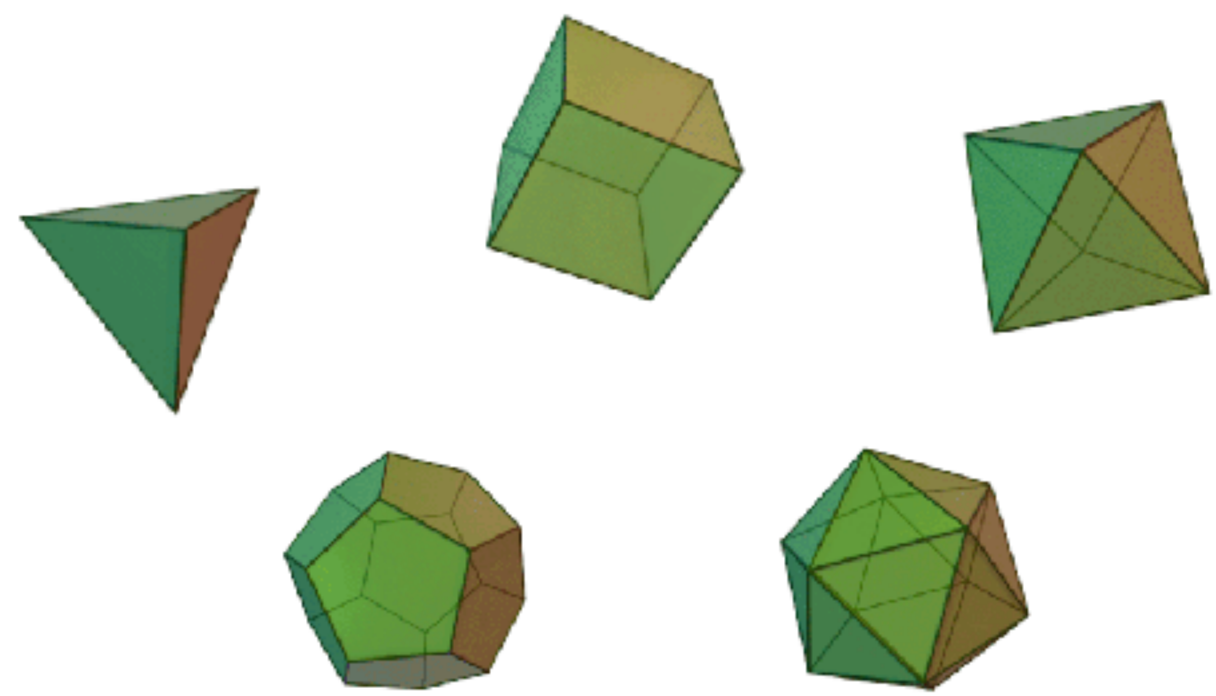
Ali so tvoji poliedri enaki mojim poliedrom?

Mesec matematike na PEF
UL Pedagoška Fakulteta
Marec 2026



Antonio Montero

UNIVERZA V LJUBLJANI

MATHEMA

Videos and graphics: wikipedia.
Figures and slides: Isabel Hubard and A.M.

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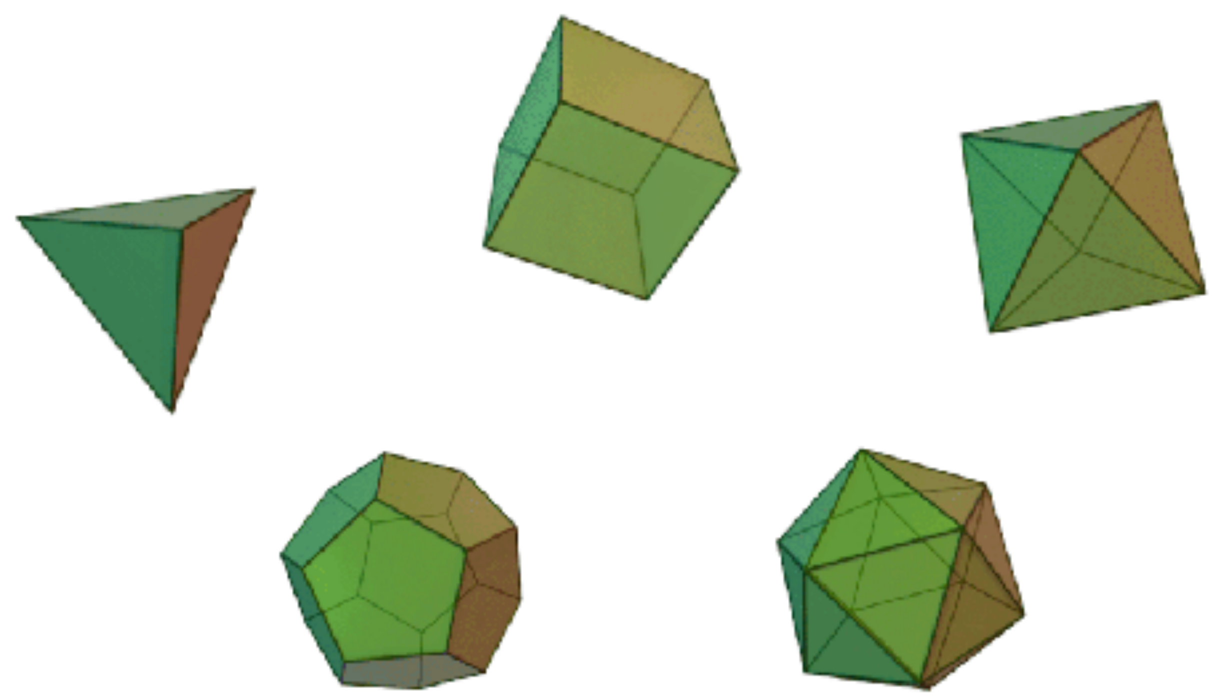

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Are your polyhedra the same as my polyhedra?

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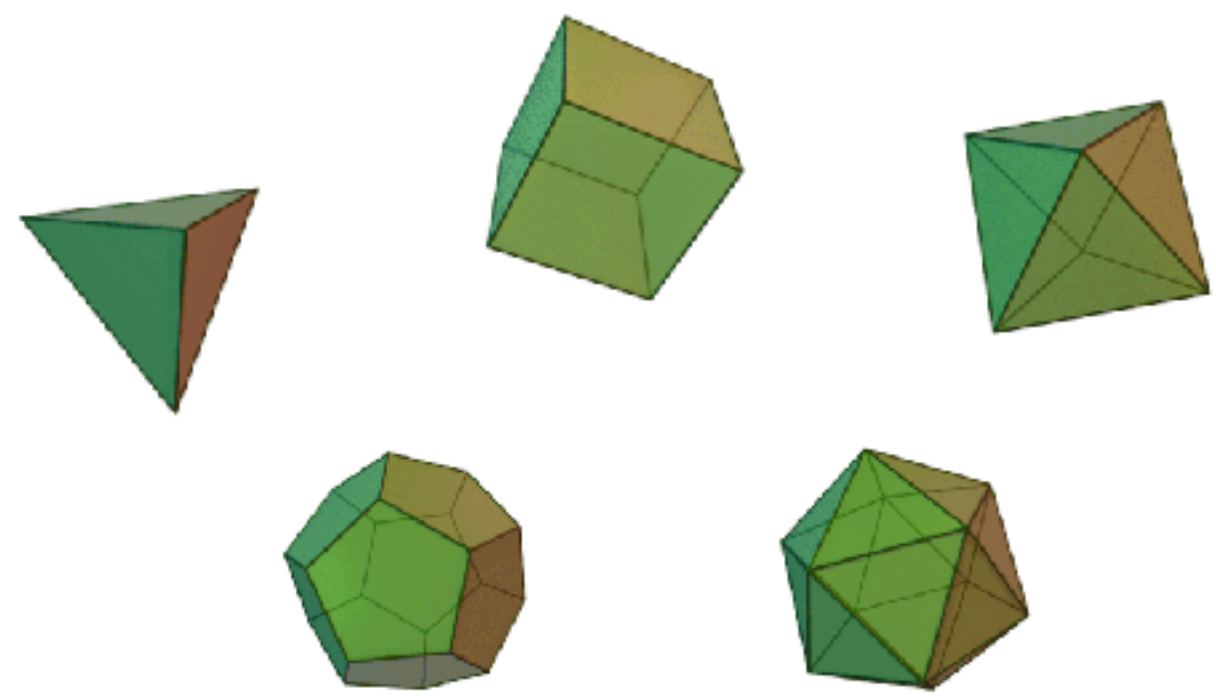

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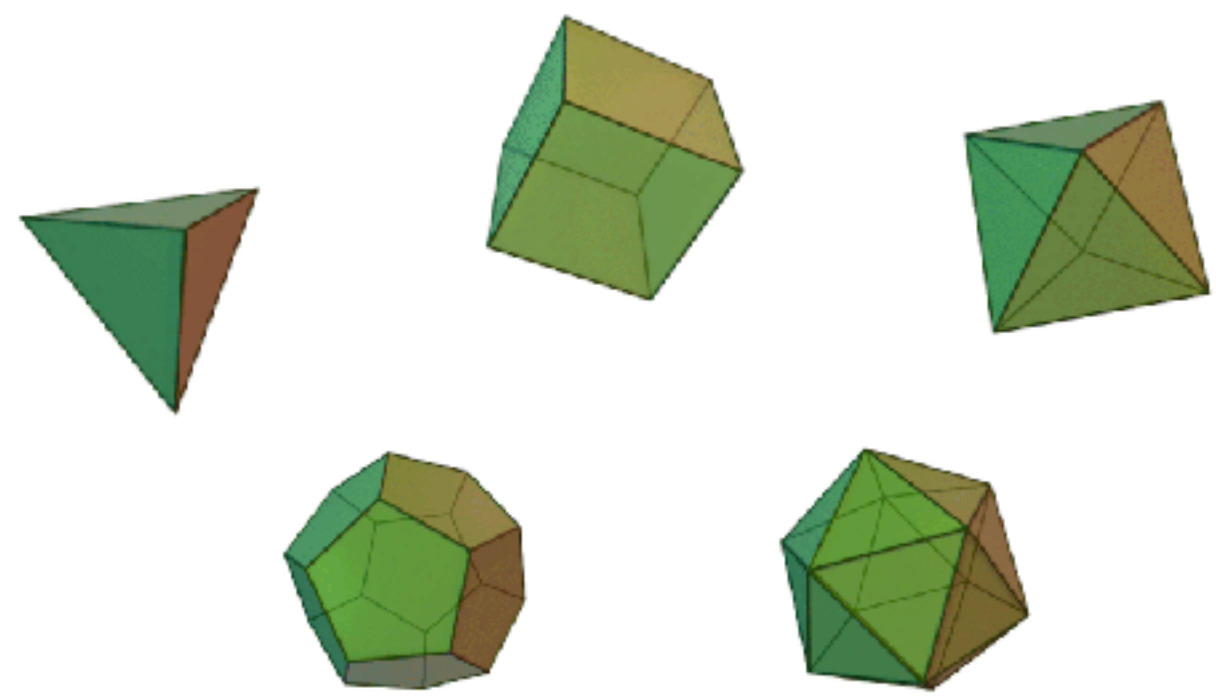

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MATHEMA

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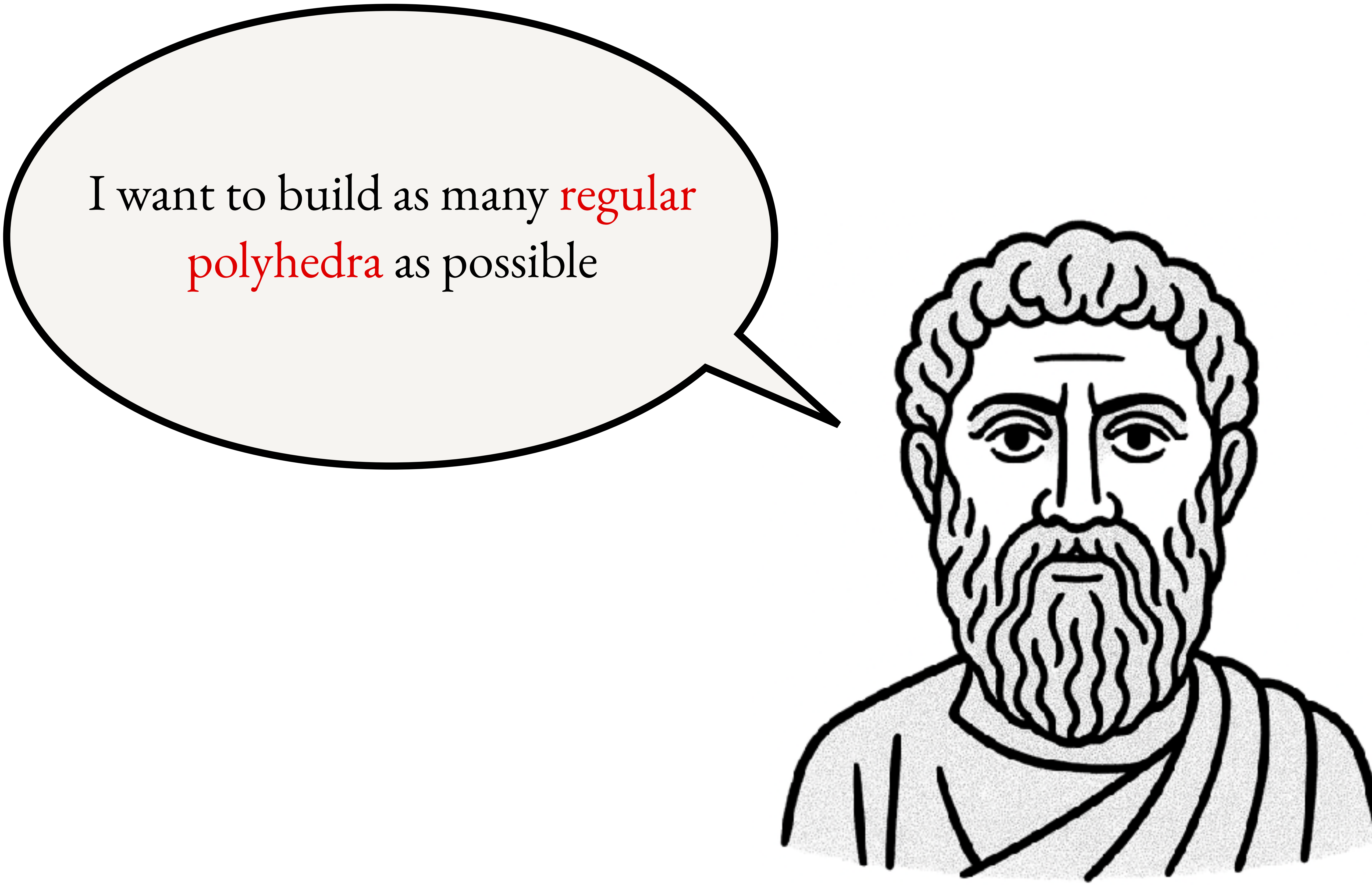
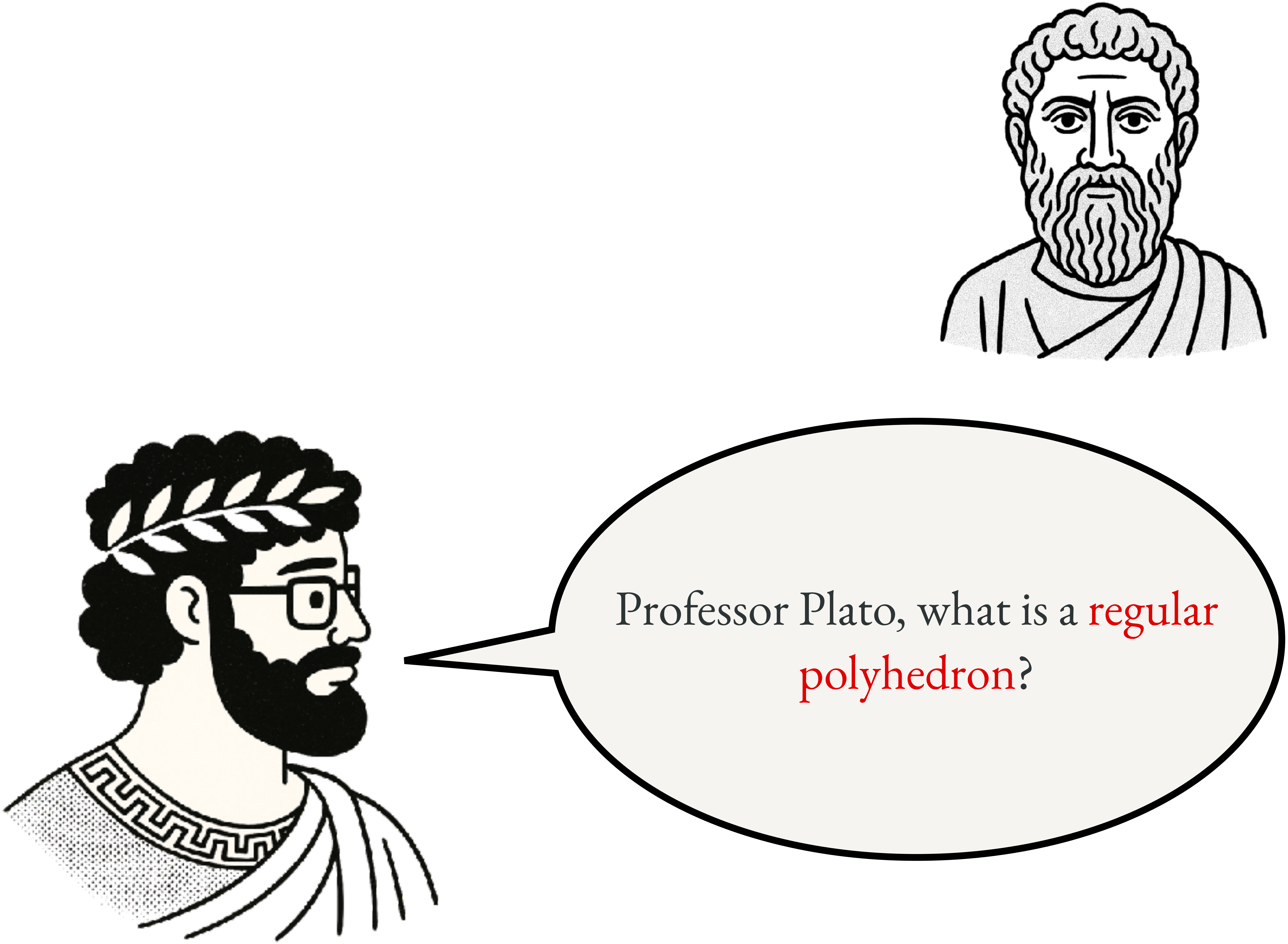
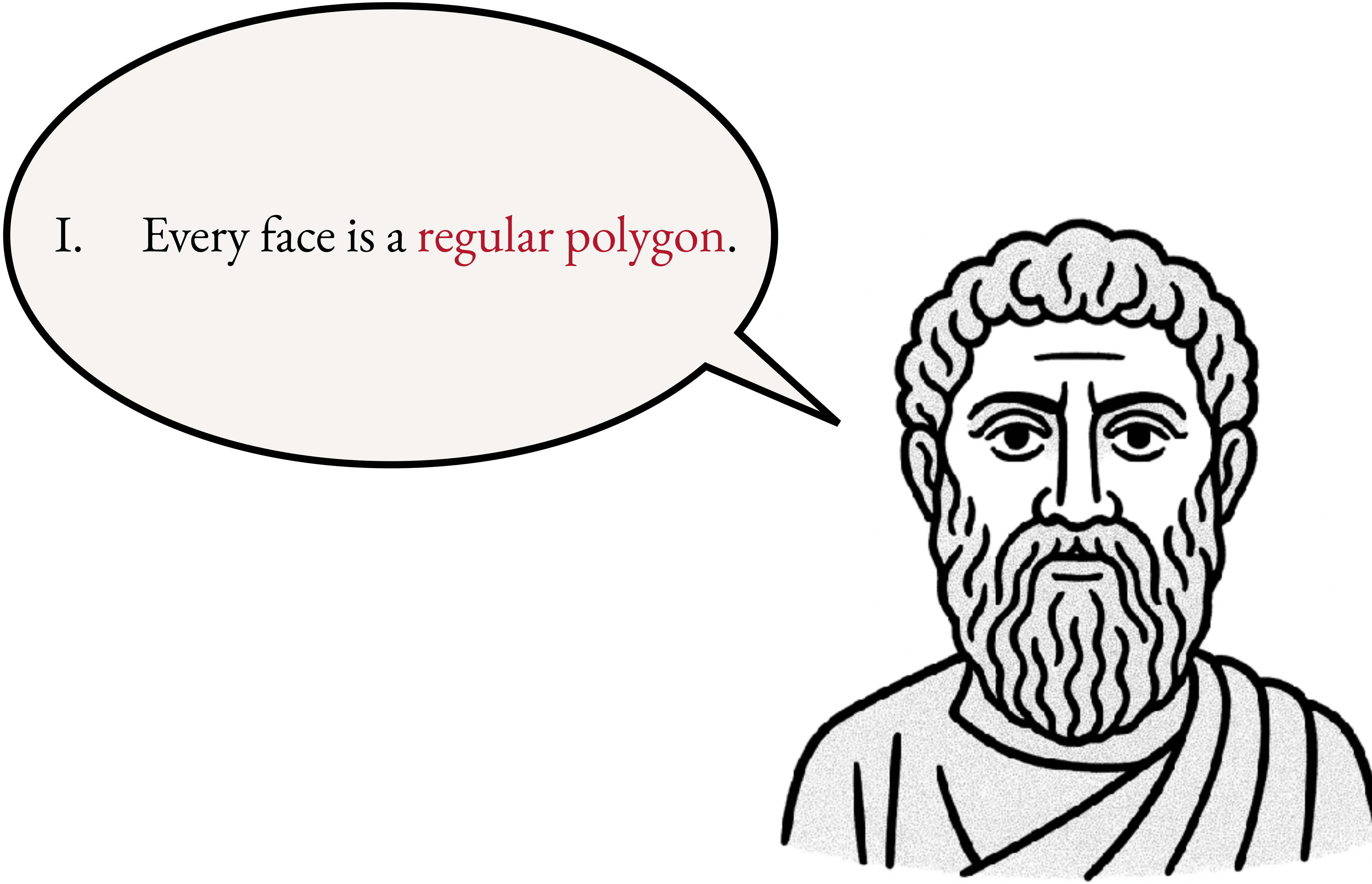
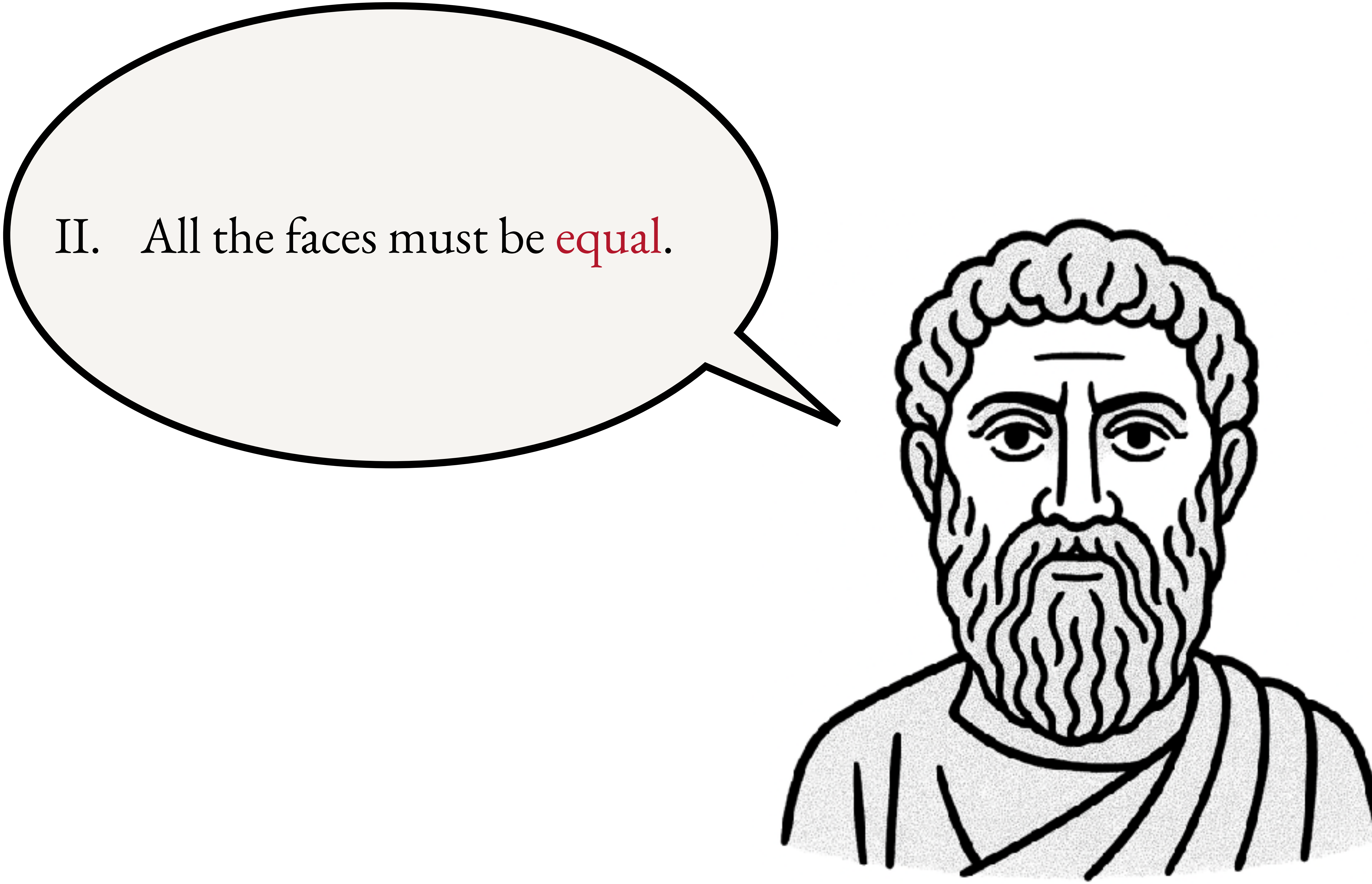
Greece | ~ 360 BC

