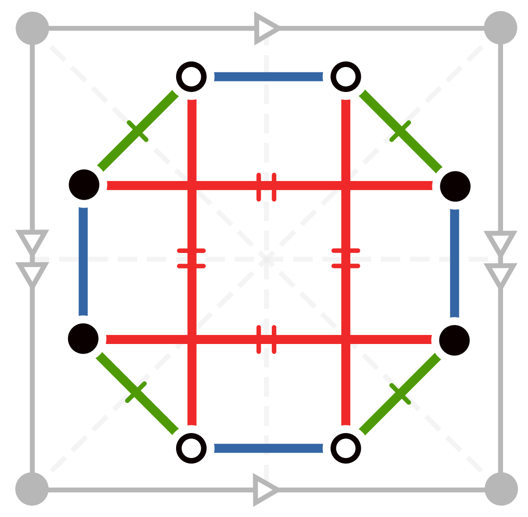

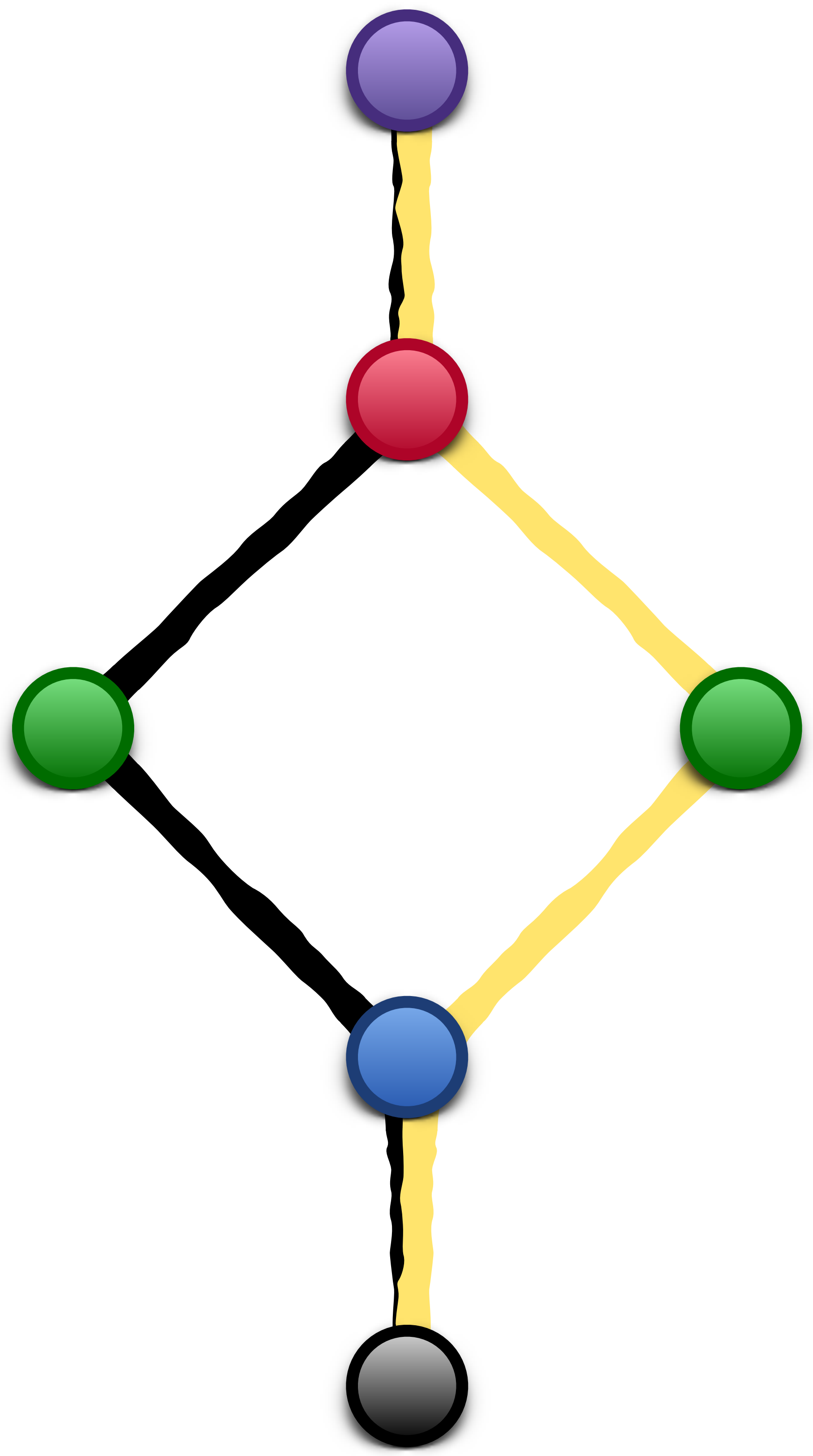

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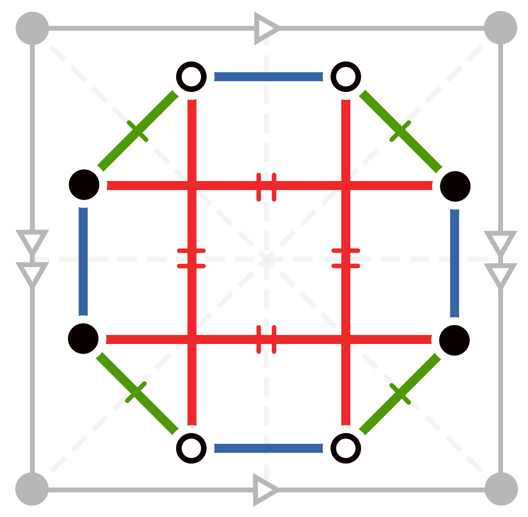