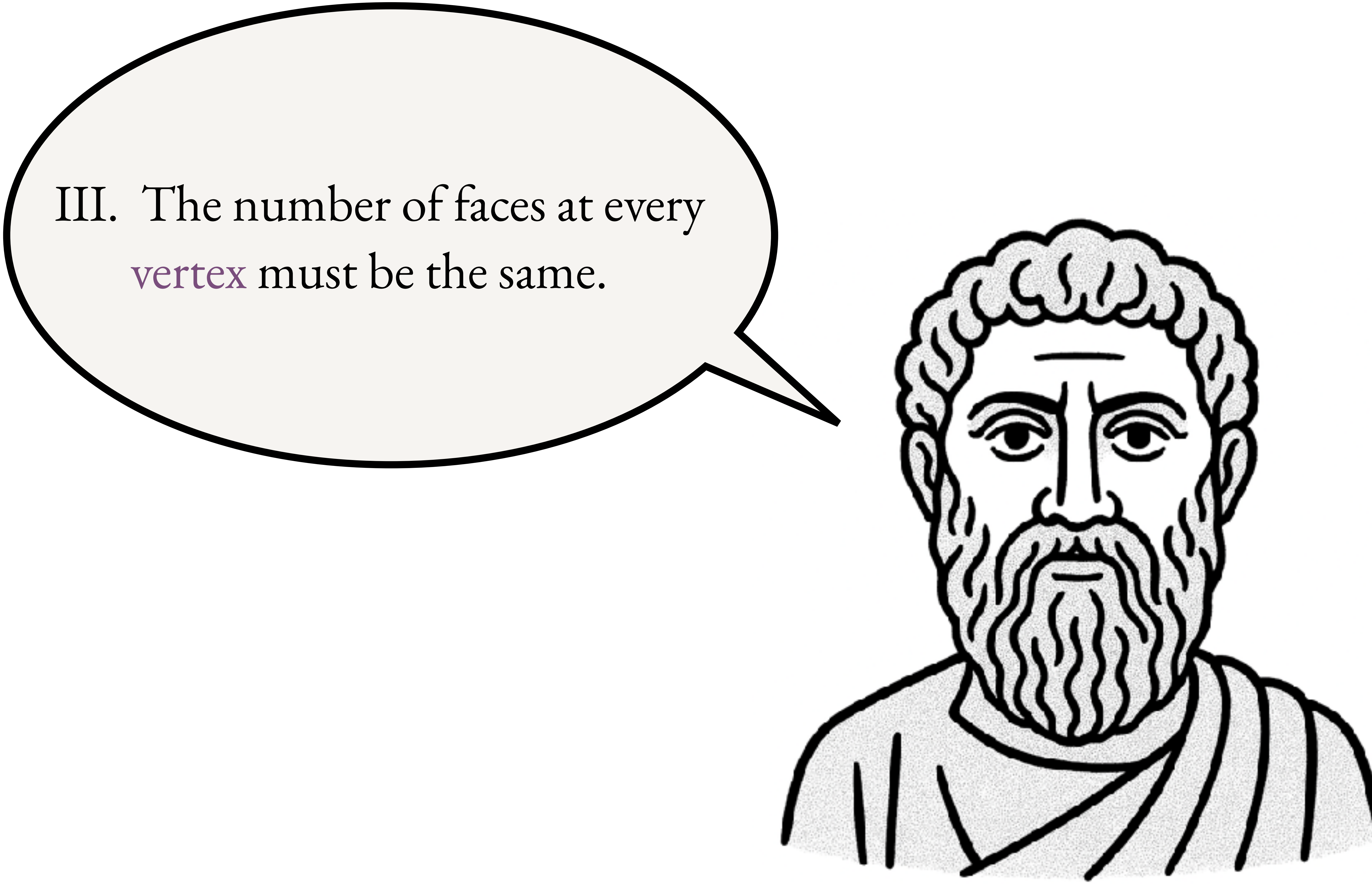

I want to build as many regular polyhedra as possible


Professor Plato, what is a regular polyhedron?

They are built by glueing polygons (faces) along their edges (two per edge)


I. Every face is a regular polygon.


II. All the faces must be equal.

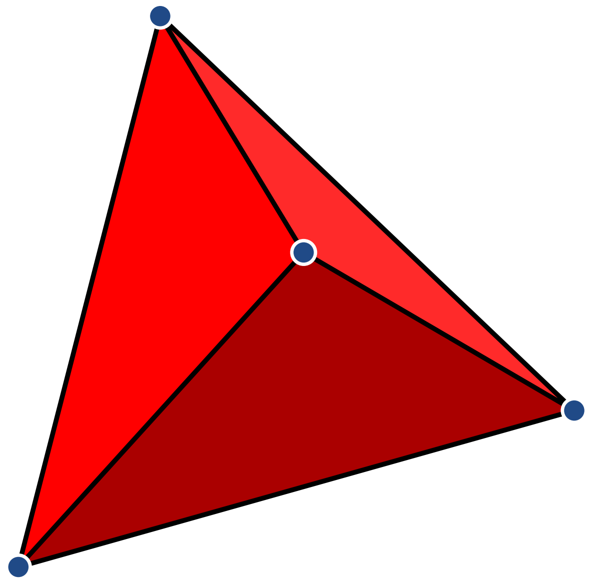

III. The number of faces at every vertex must be the same.

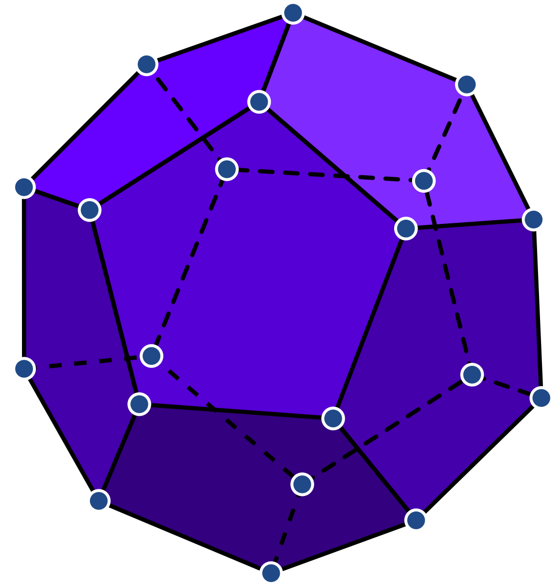
They are built by glueing polygons (faces) along their edges (two per edge)

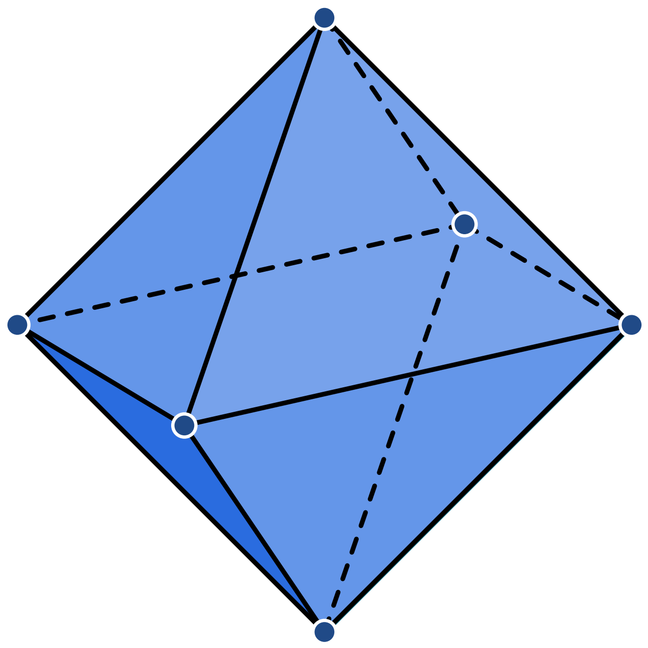
I. All the faces must be equal.
II. Every face is a regular polygon.
III. The number of faces at every vertex must be the same.

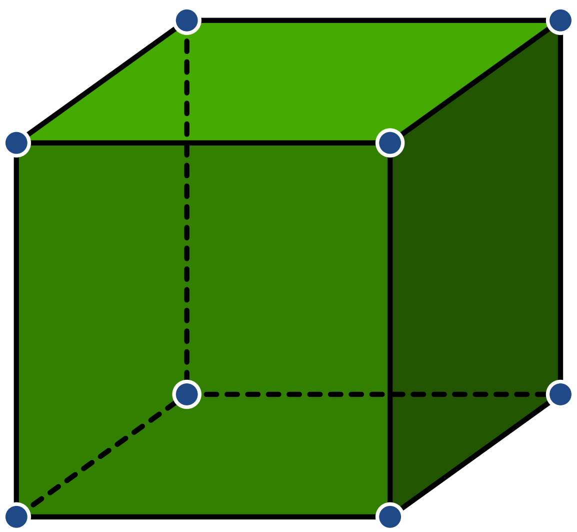
The Platonic solids

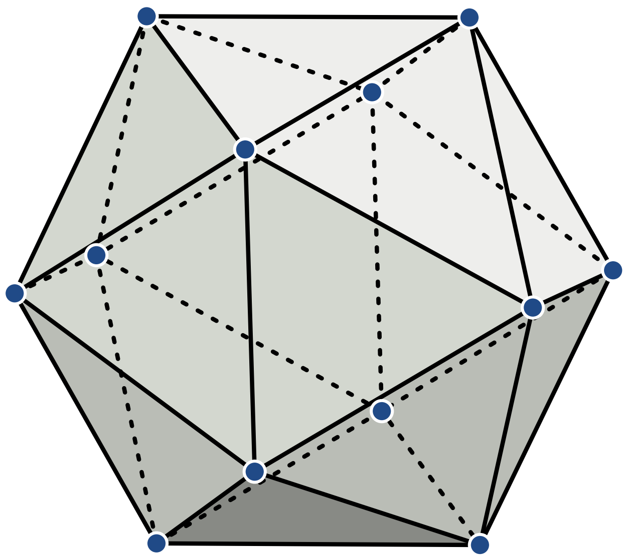
The Platonic solids

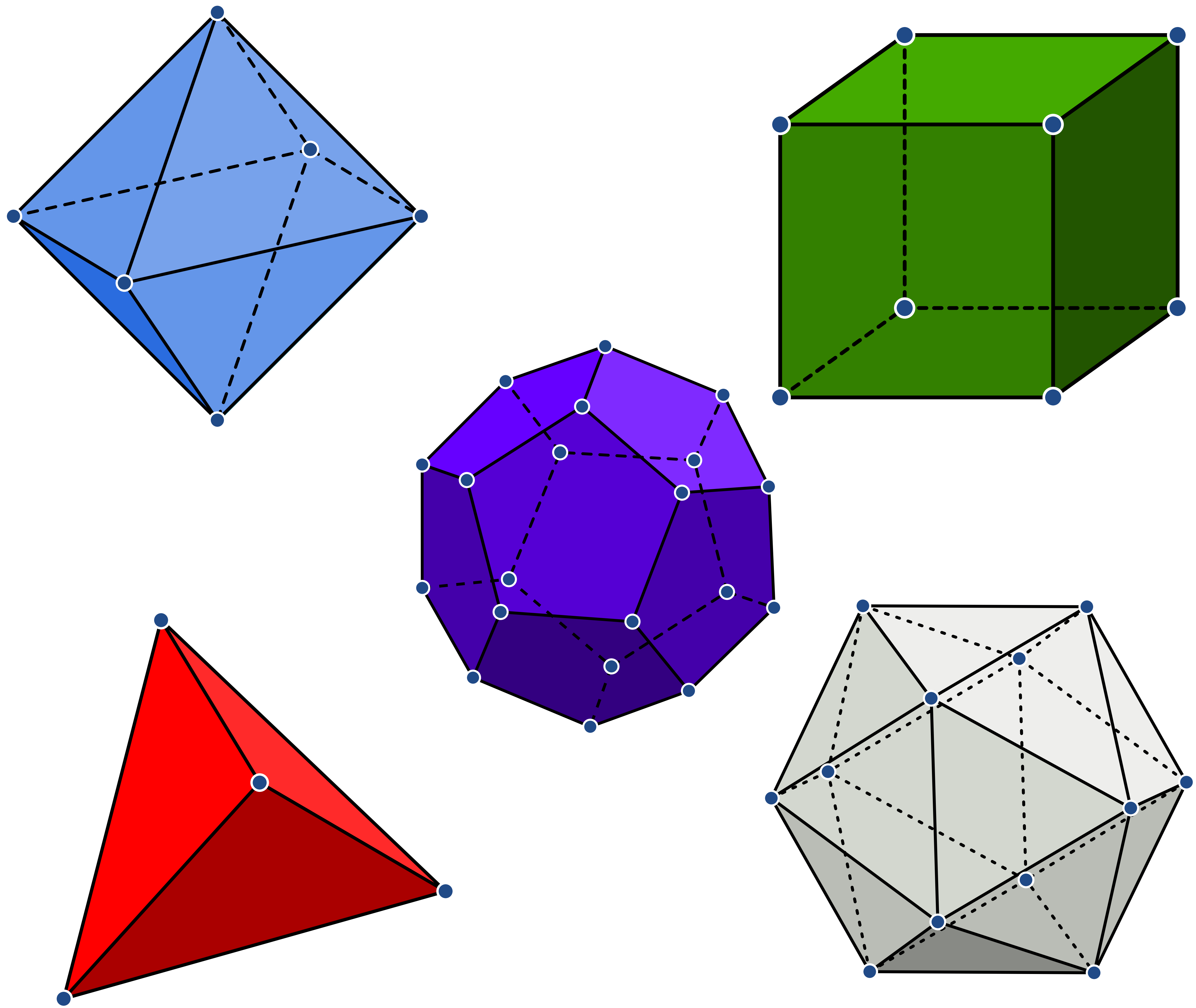


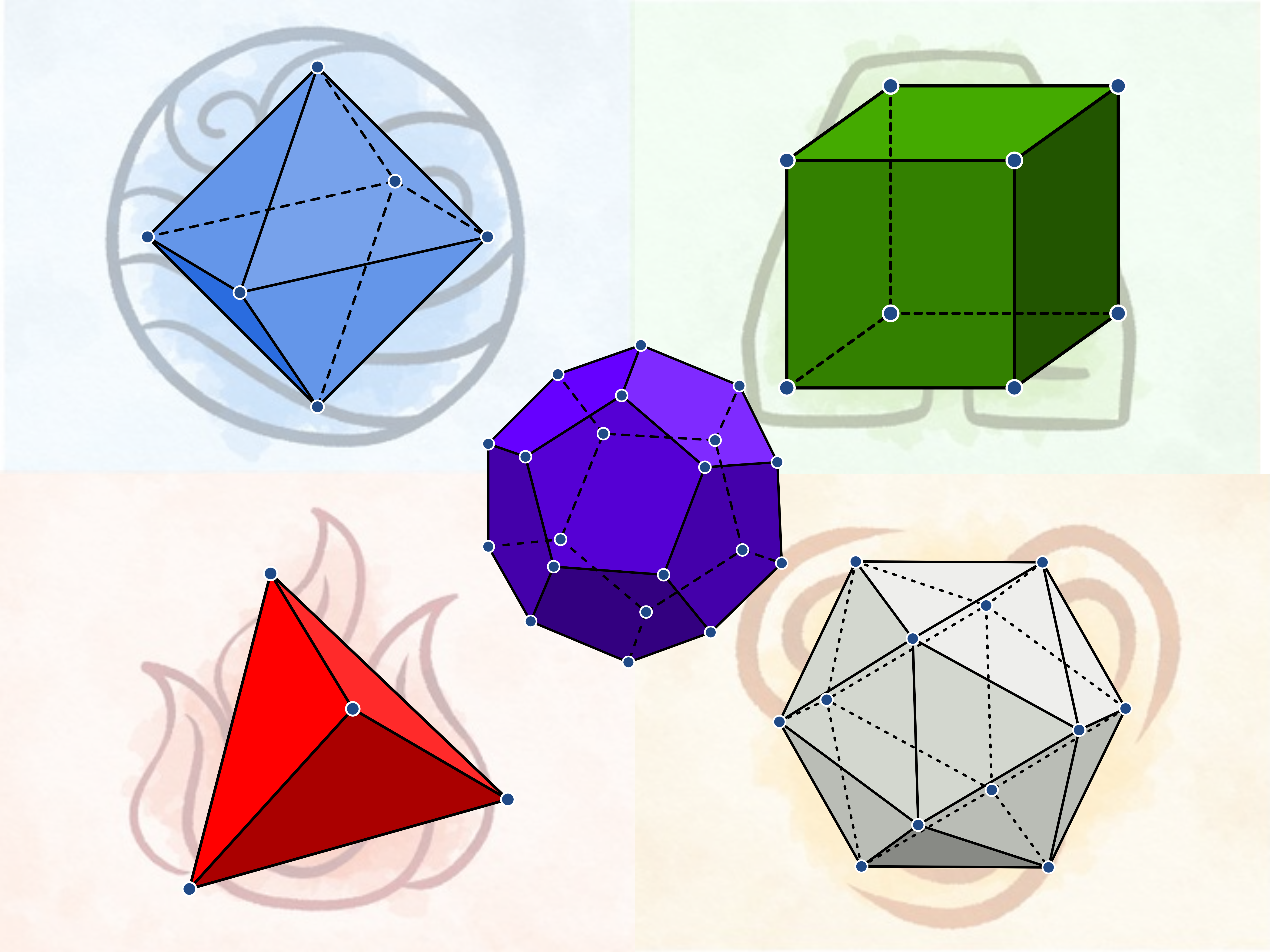


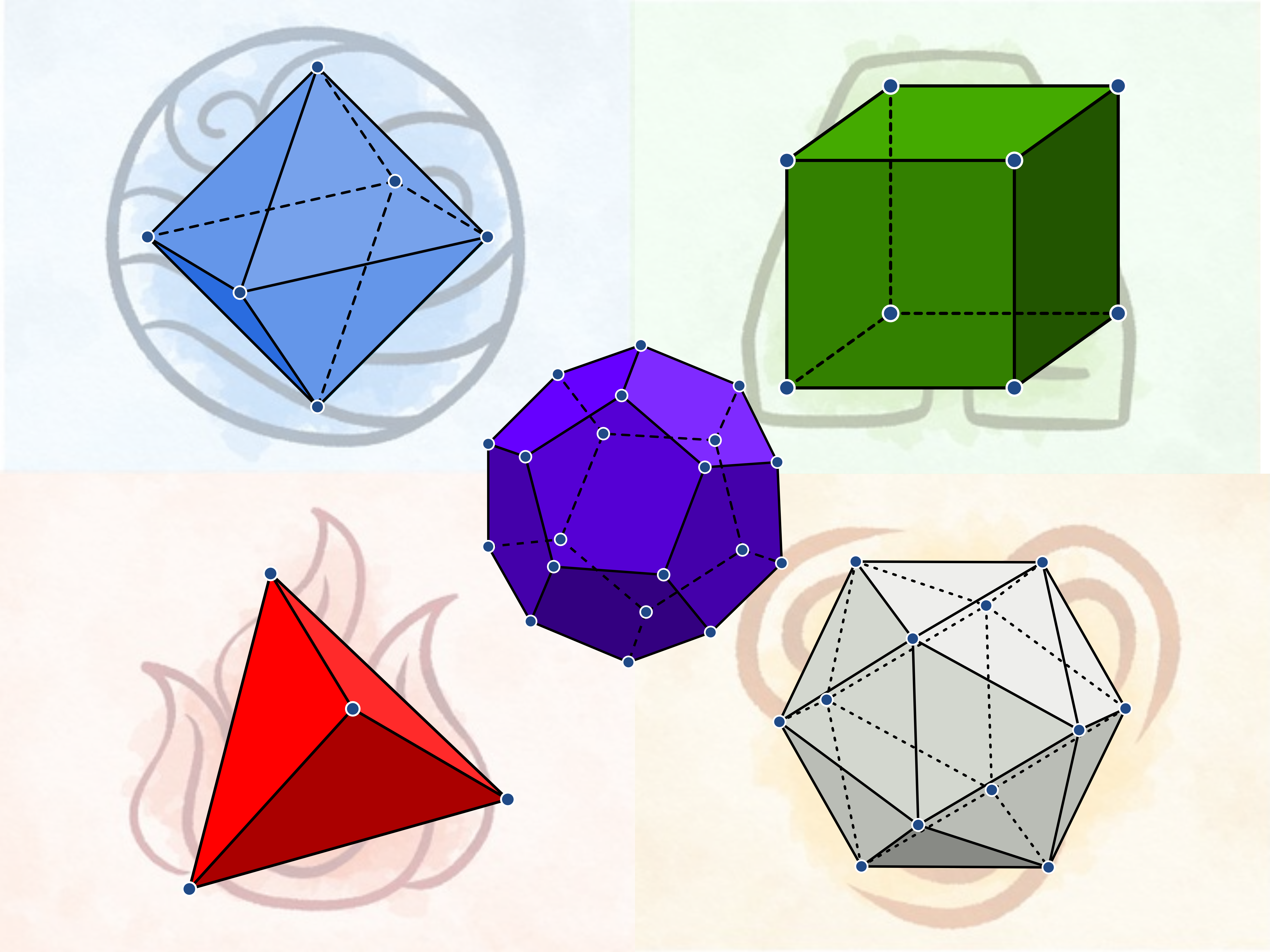








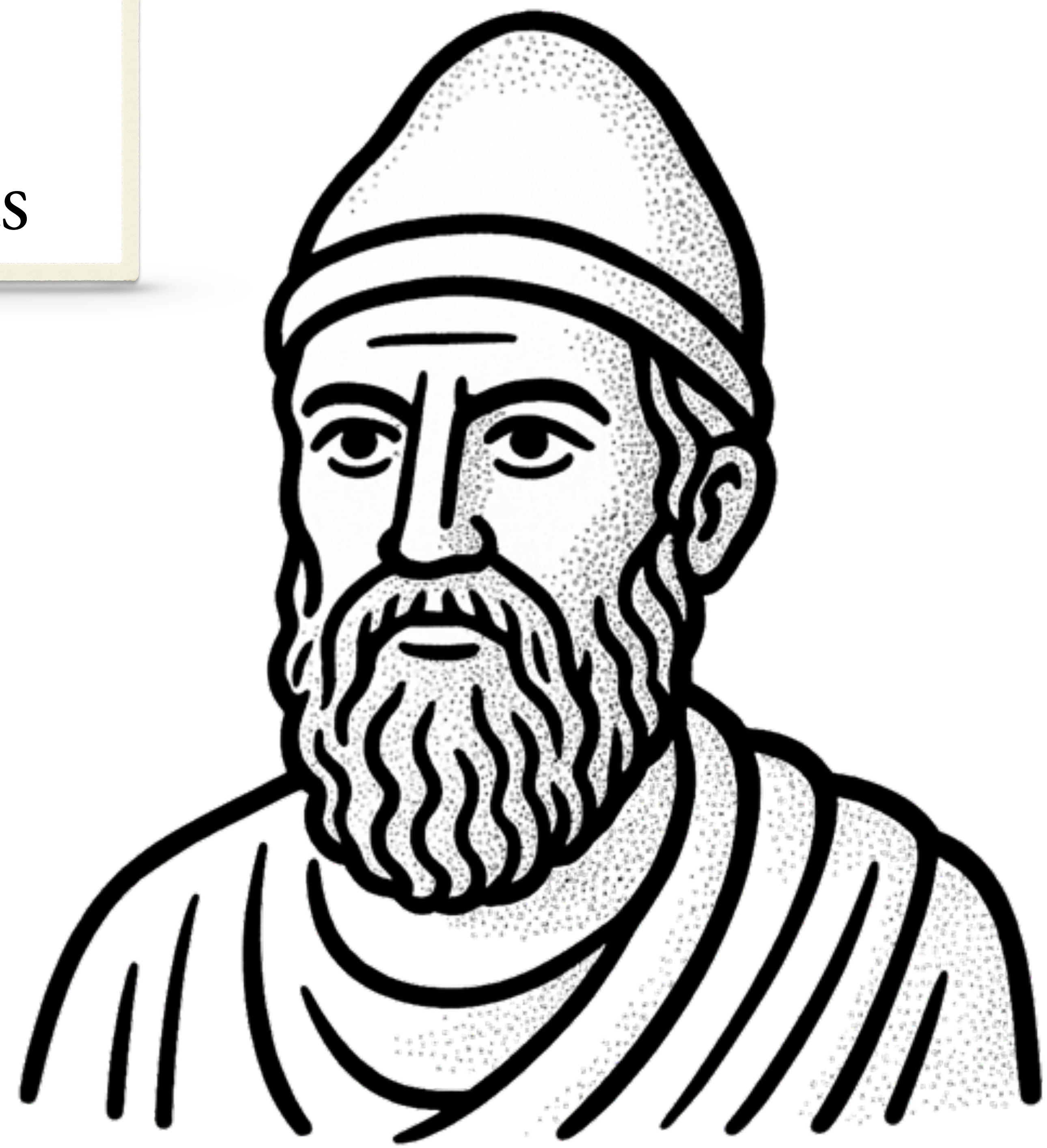




Greece | ~ 360 BC

Greece | ~ 300 BC

Theorem (Euclid's elements)
There exist exactly 5 regular polyhedra: the Platonic solids

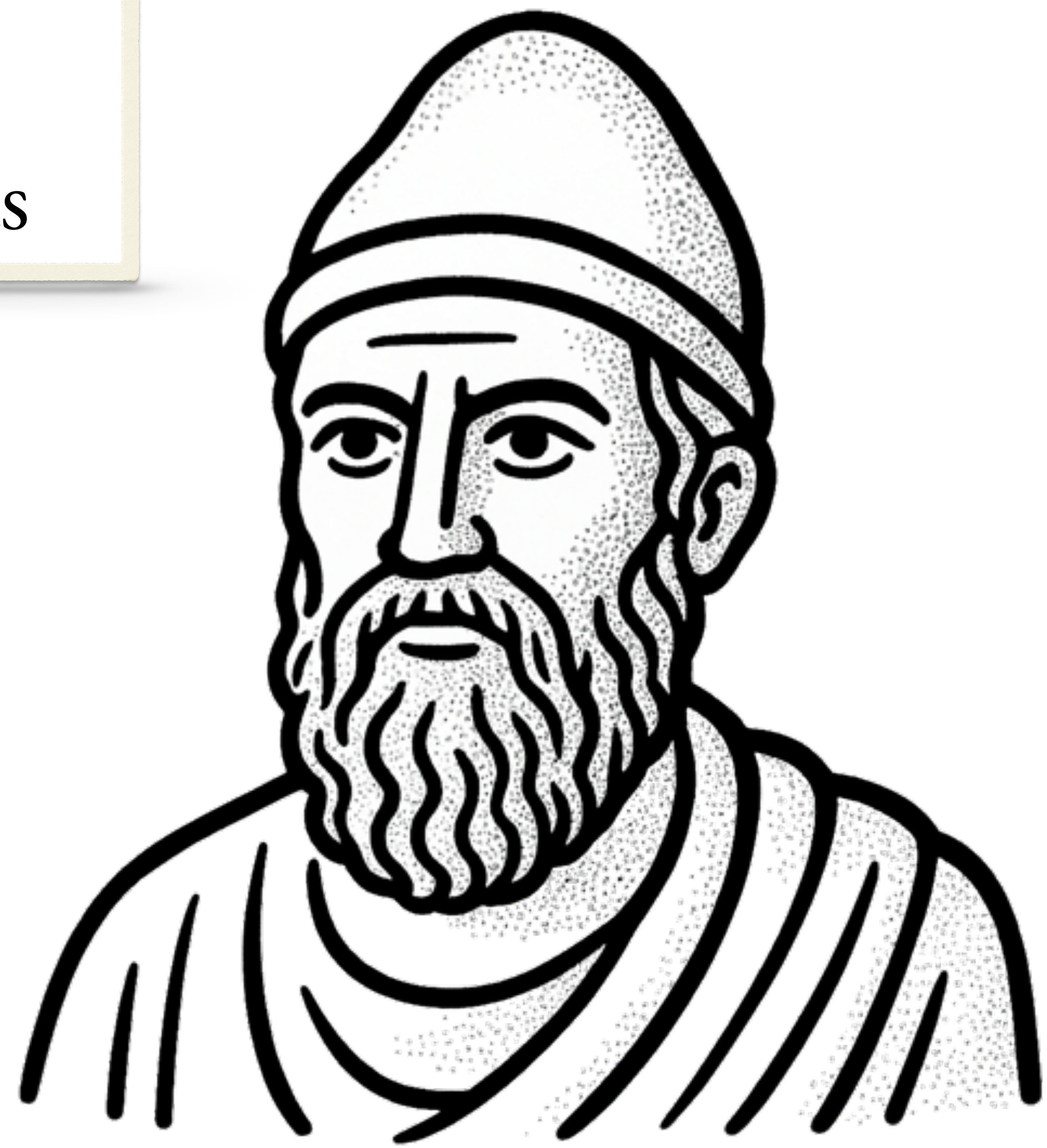


Euclid

Theorem (Euclid's elements)

There exist exactly 5 regular polyhedra: the Platonic solids

Theaetetus



Euclid

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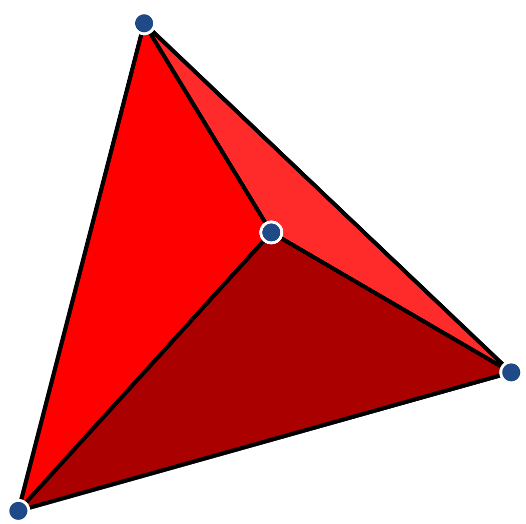

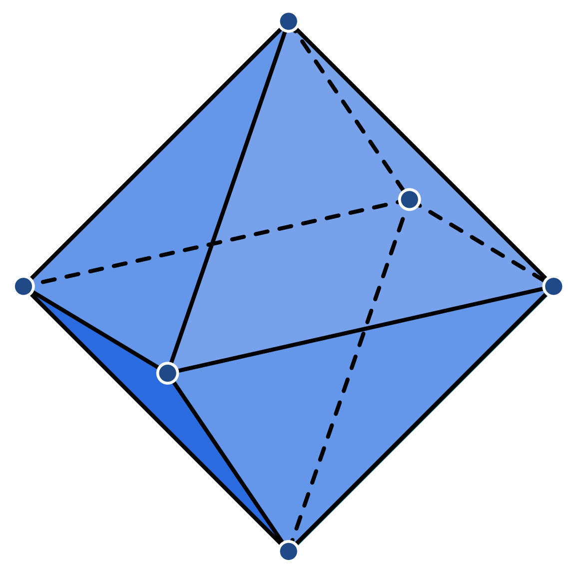

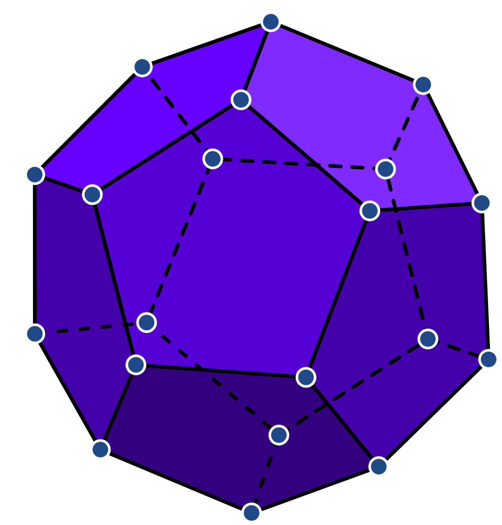

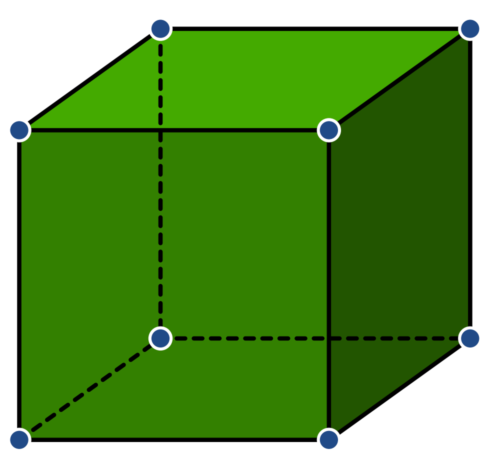

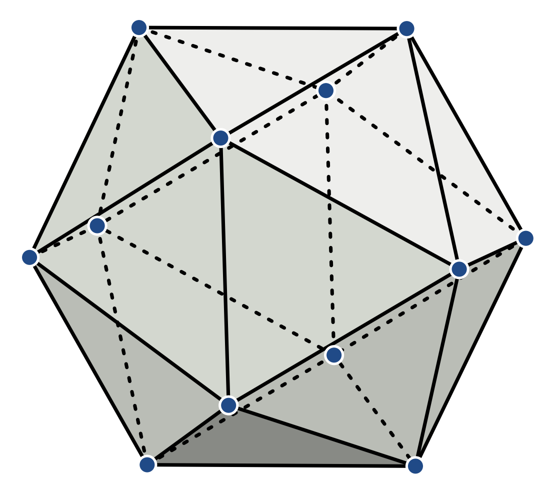

Theorem (Euclid's elements)

There exist exactly 5 regular polyhedra: the Platonic solids

How do we prove this?

Theorem (Euclid's elements)

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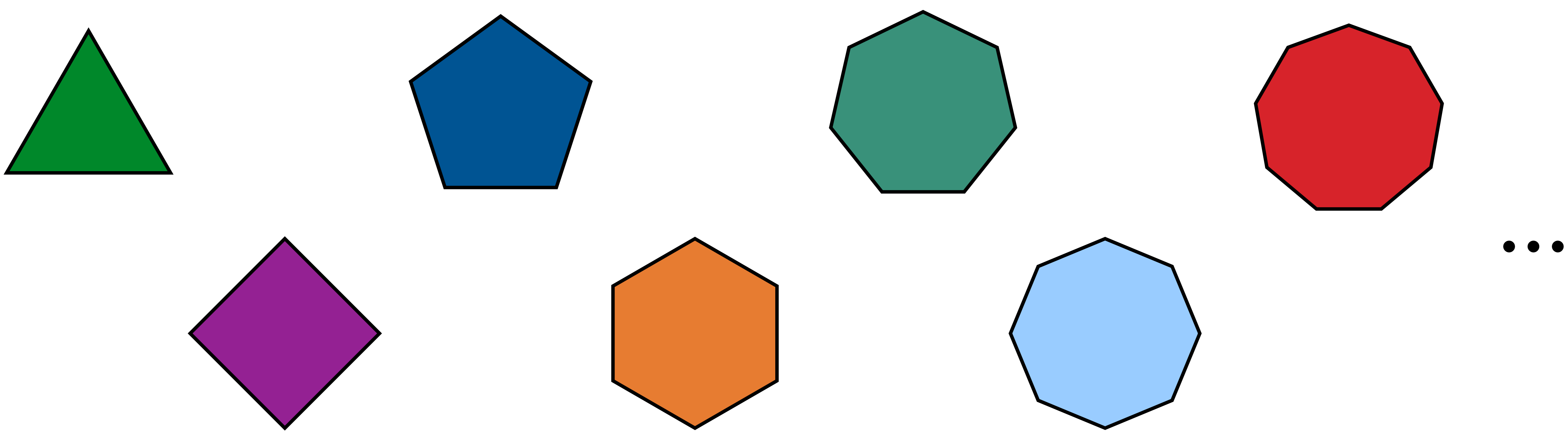
How do we prove this?

Faces? $\rightarrow$ regular polygons



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Faces? $\rightarrow$ regular polygons

How many around each vertex?

Faces?

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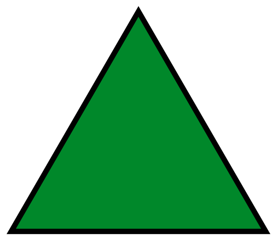
Faces?



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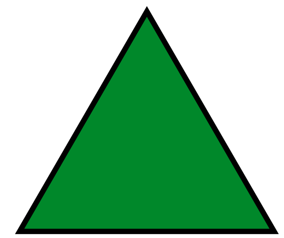

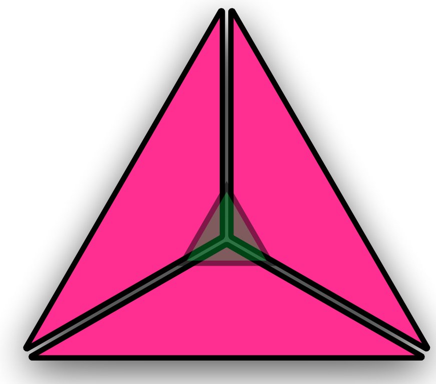

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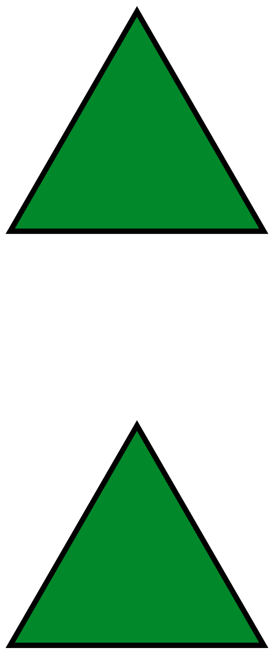

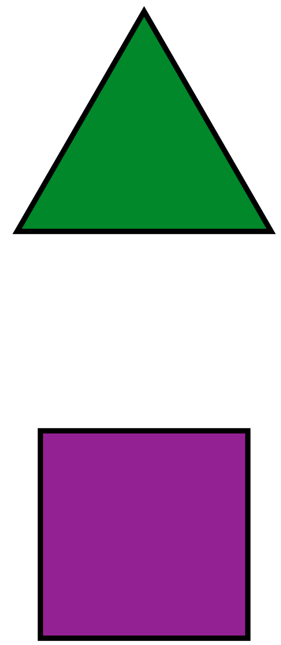

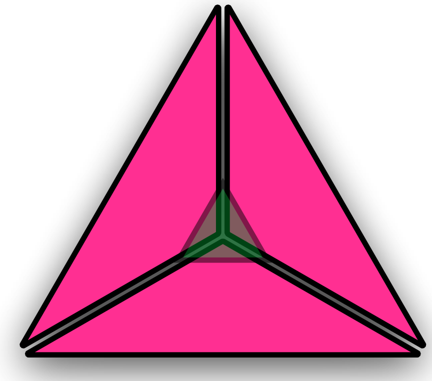

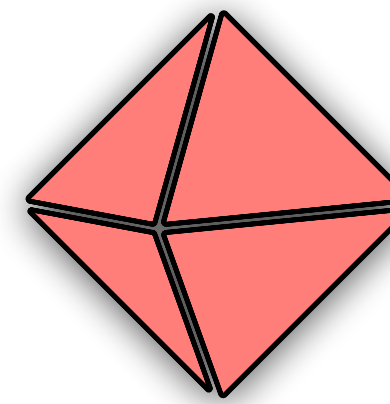

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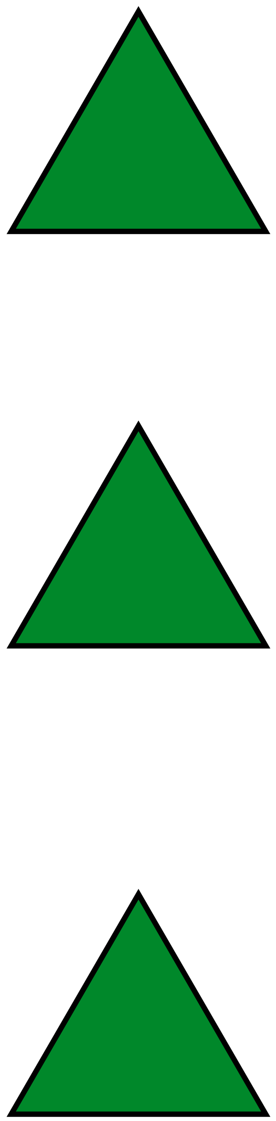