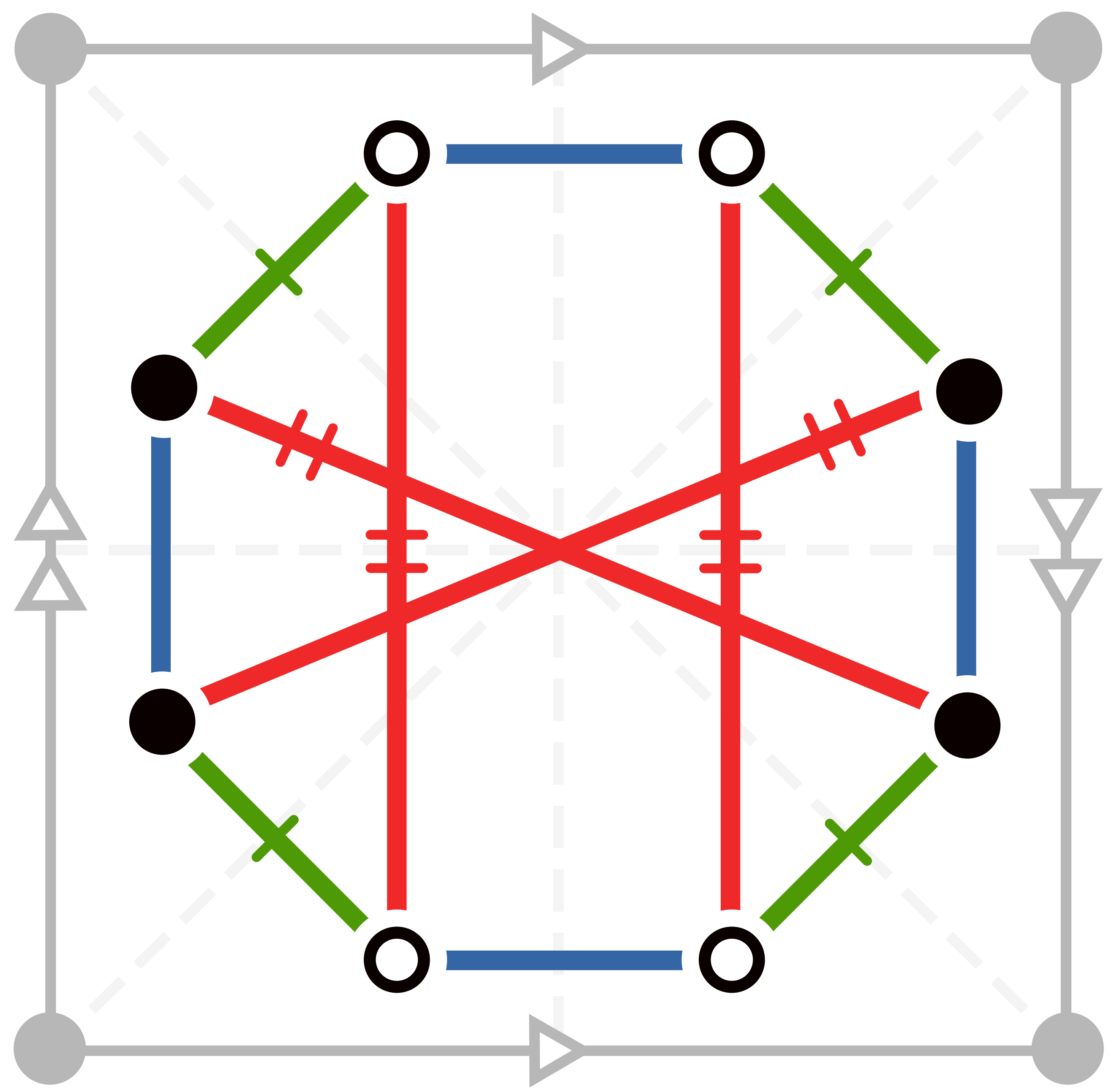

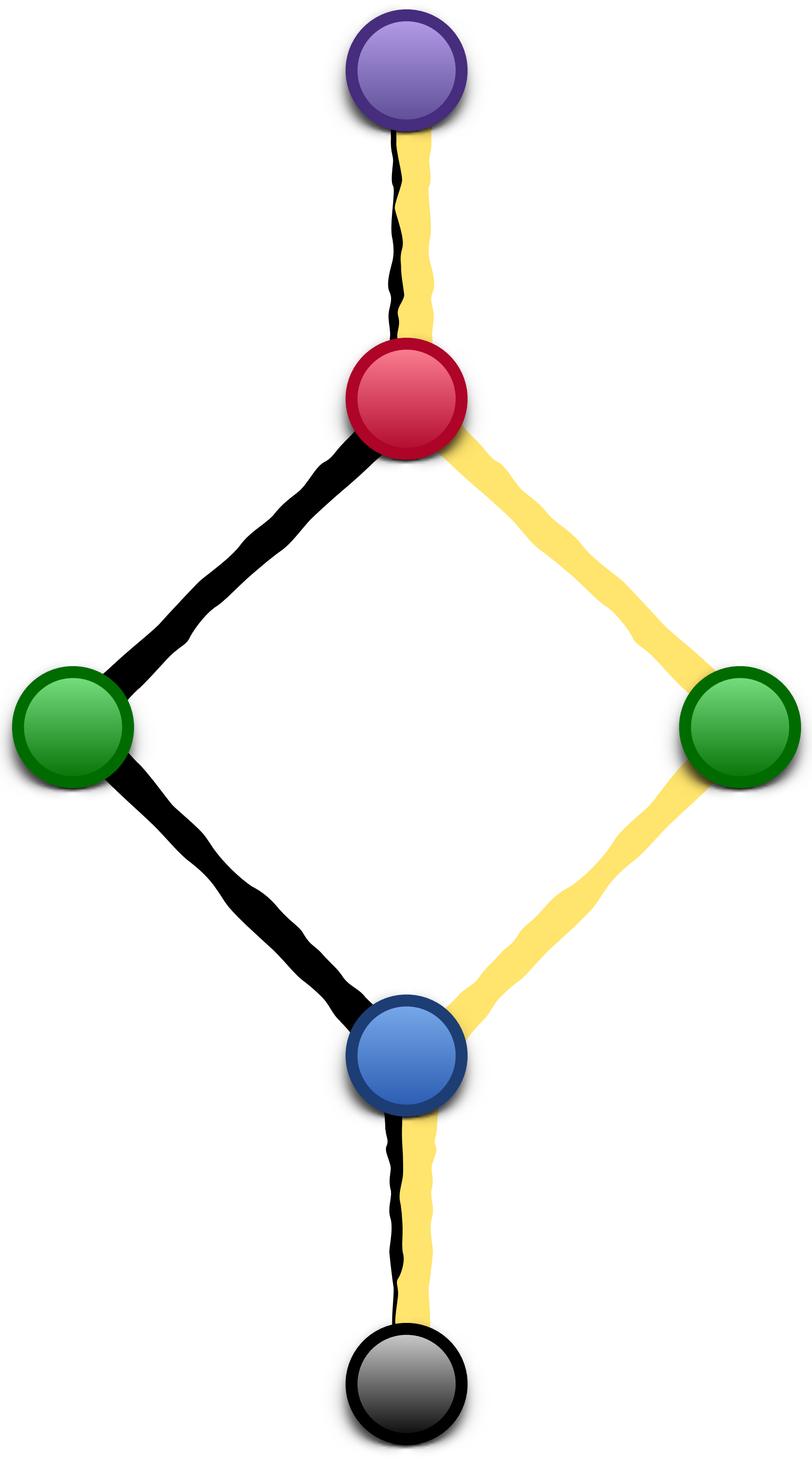

Faithfulness in maniplexes

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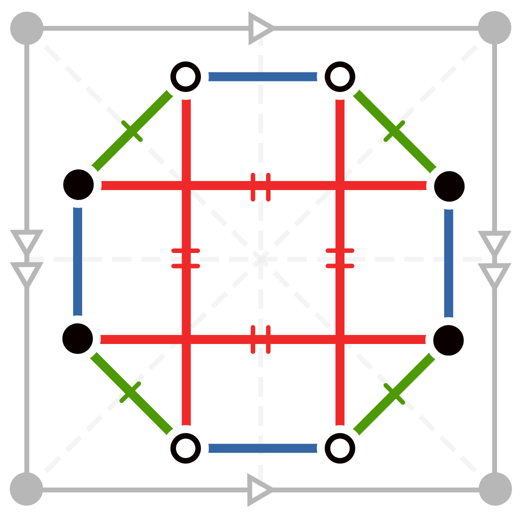

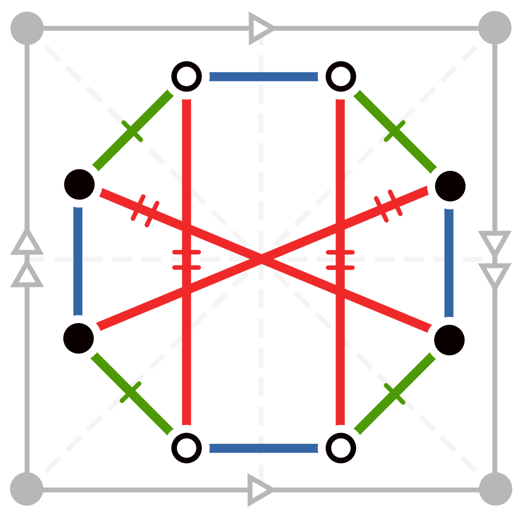

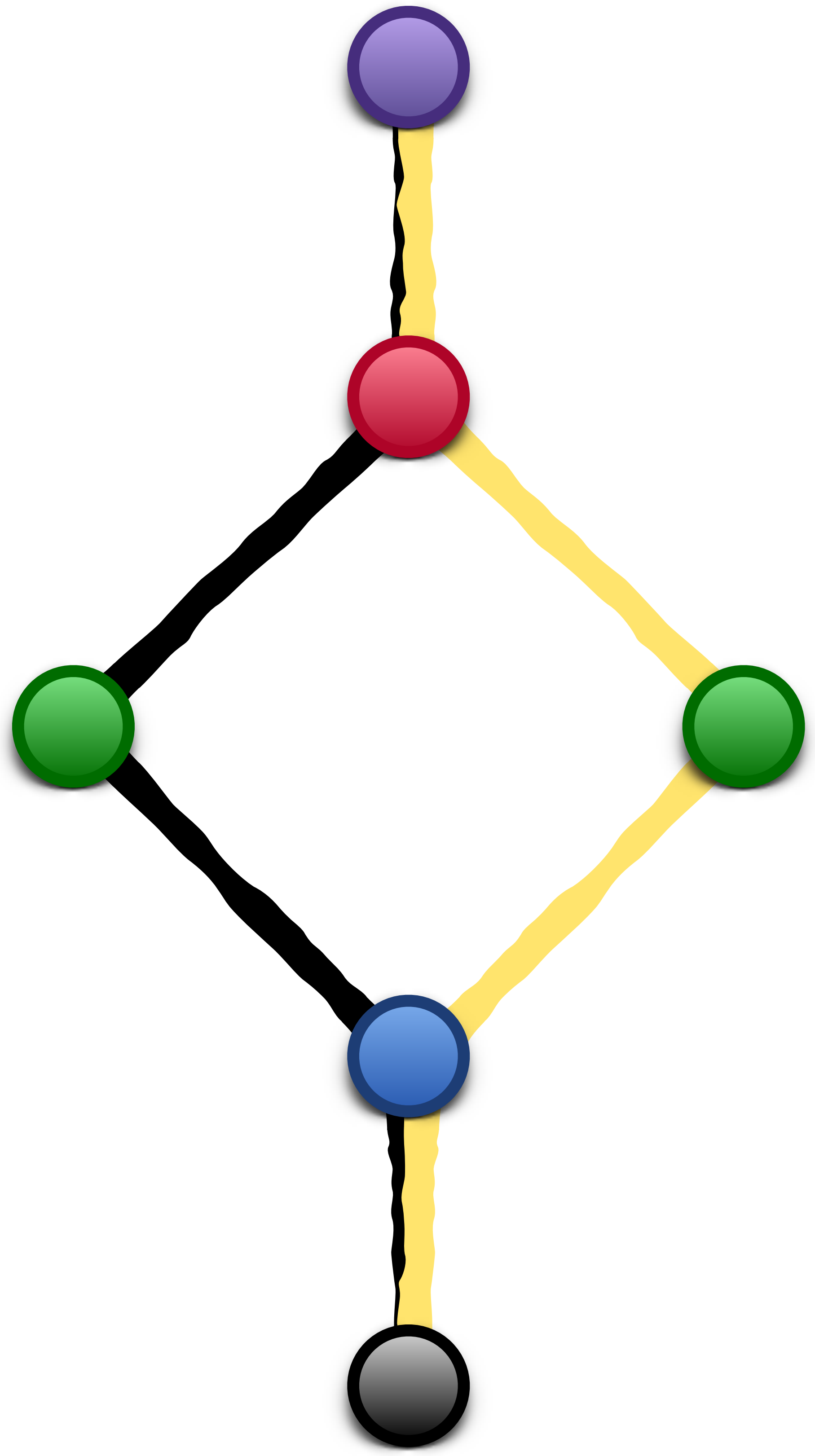

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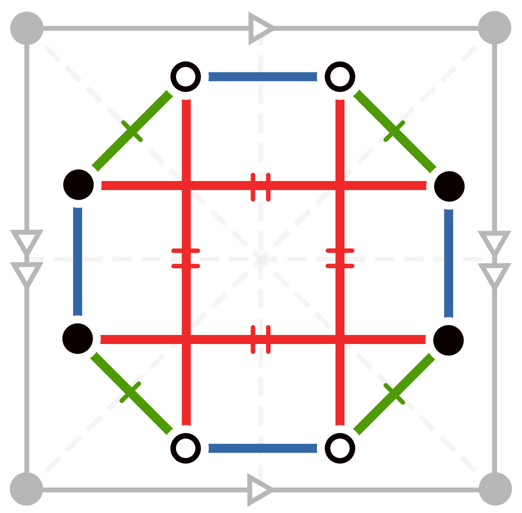