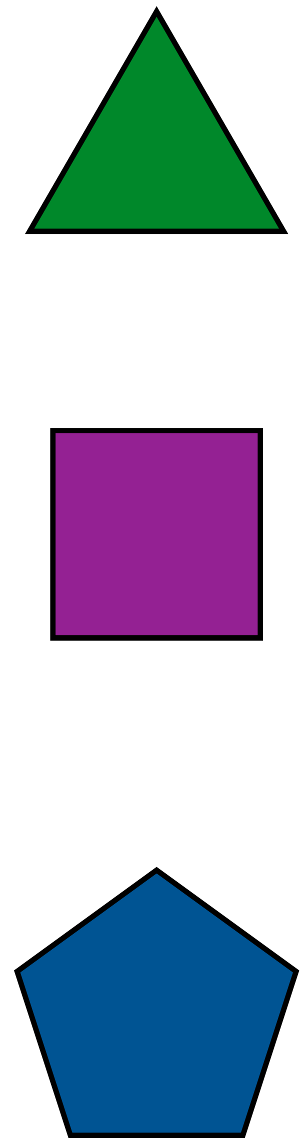

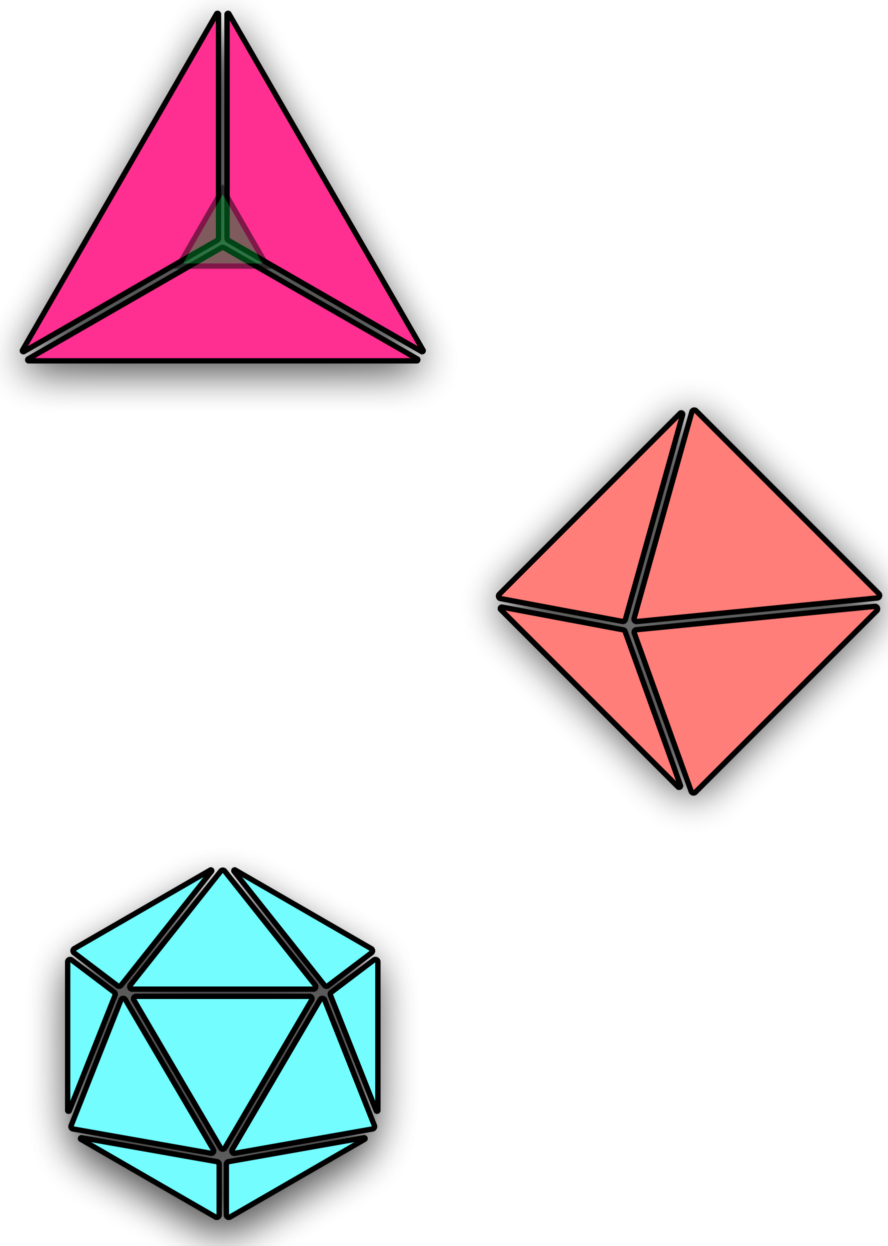

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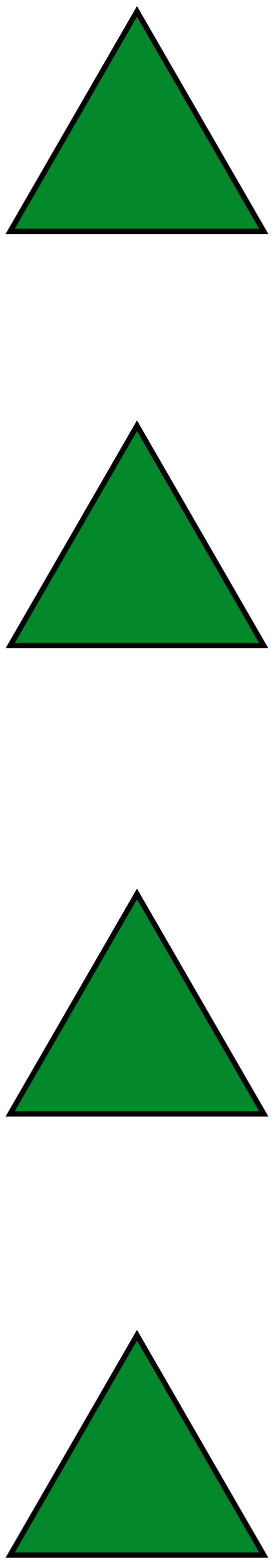

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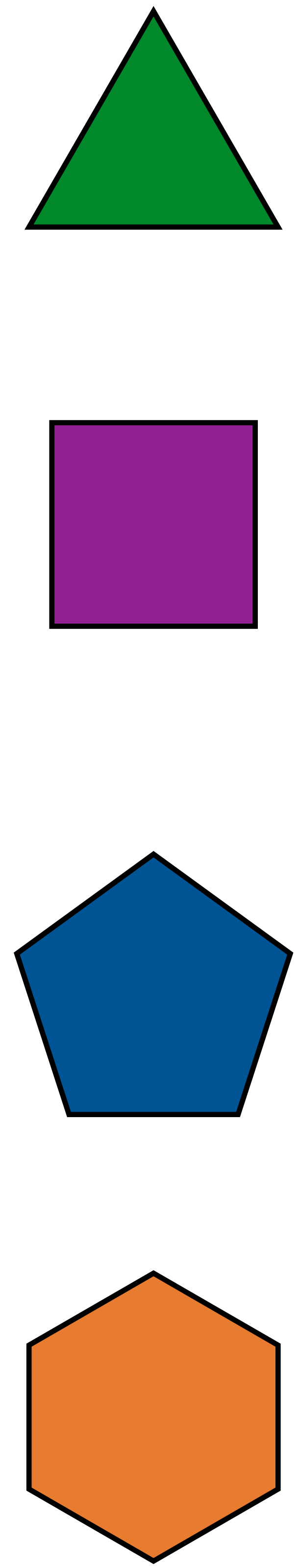

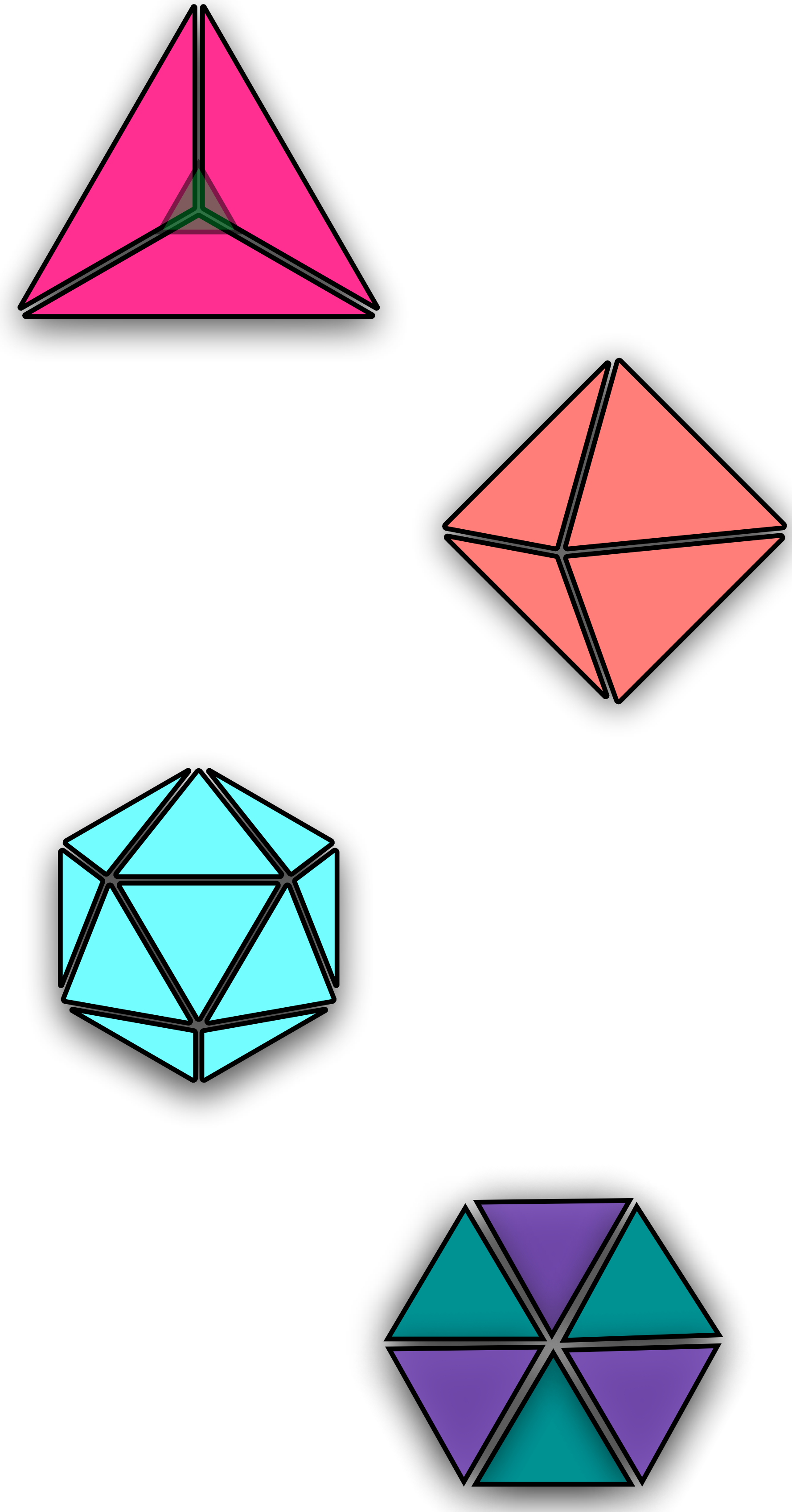

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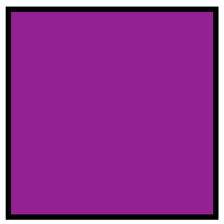

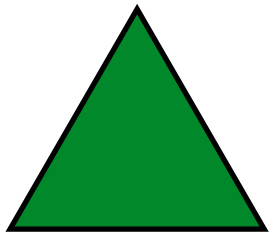

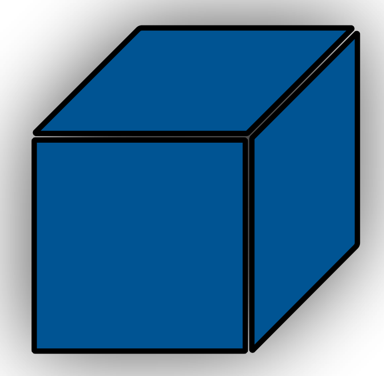

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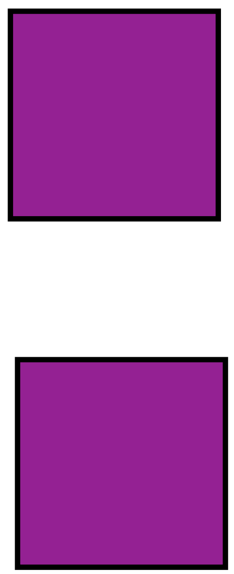

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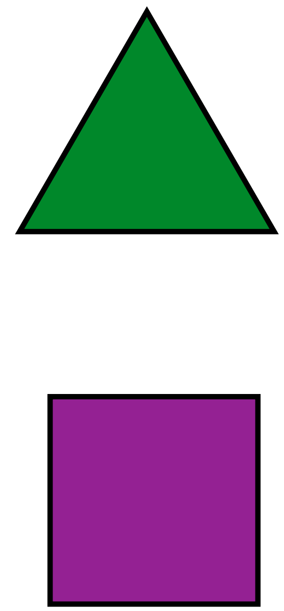

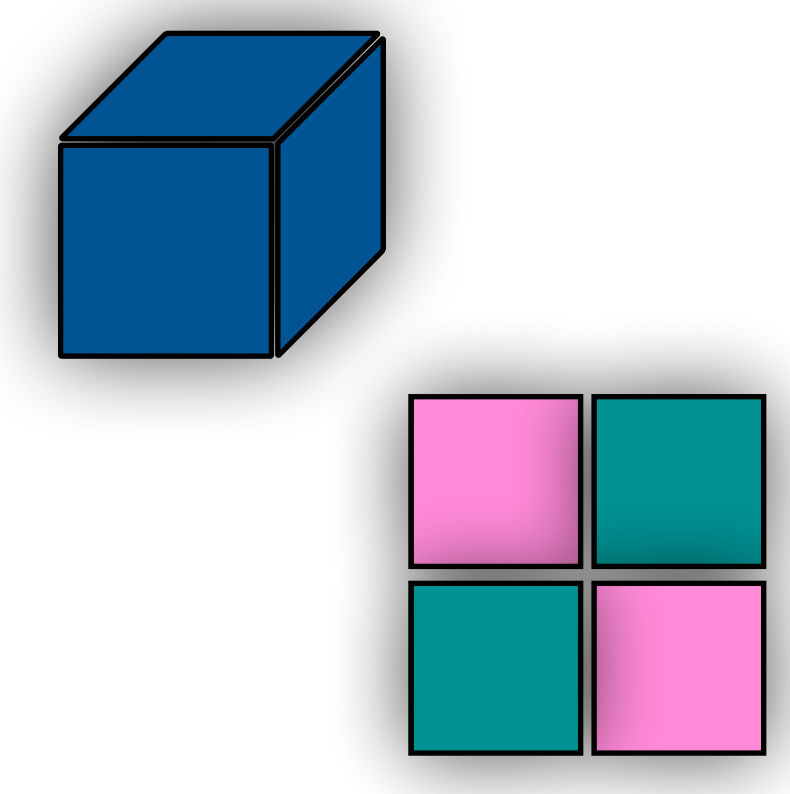

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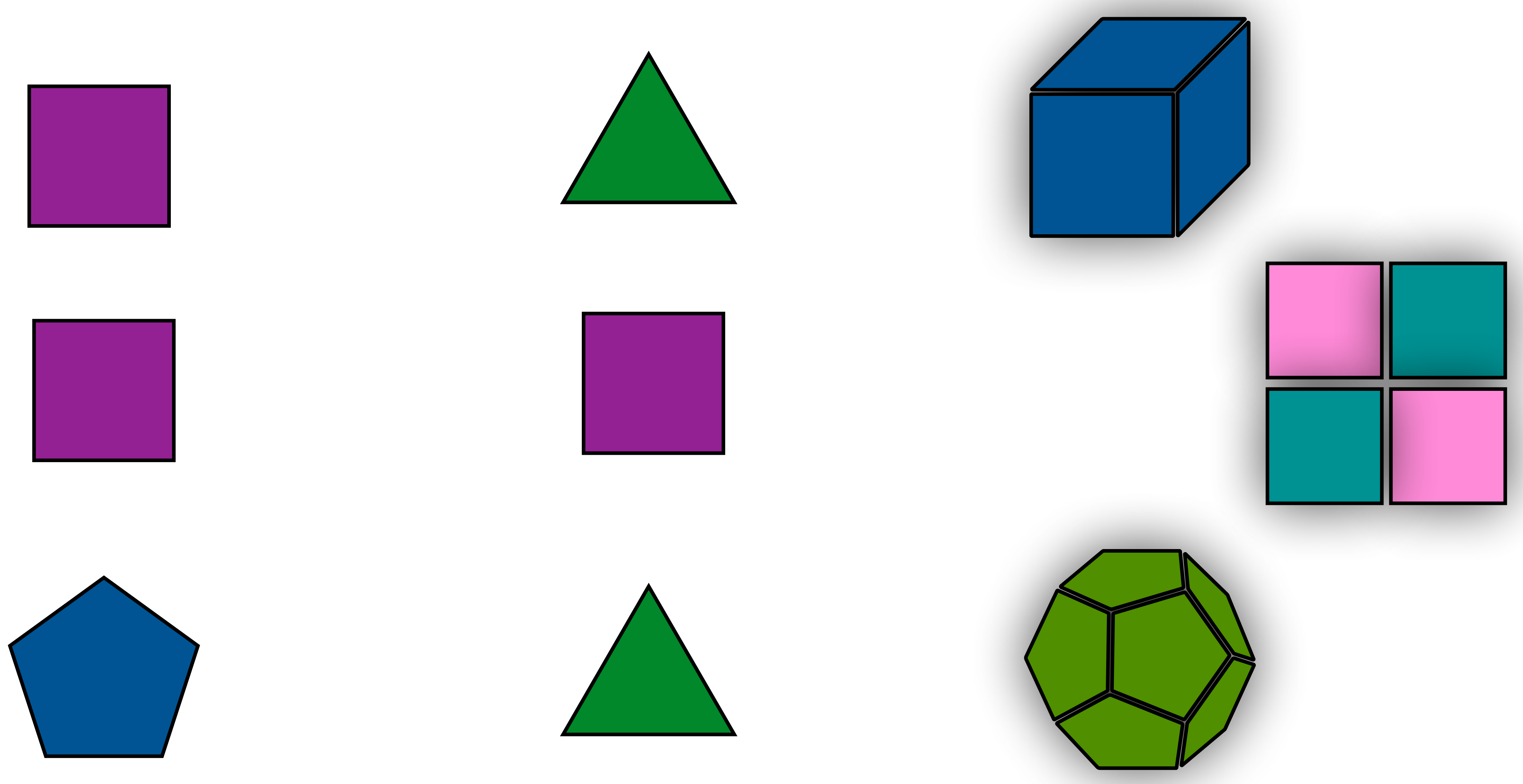

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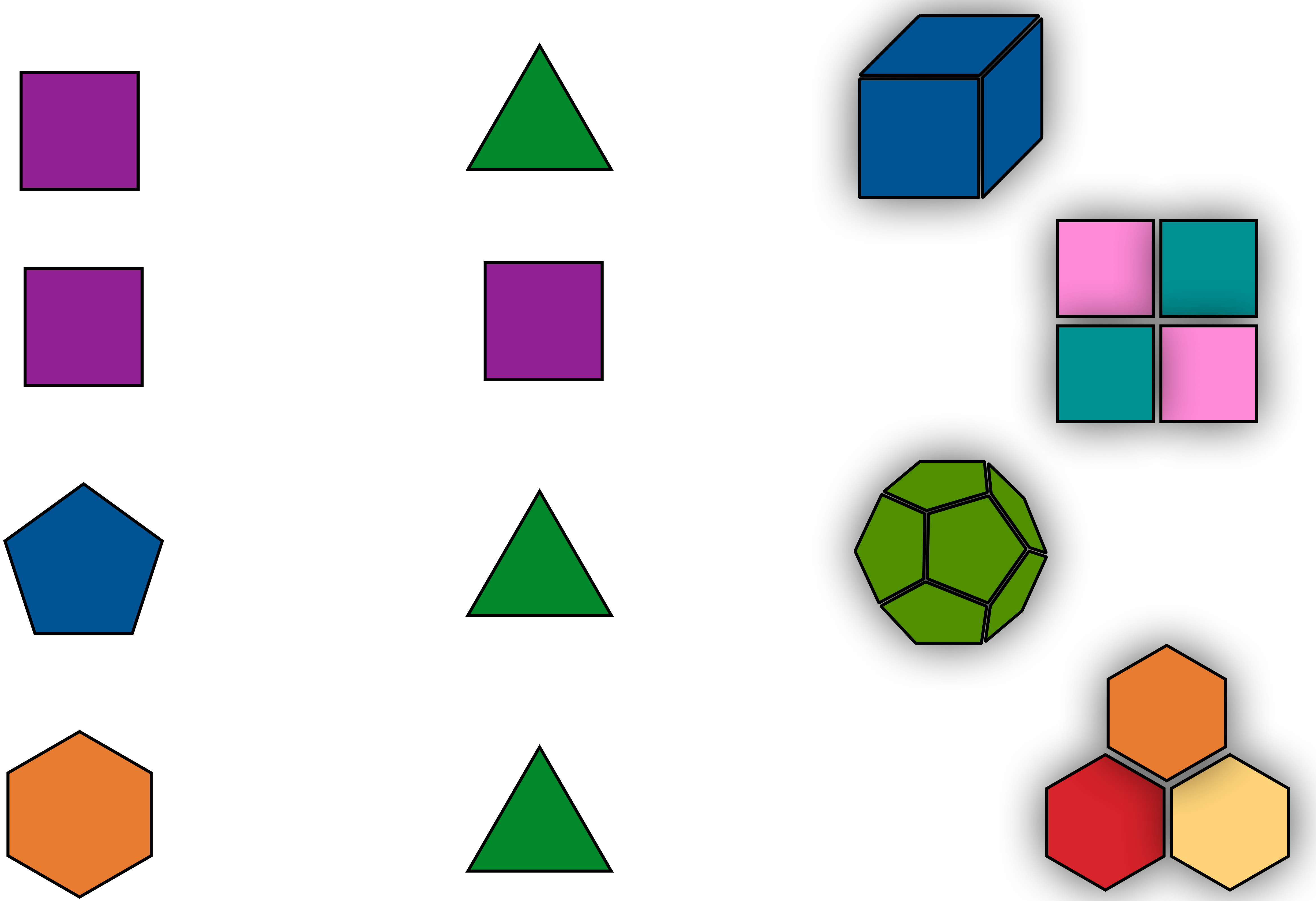
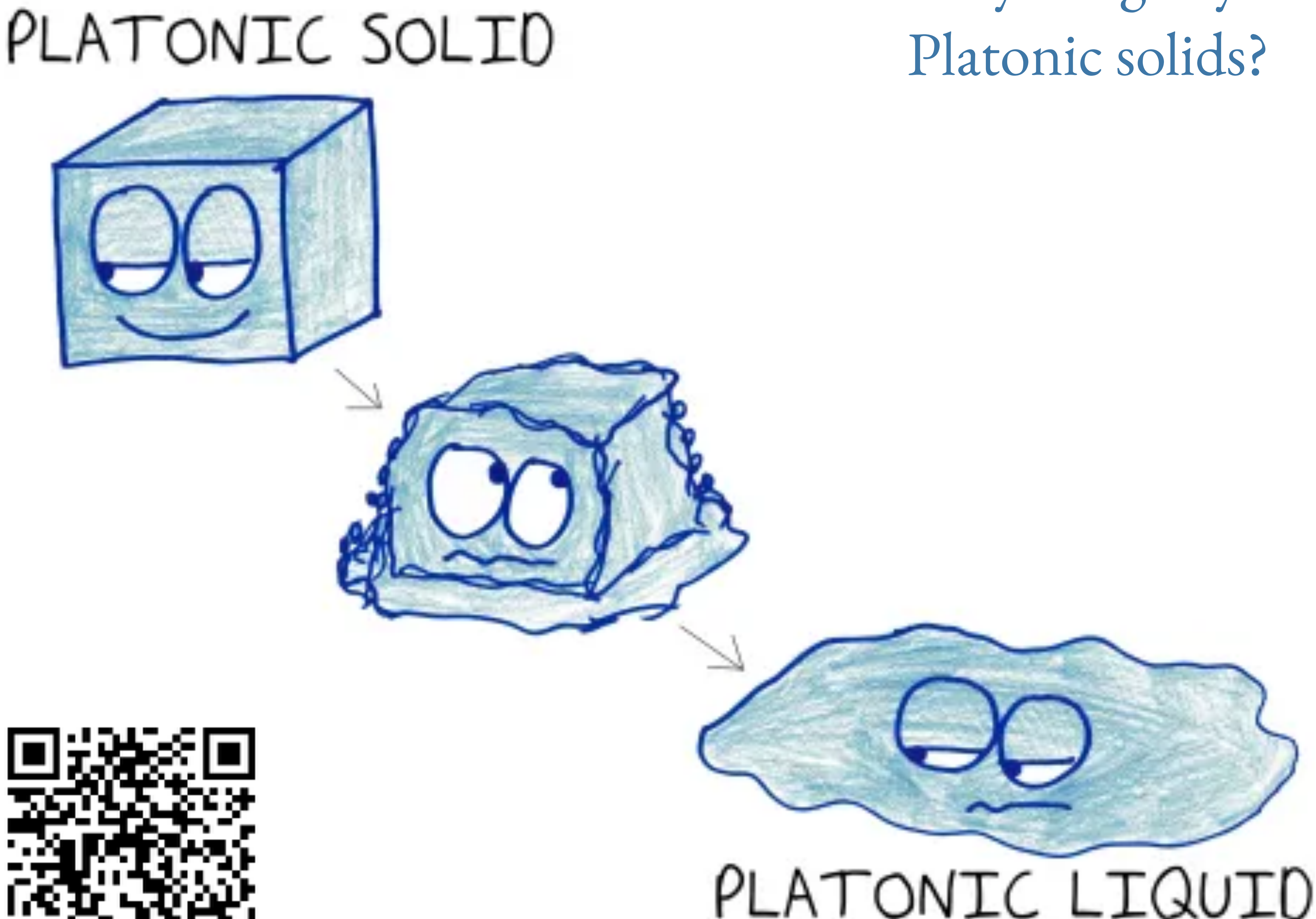


Is there anything beyond the Platonic solids?