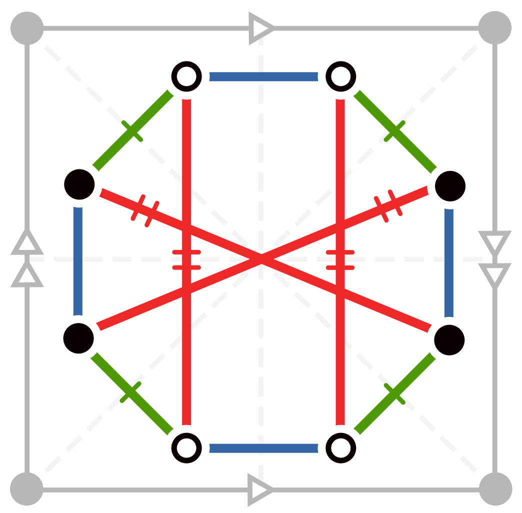

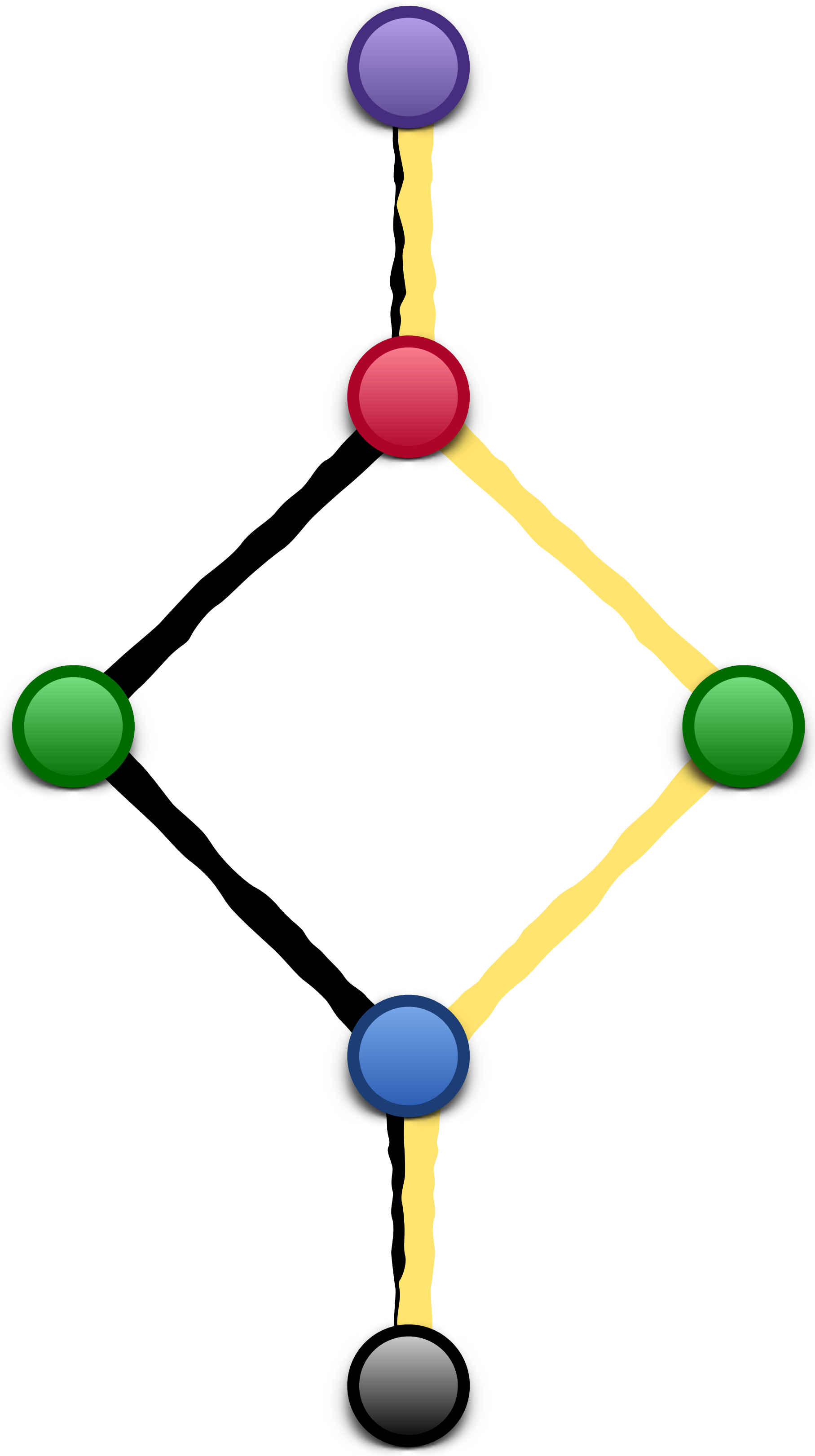

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Leemans, Toledo (2025):

- If $\mathscr{M}$ is faithful, thin and of rank 3, then it has the PIP (is polytopal)
- There exist infinitely many faithful and thin but non-polytopal maniplexes (of rank $n \geq 4$)

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Faithfulness in maniplexes

$$\begin{array}{ccc}
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\cong ? & & ✅ \\
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All of them are unfaithful.

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Faithfulness and covers

- If $\pi : \mathscr{M} \to \mathscr{M}'$ is a cover, and $\Phi \approx \Psi$, then $\pi(\Phi) \approx \pi(\Psi)$