Is there anything beyond the Platonic solids?

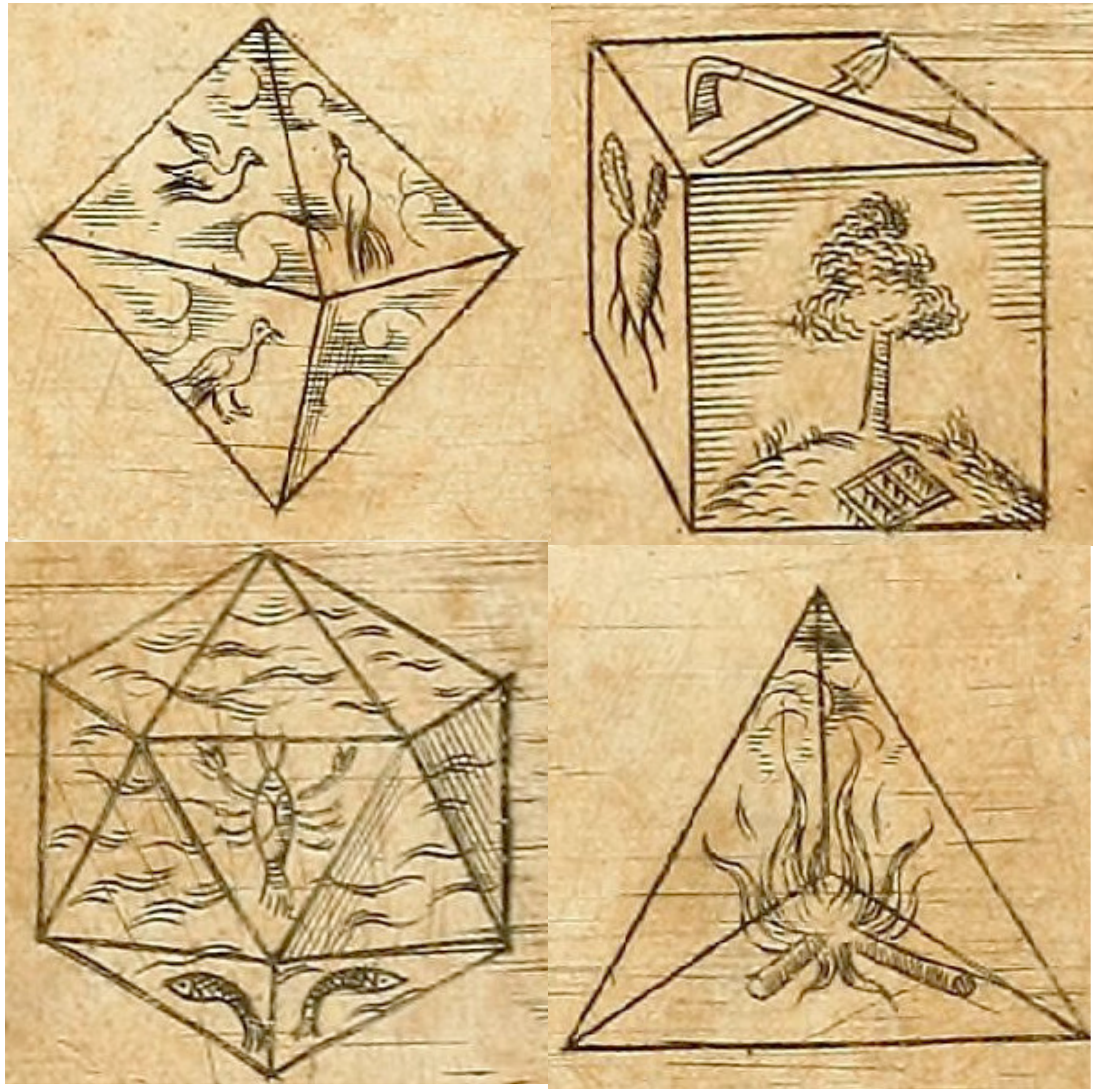


Credits: Math with bad drawings

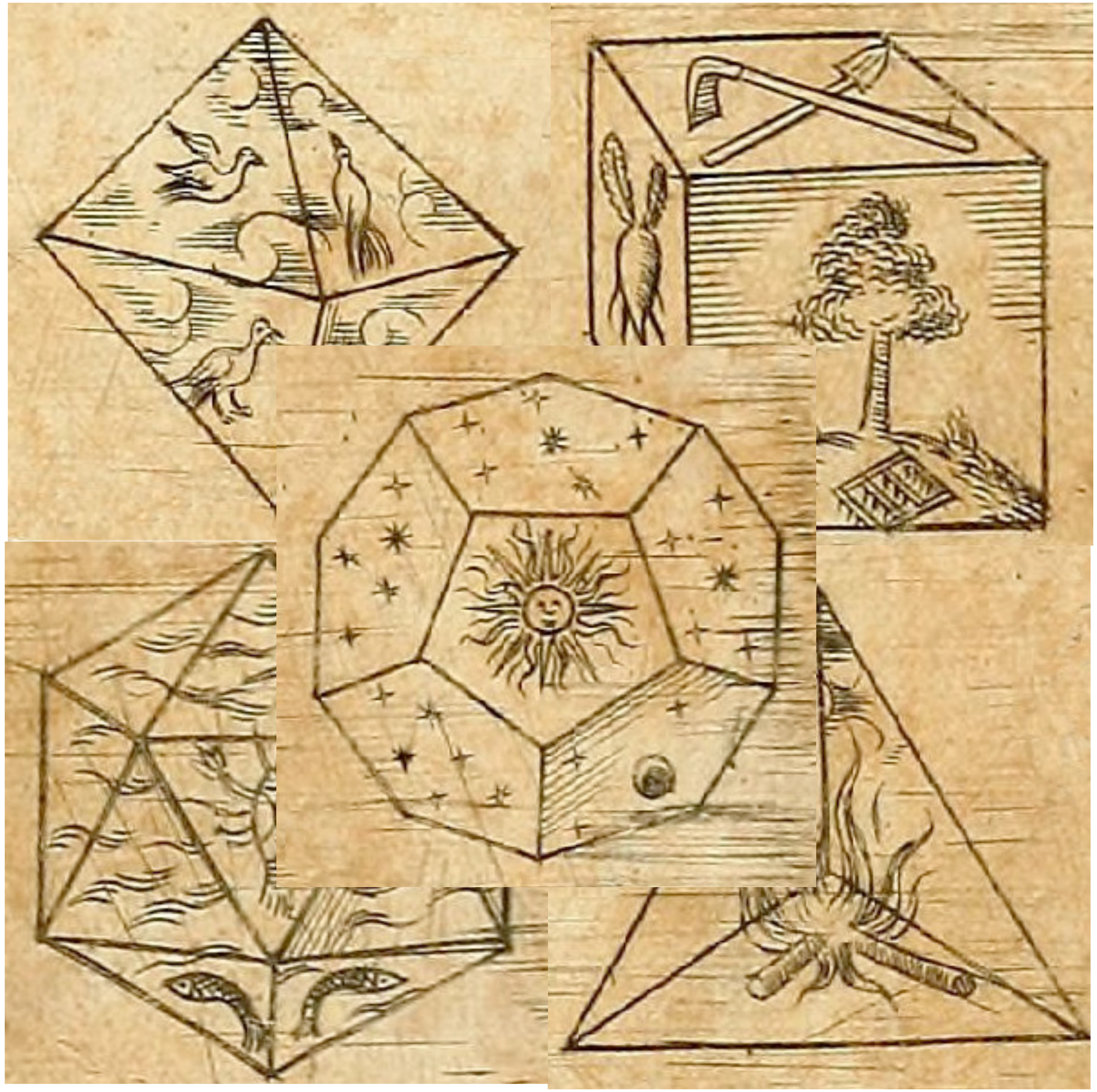
Greece | ~ 300 BC

Greece | ~ 300 BC

Europe | 1600s - 1800s



Harmonices Mundi, Johannes Kepler (1619)



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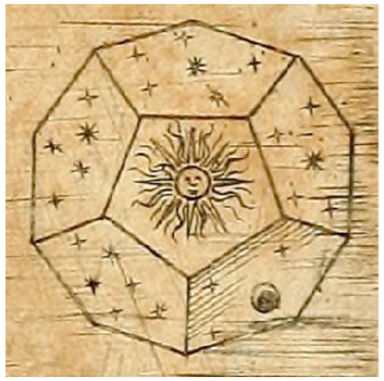
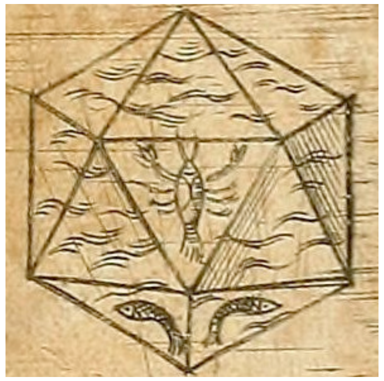
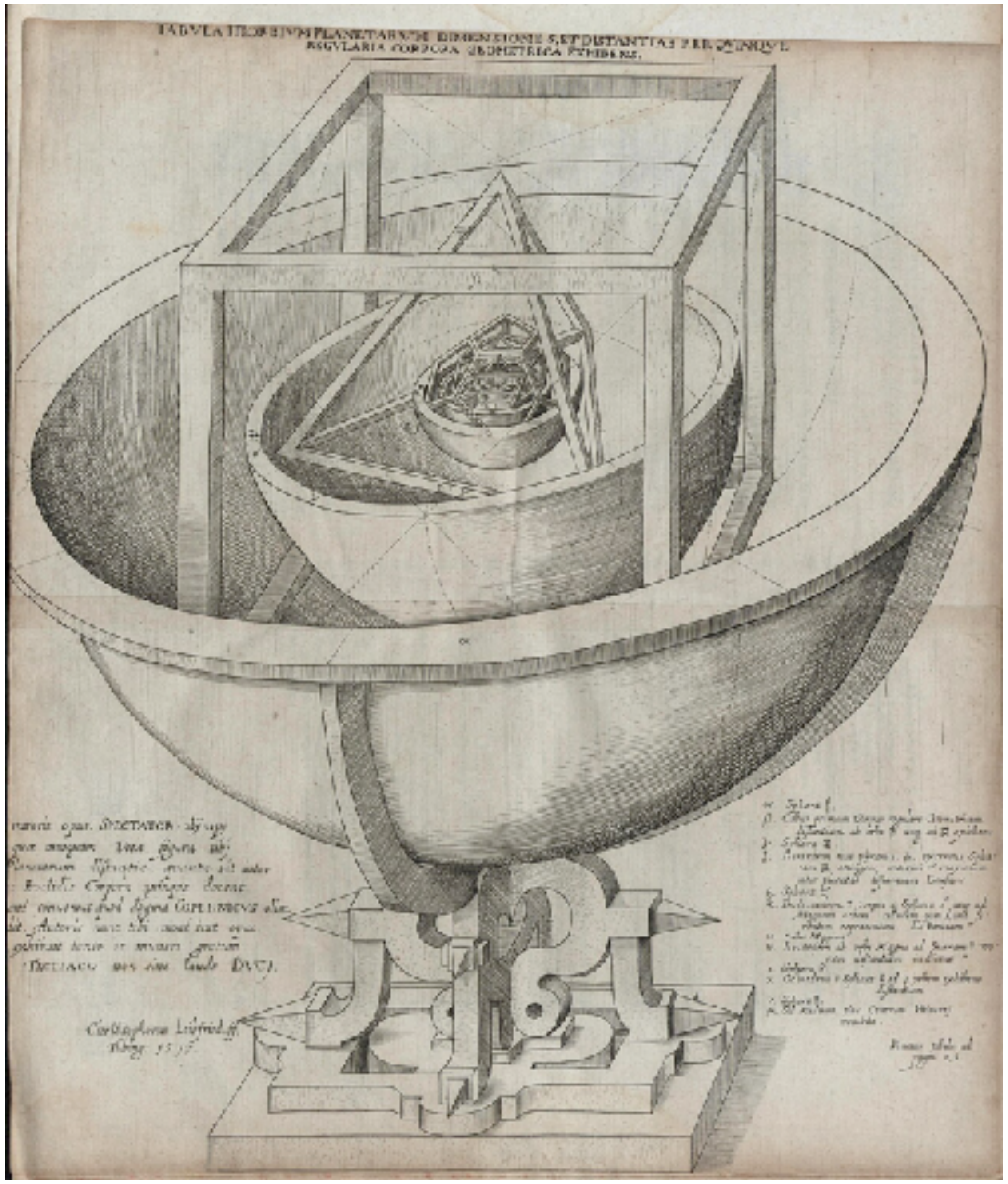
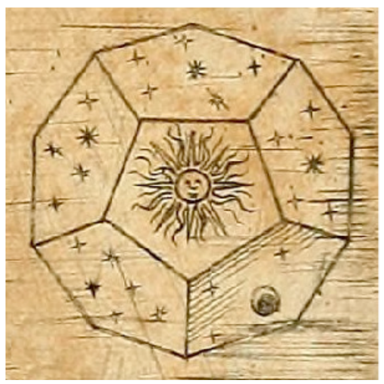
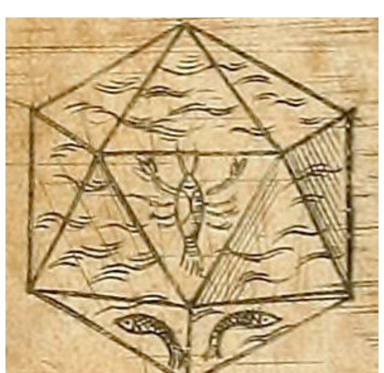
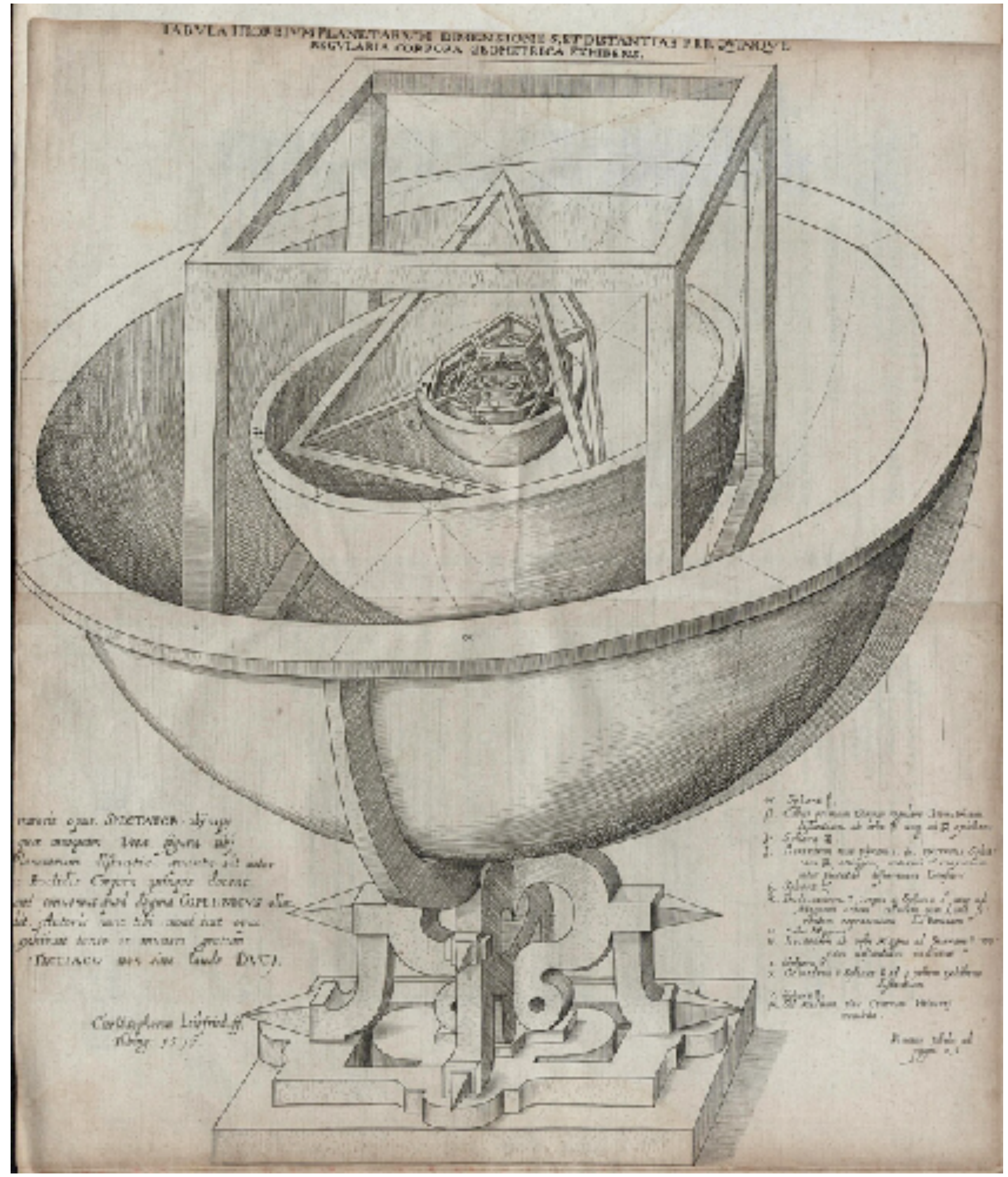


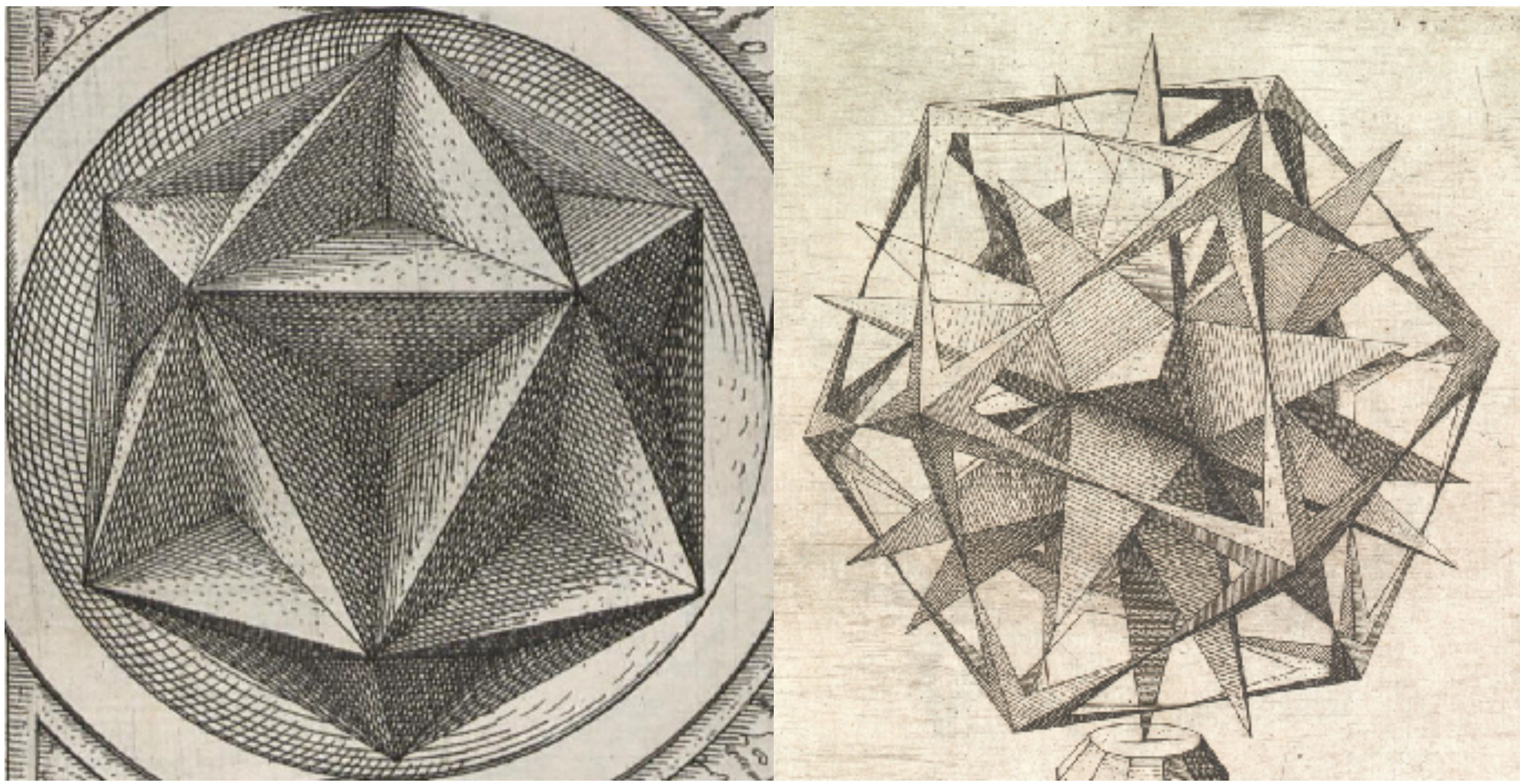




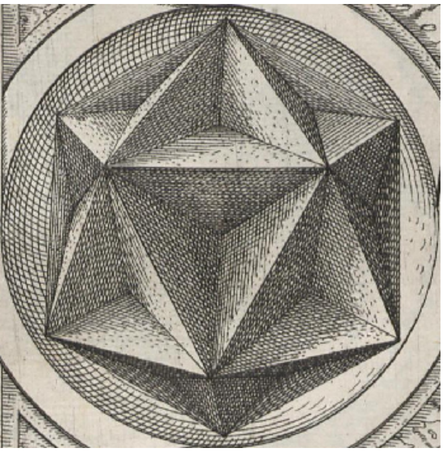


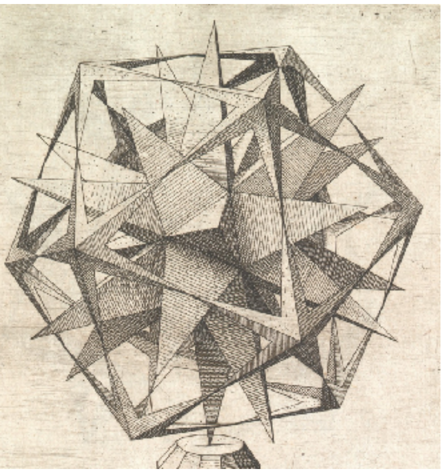
Mysterium Cosmographicum,
Johannes Kepler (1596)

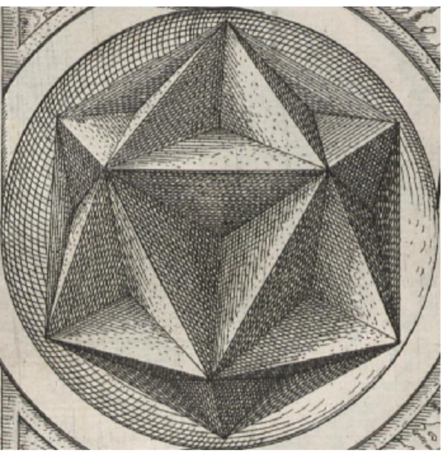
Mosaic in St. Mark's Venice,
Paolo Uccello, 15th century

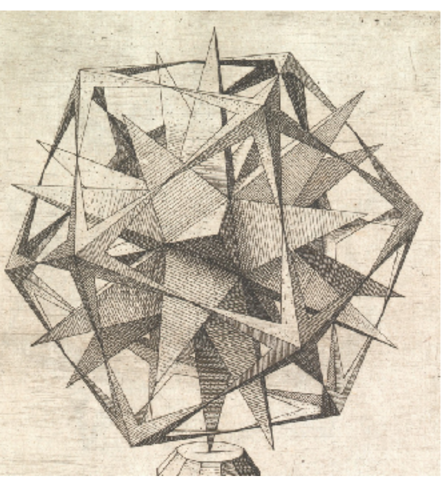


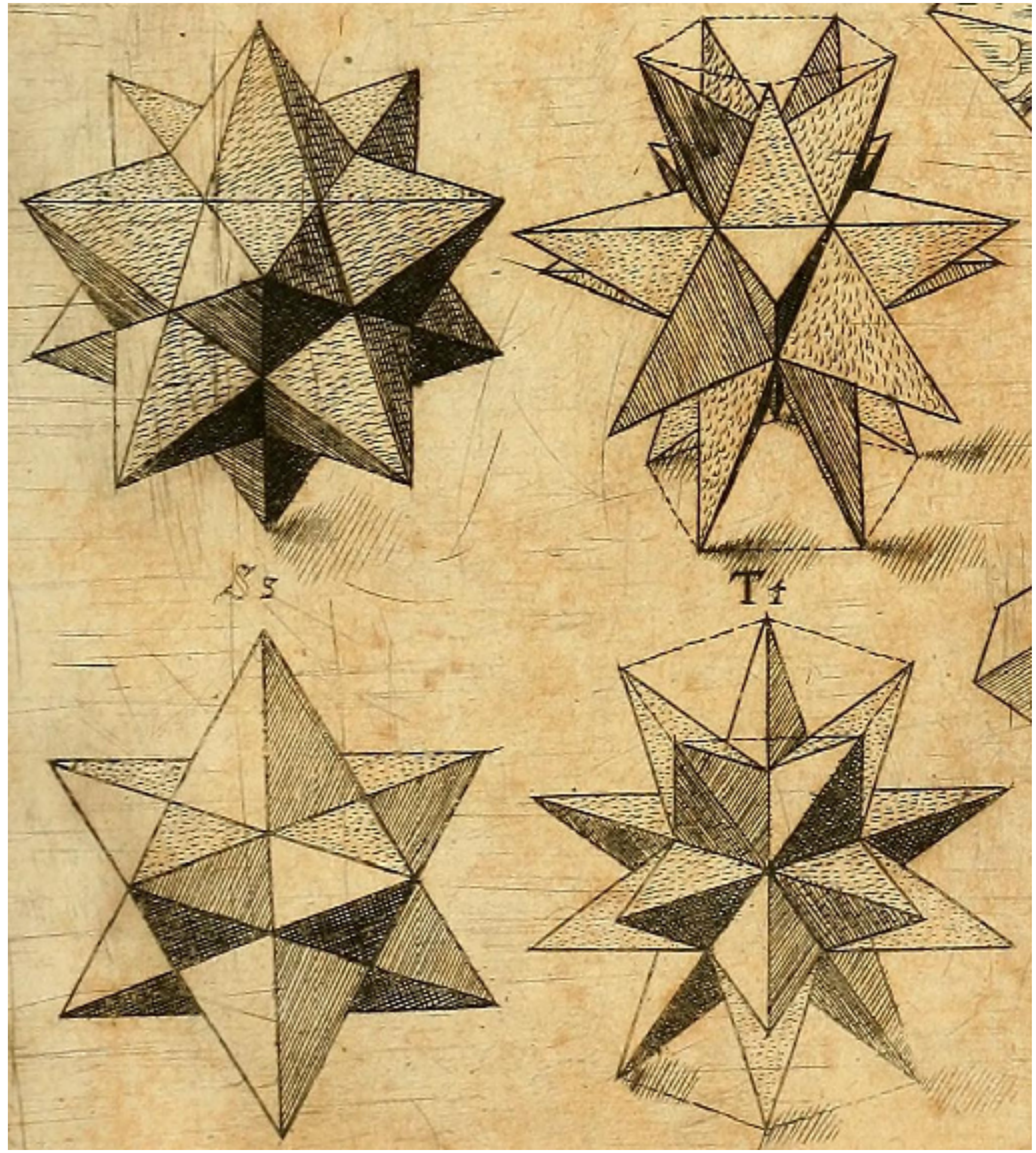
Perspectiva corporum regularium,
Wenzel Jamnitzer (1568)











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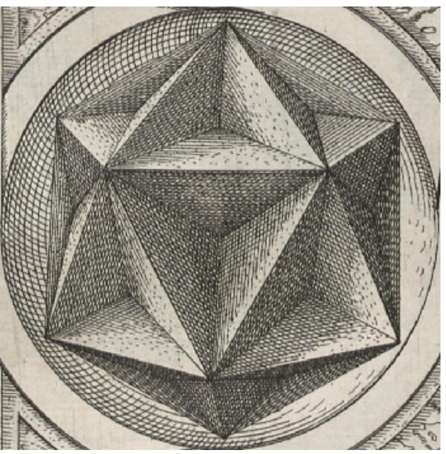
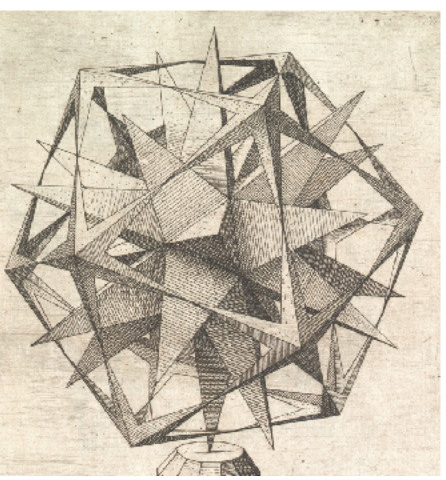
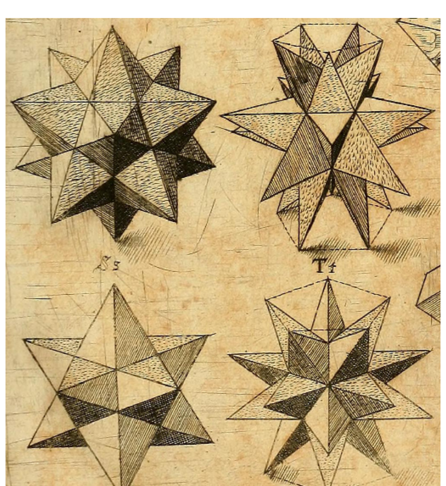
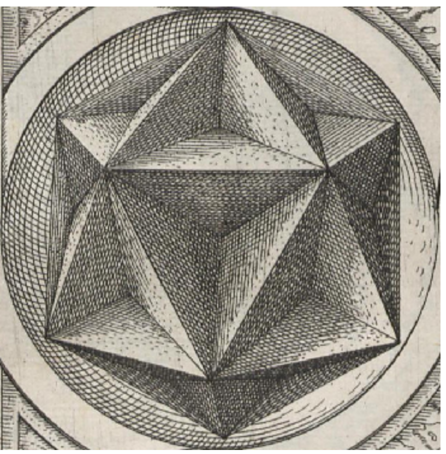
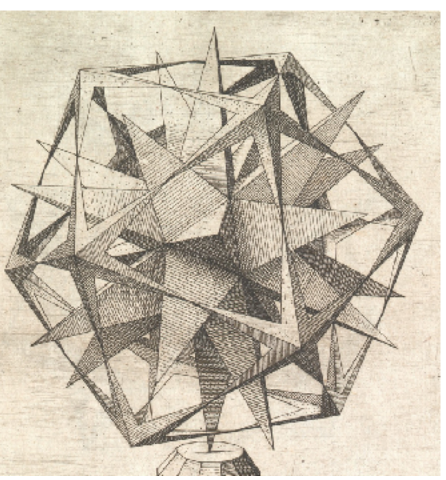
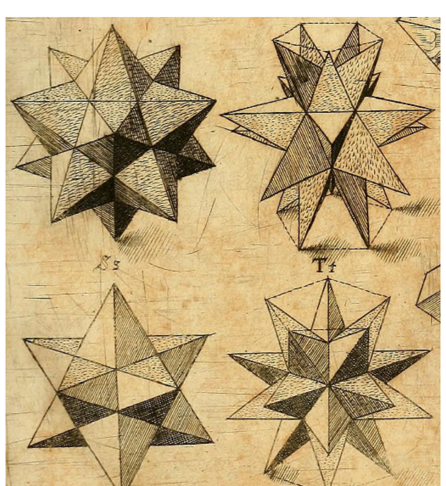
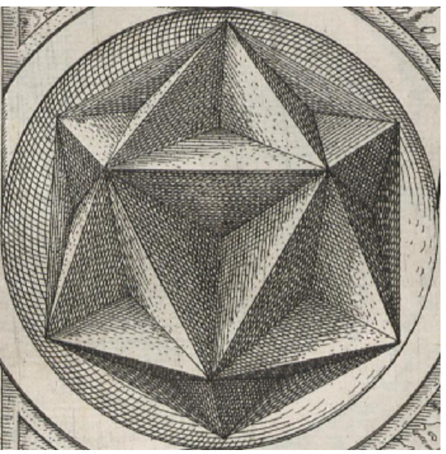


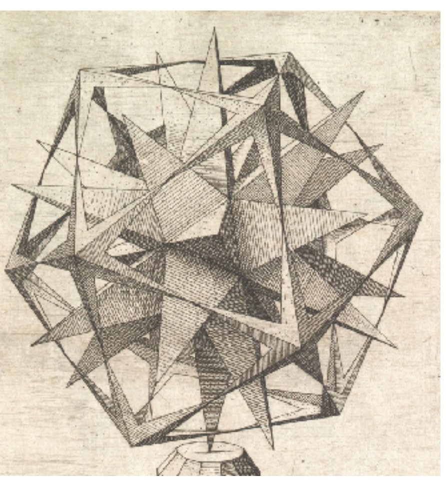


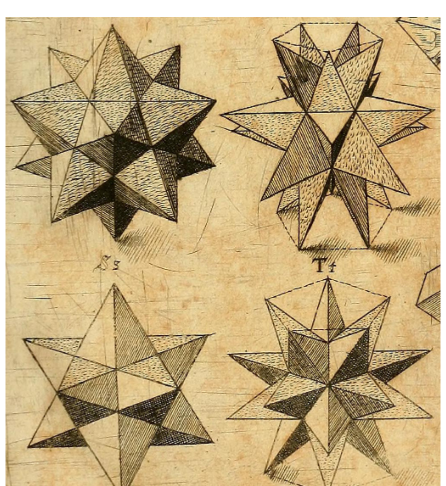



A Mexican tourist being a weirdo in Venice,
The weirdo, 2023







Regular polyhedra

They are built by glueing polygons (faces) along their edges (two per edge)

I. All the faces must be equal.
II. Every face is a regular polygon.
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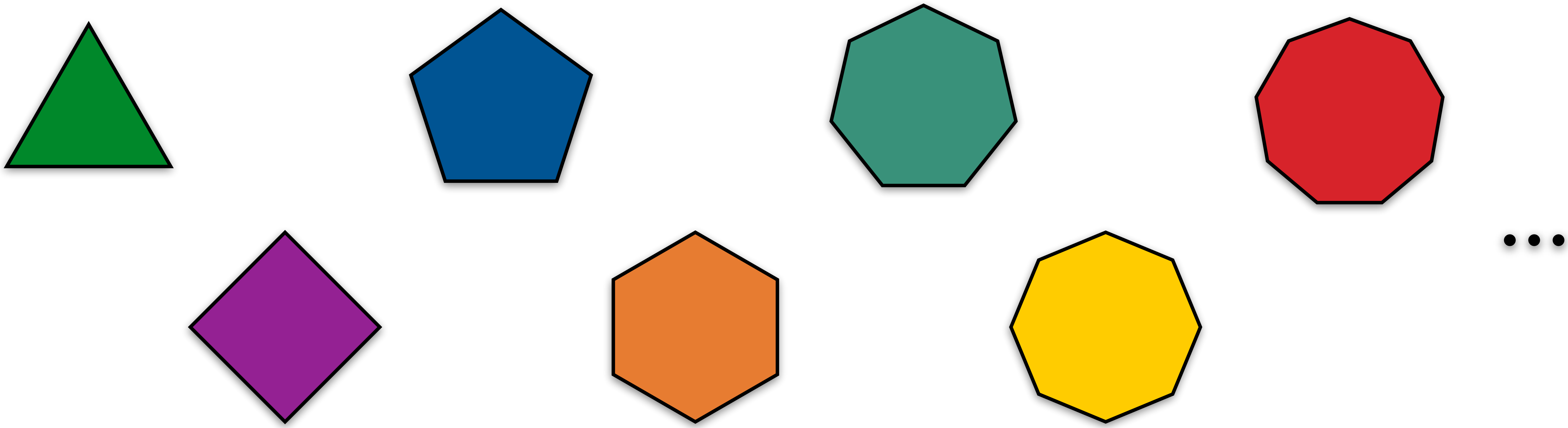
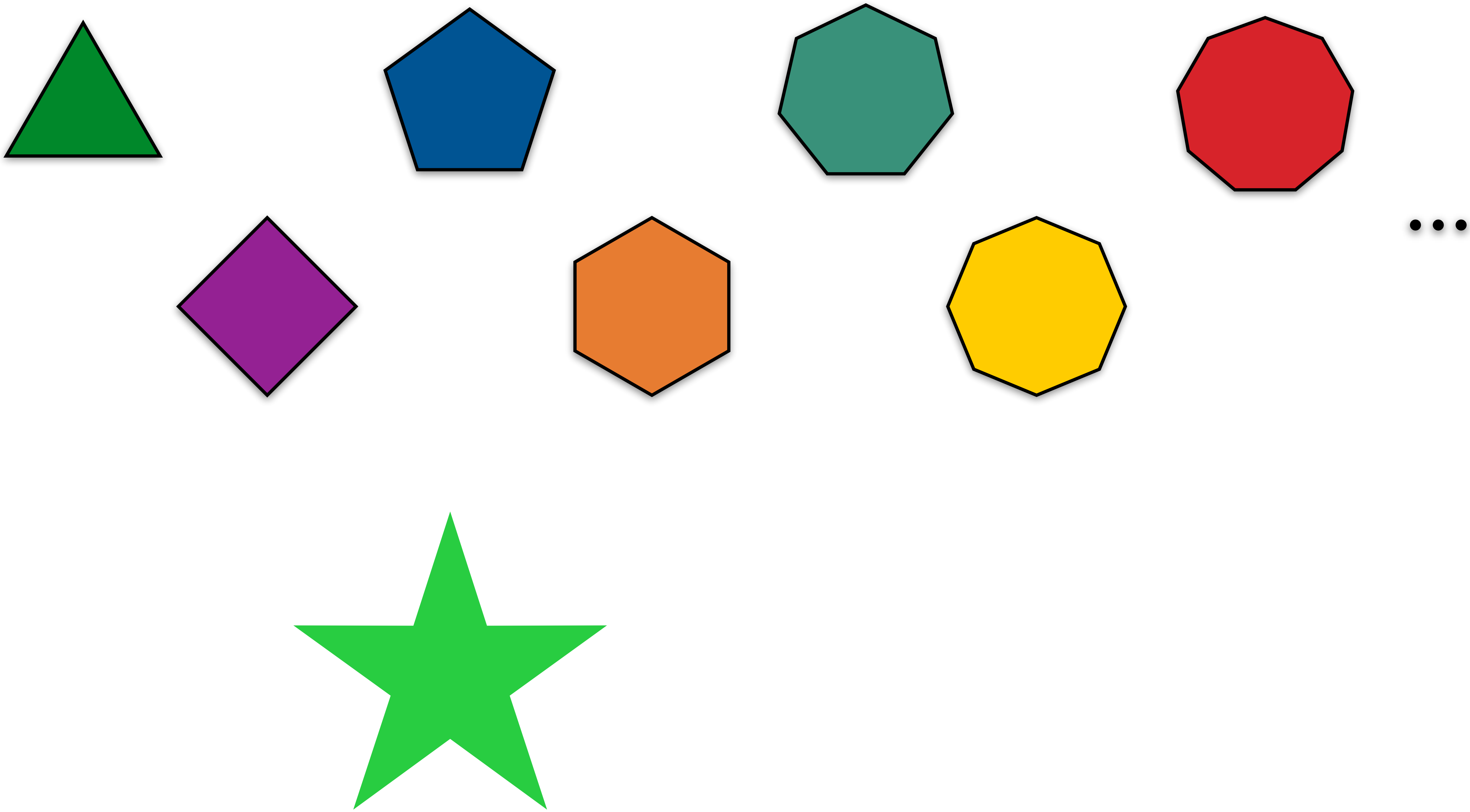
Regular polygons