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Faithfulness and symmetries

- If $\alpha \in \mathrm{Aut}(\mathscr{M})$ and $\Phi \approx \Psi$, then $\Phi\alpha \approx \Psi\alpha$.
- Assume $\Phi \approx \Psi$, then $\Phi \approx \Psi\alpha$ if and only if α is in the intersection of the face stabilisers of Φ.

Faithfulness in maniplexes

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Classification of regular and 2-orbit unfaithful maps:

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- All the other symmetry classes (of 2 flag-orbits) admit unfaithful maps.

Faithfulness in maniplexes

Classification of unfaithful rotary 4-maniplexes:

graphsym.net

(Primož Potočnik)

Census of Rot 4 Maniplexes

(Cun, Pot, Tol, Moc, M.)

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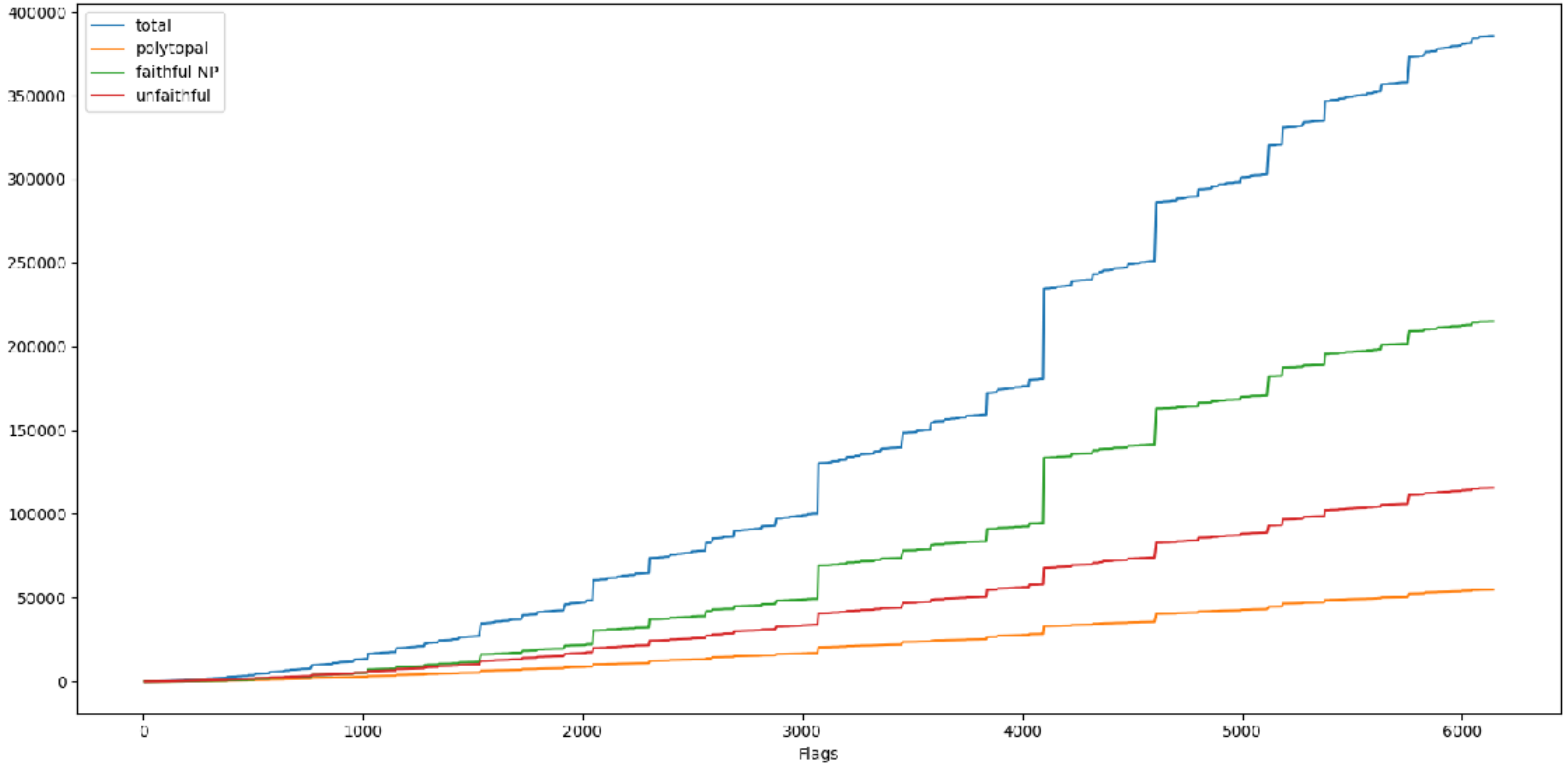

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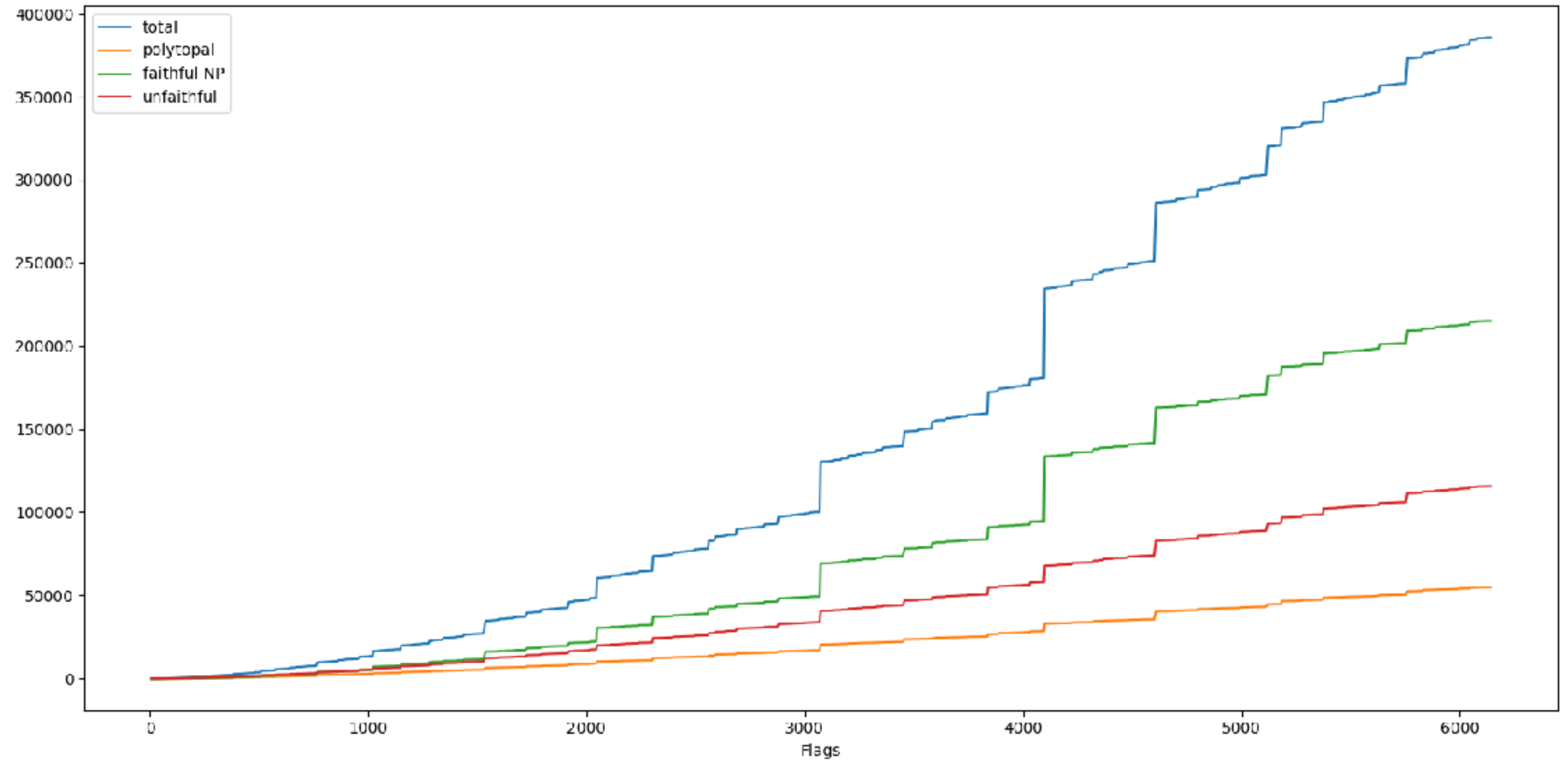