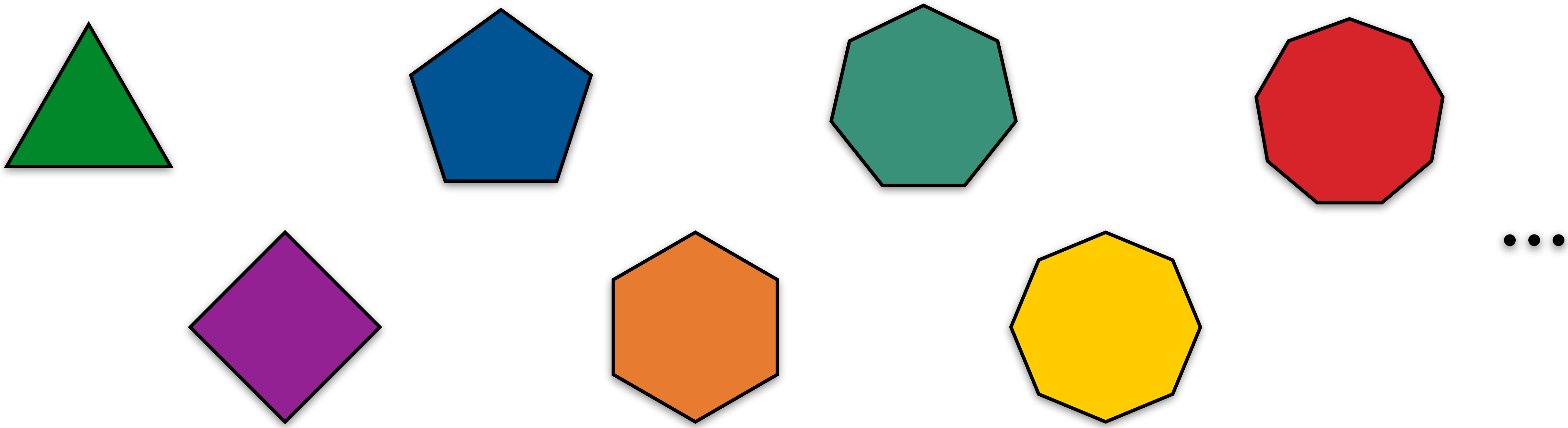
Regular polygons



Regular polygons

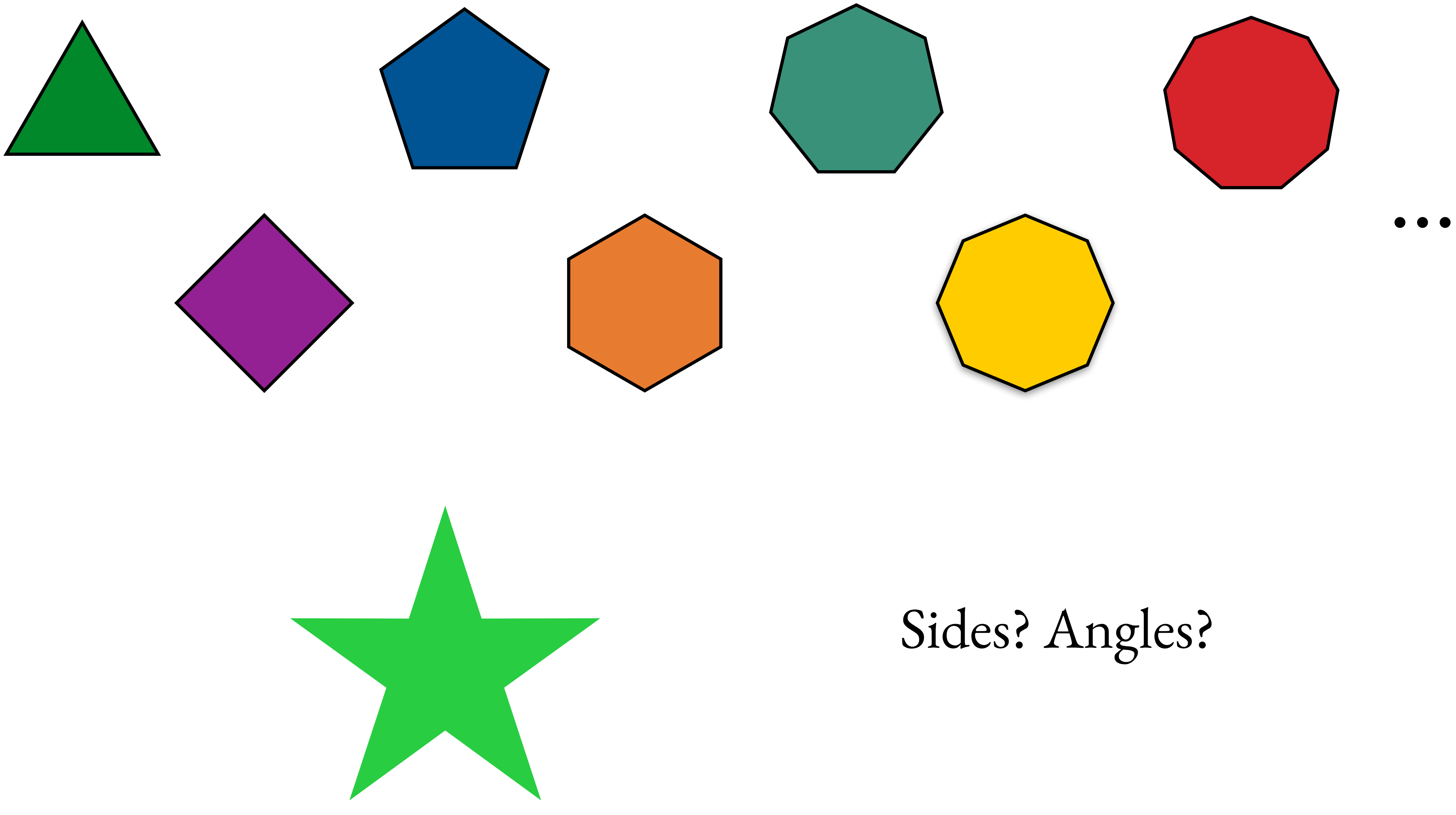


Regular polygons



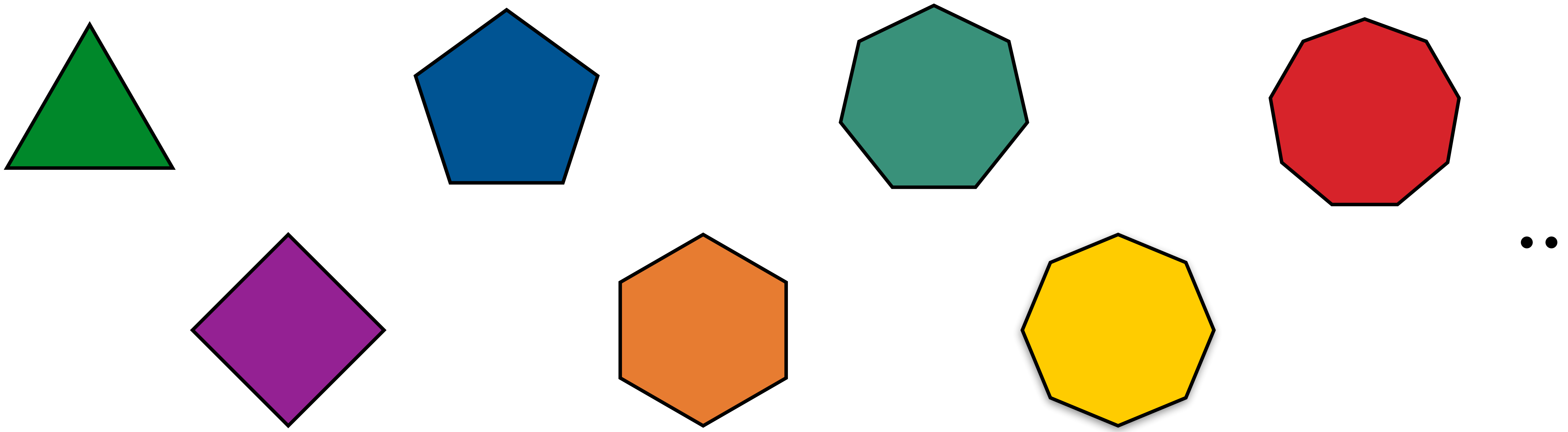
Sides? Angles?

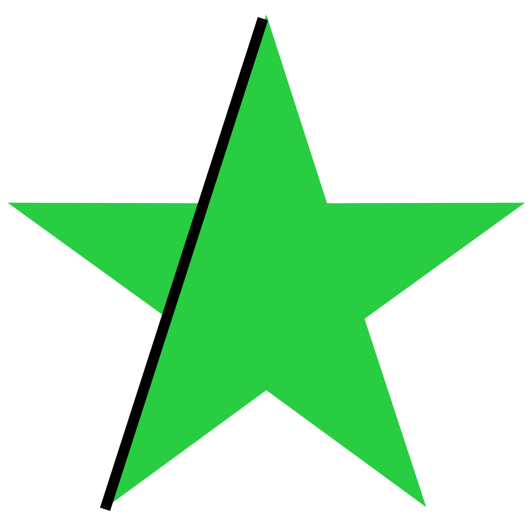
Regular polygons



Sides? Angles?

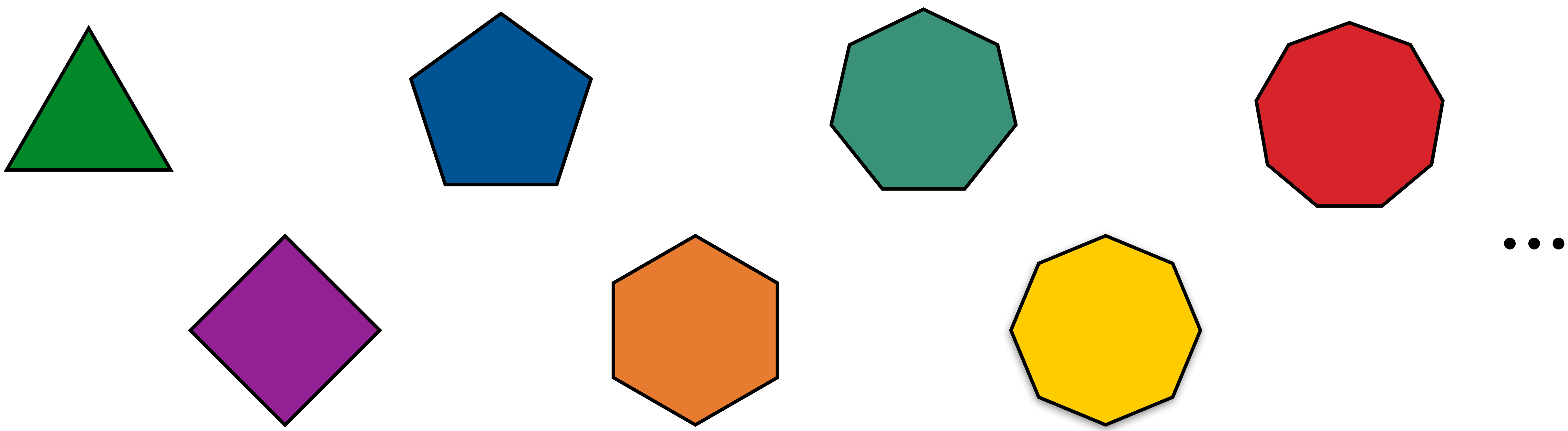
Regular polygons

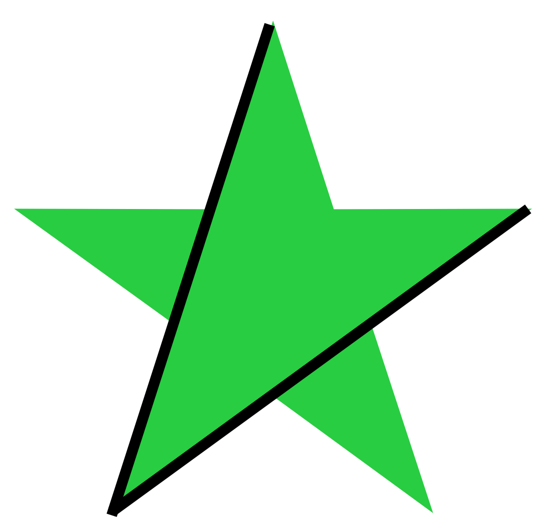




Sides? Angles?

Regular polygons





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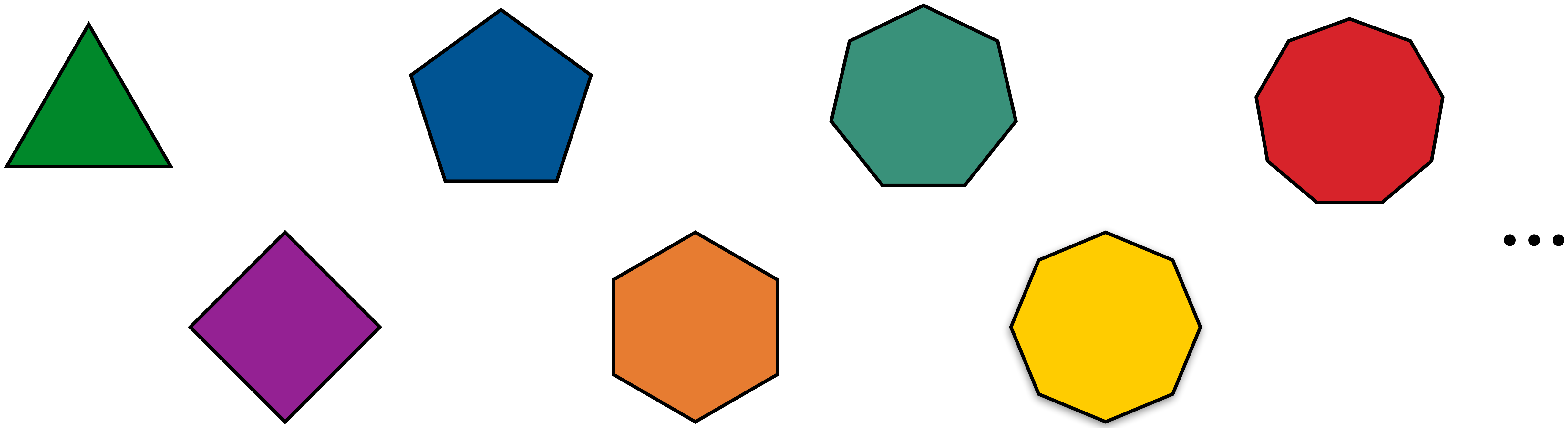

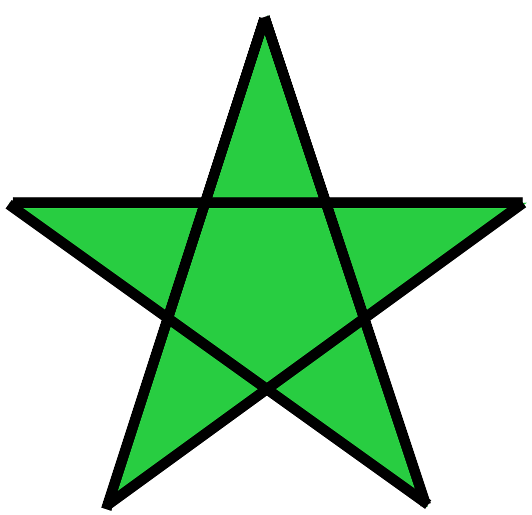

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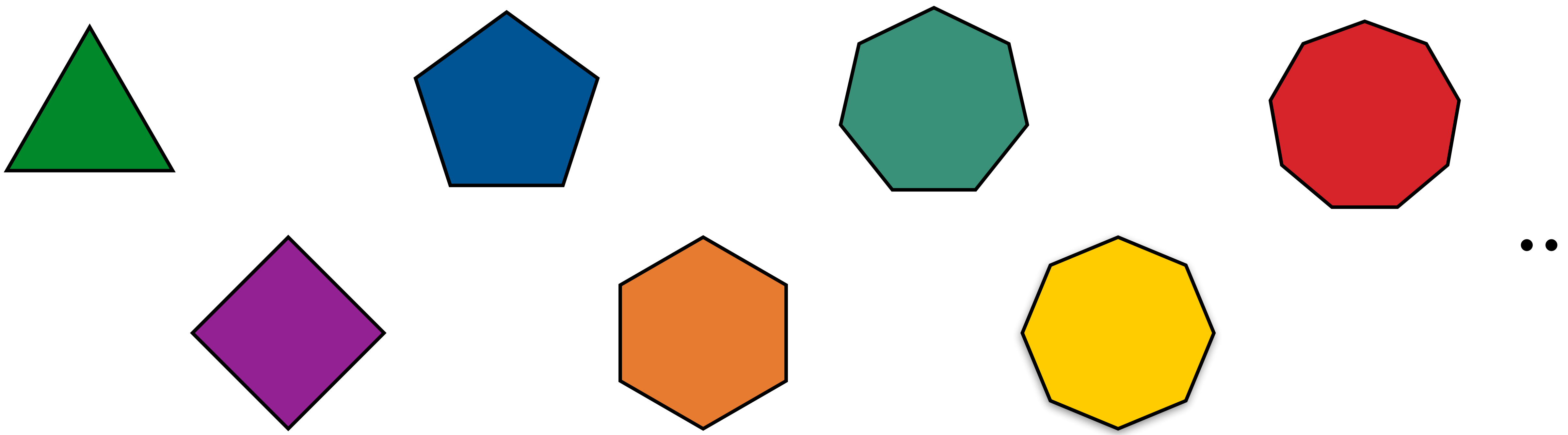
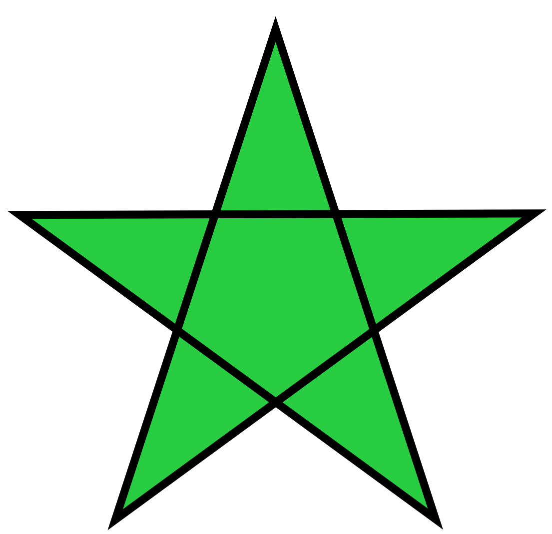
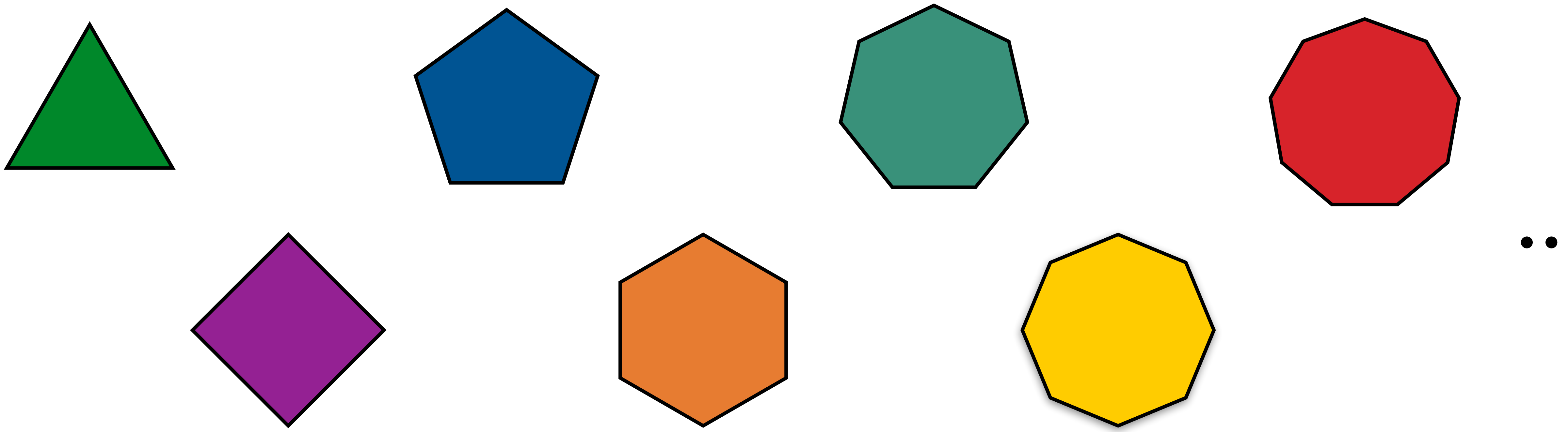
Regular polygons

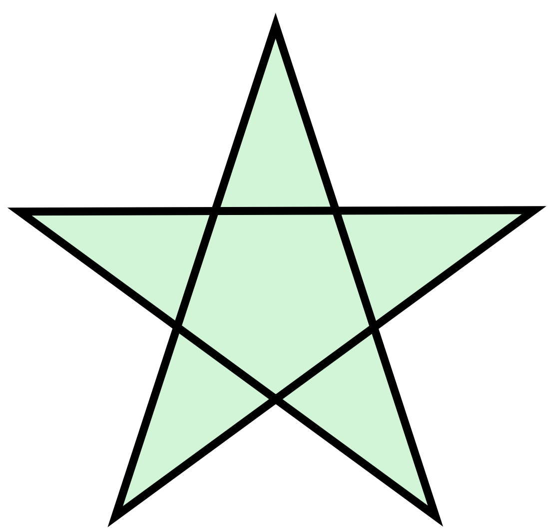




Sides? Angles?