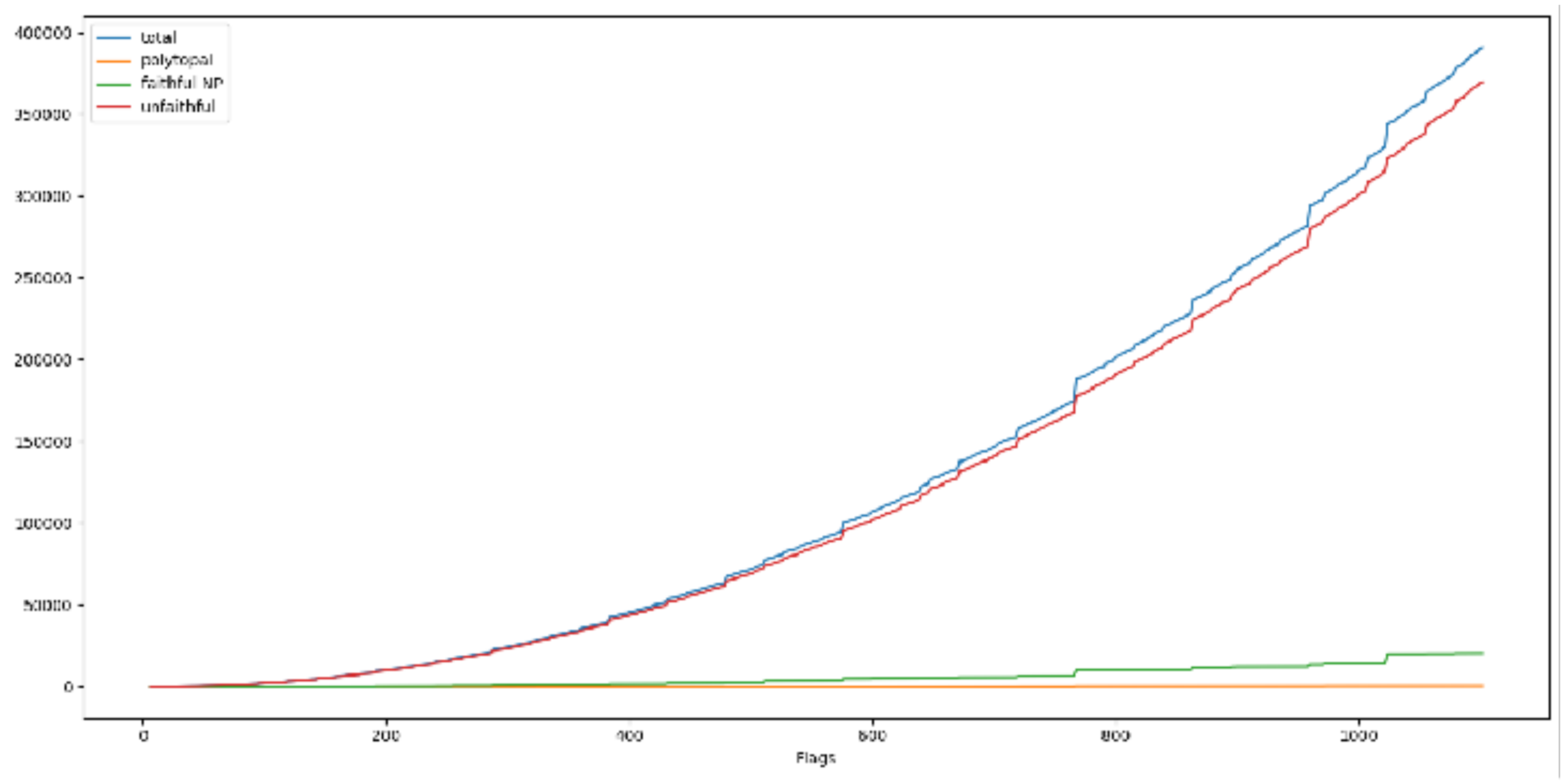

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RAMP
Research Assistant for Maps and Polytopes
(GAP Package)

Gap.app
Russ Woodroofe

Thank you!

RAMP

graphsym.net

Slides

Census

Gap.app