Regular polygons





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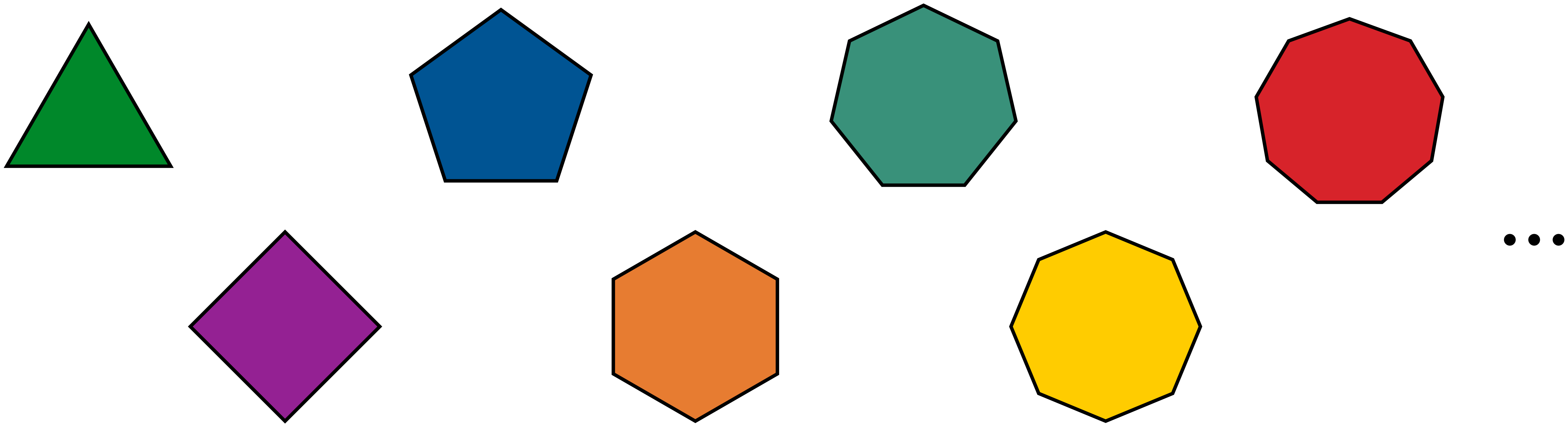
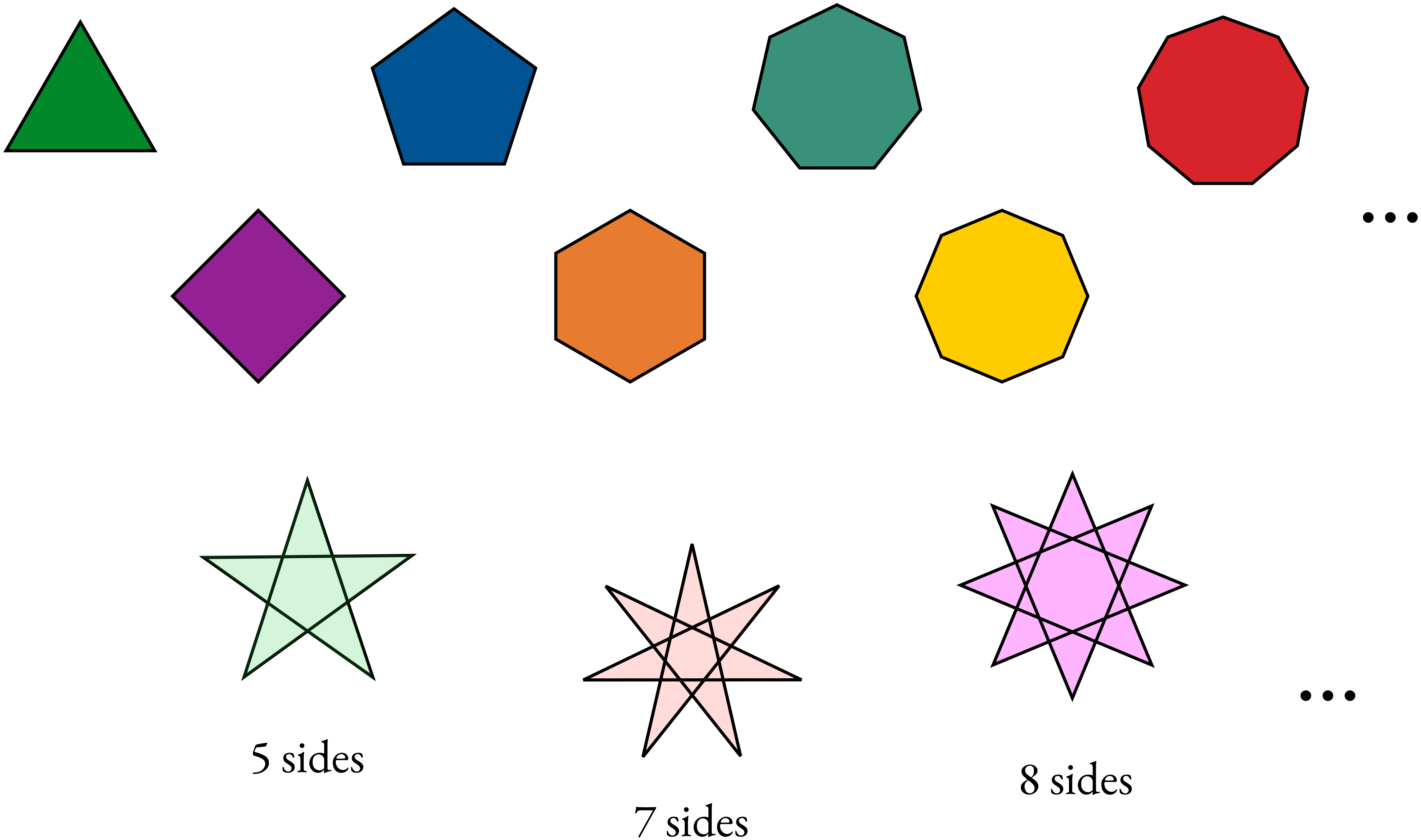
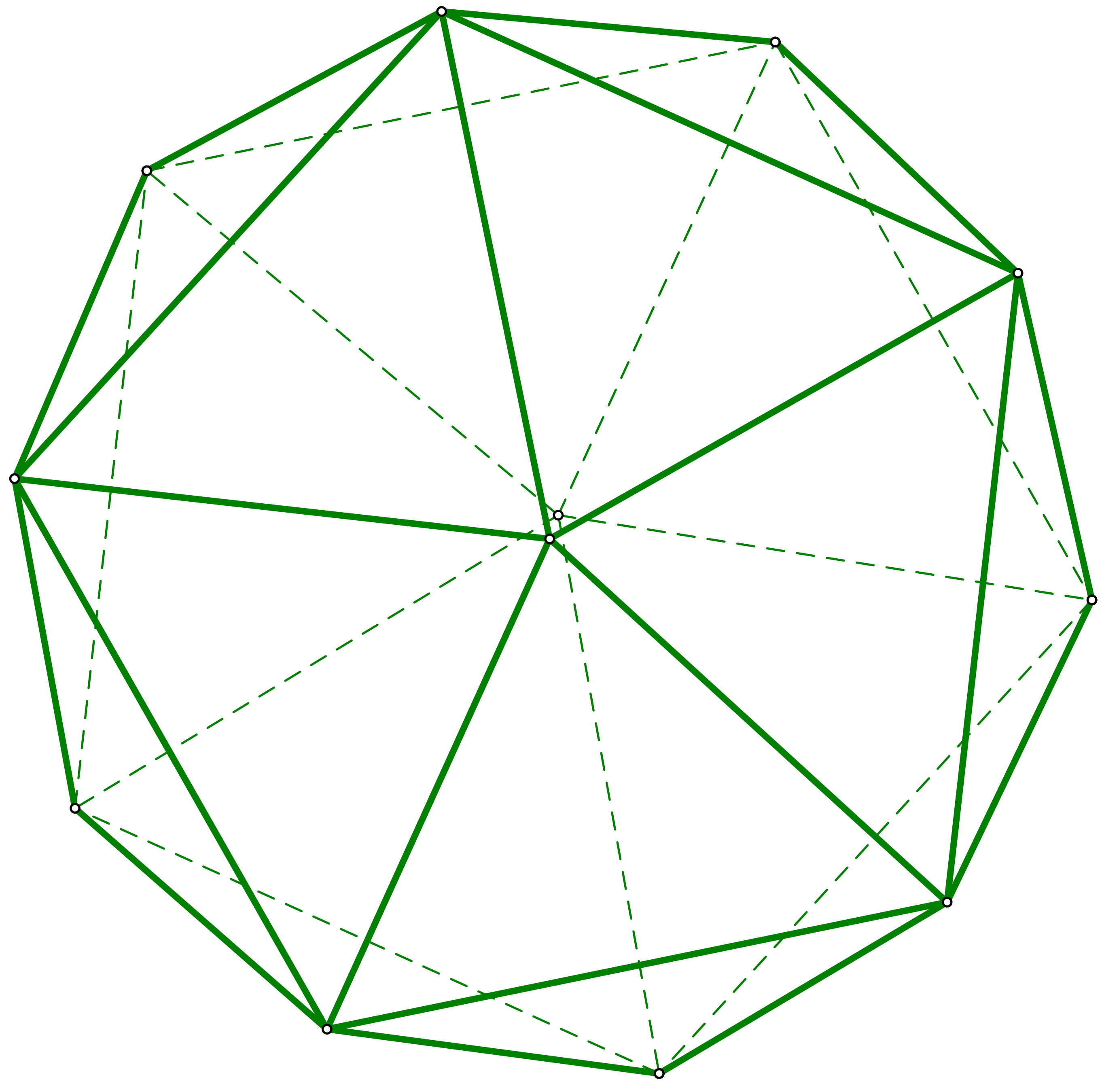


Regular polygons



Regular polyhedra



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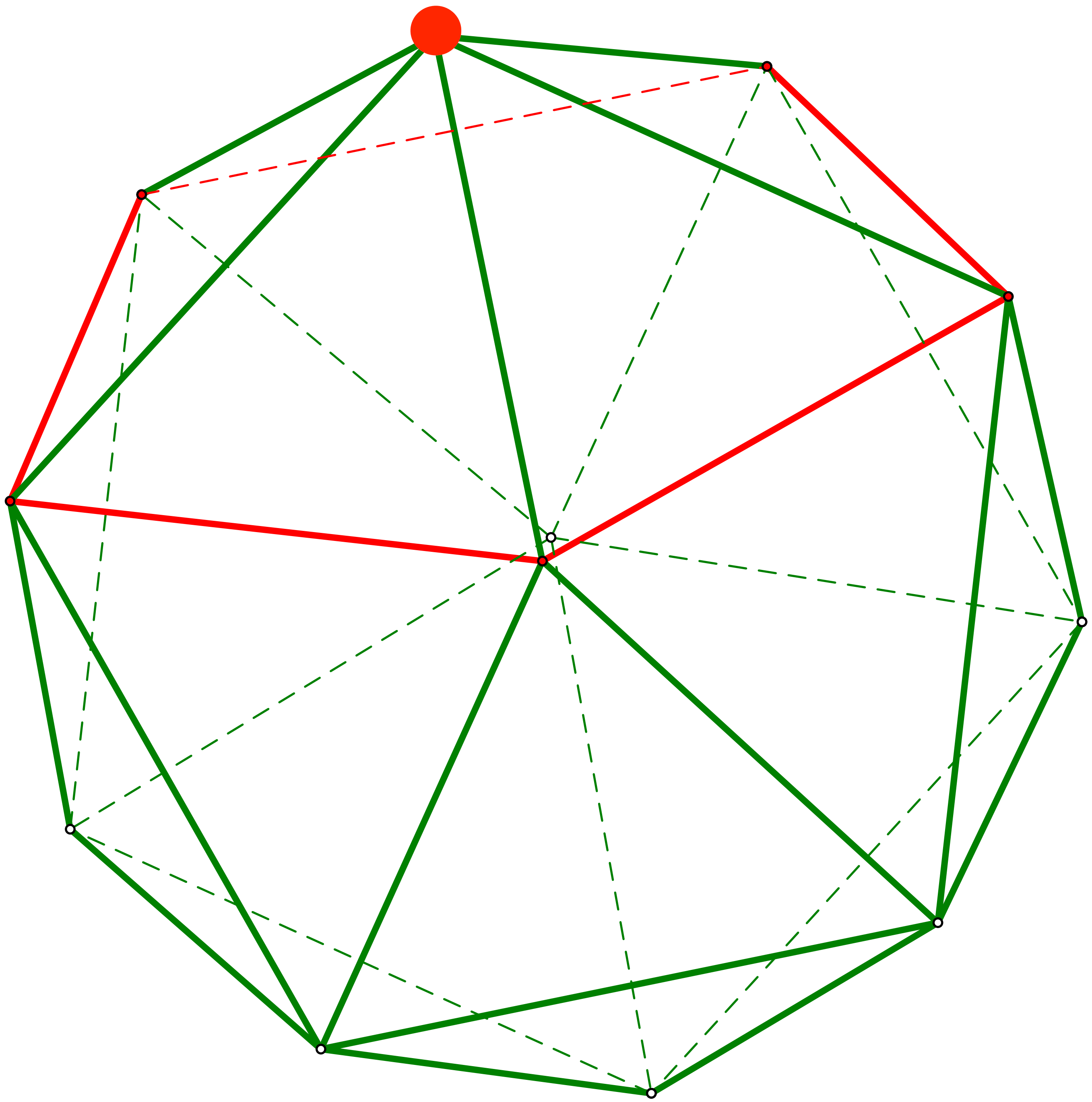

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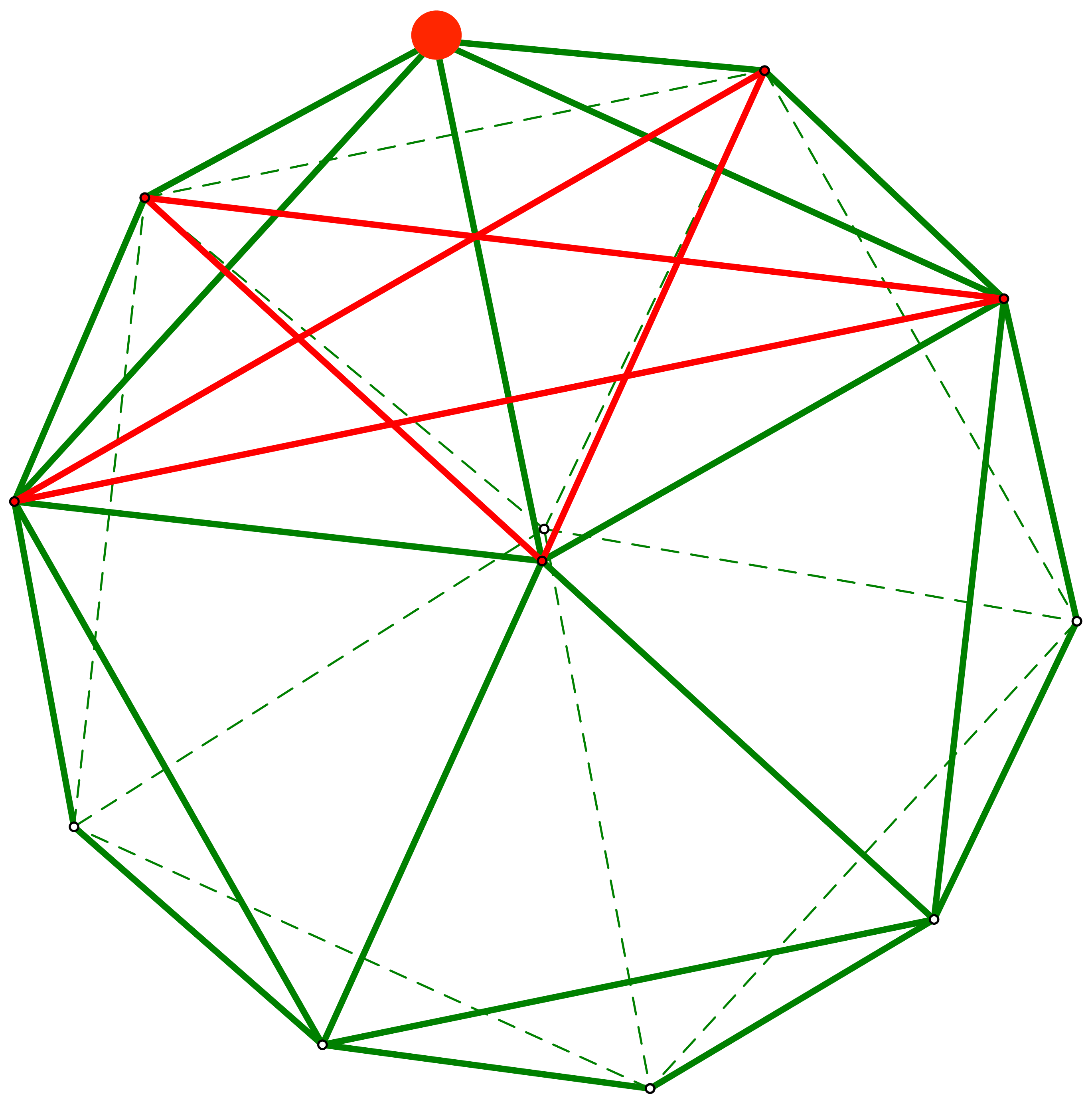

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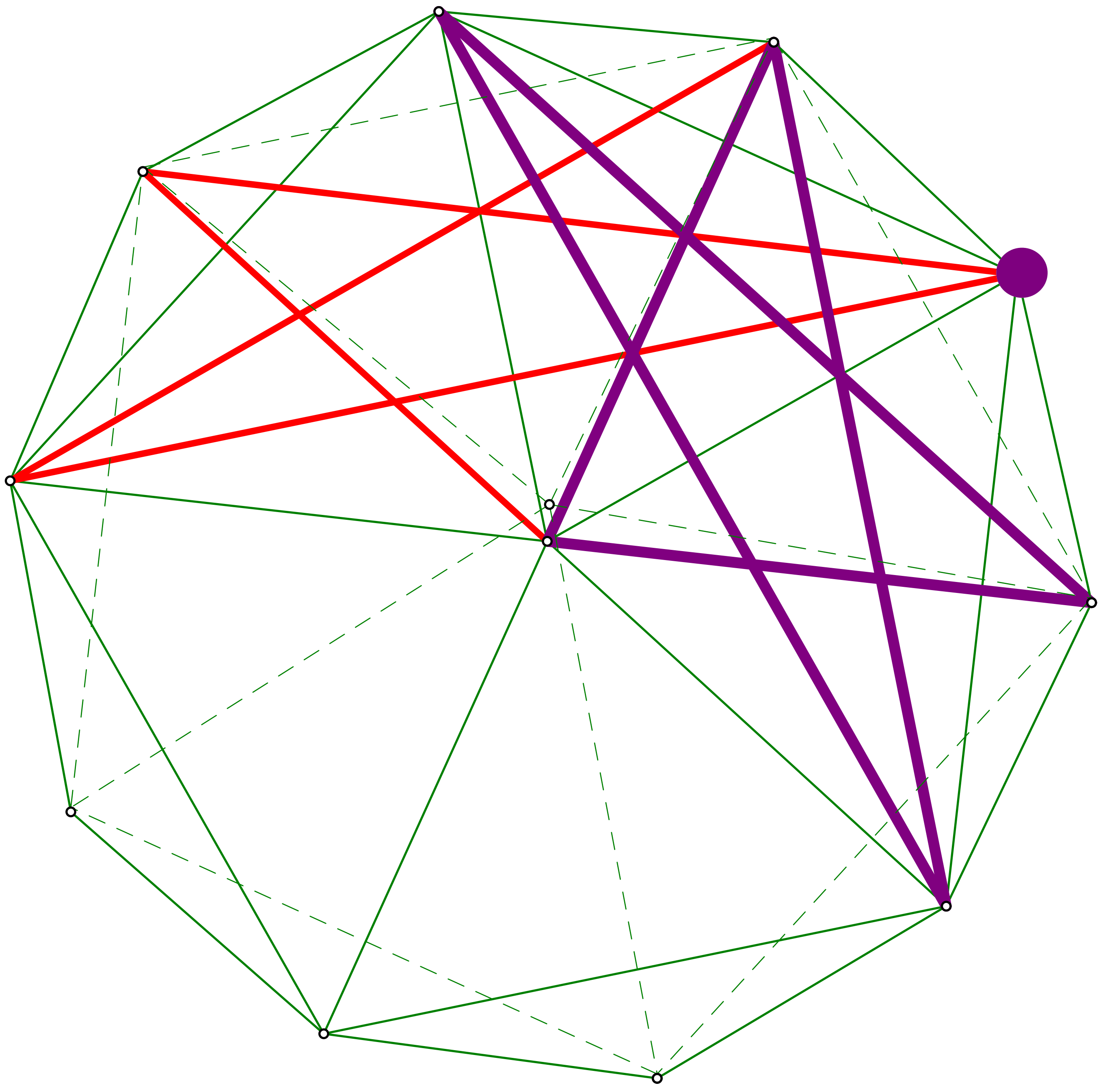

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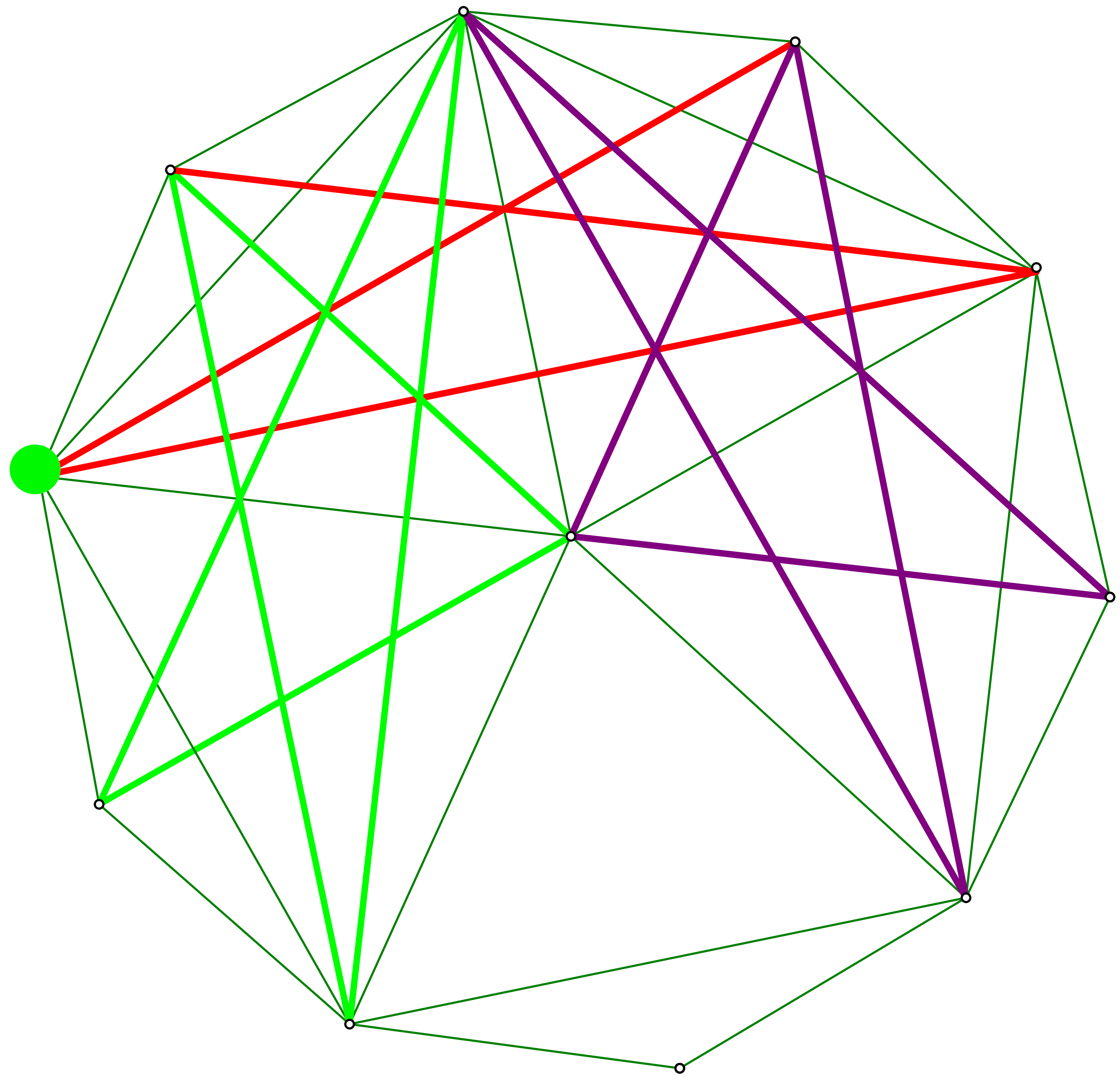

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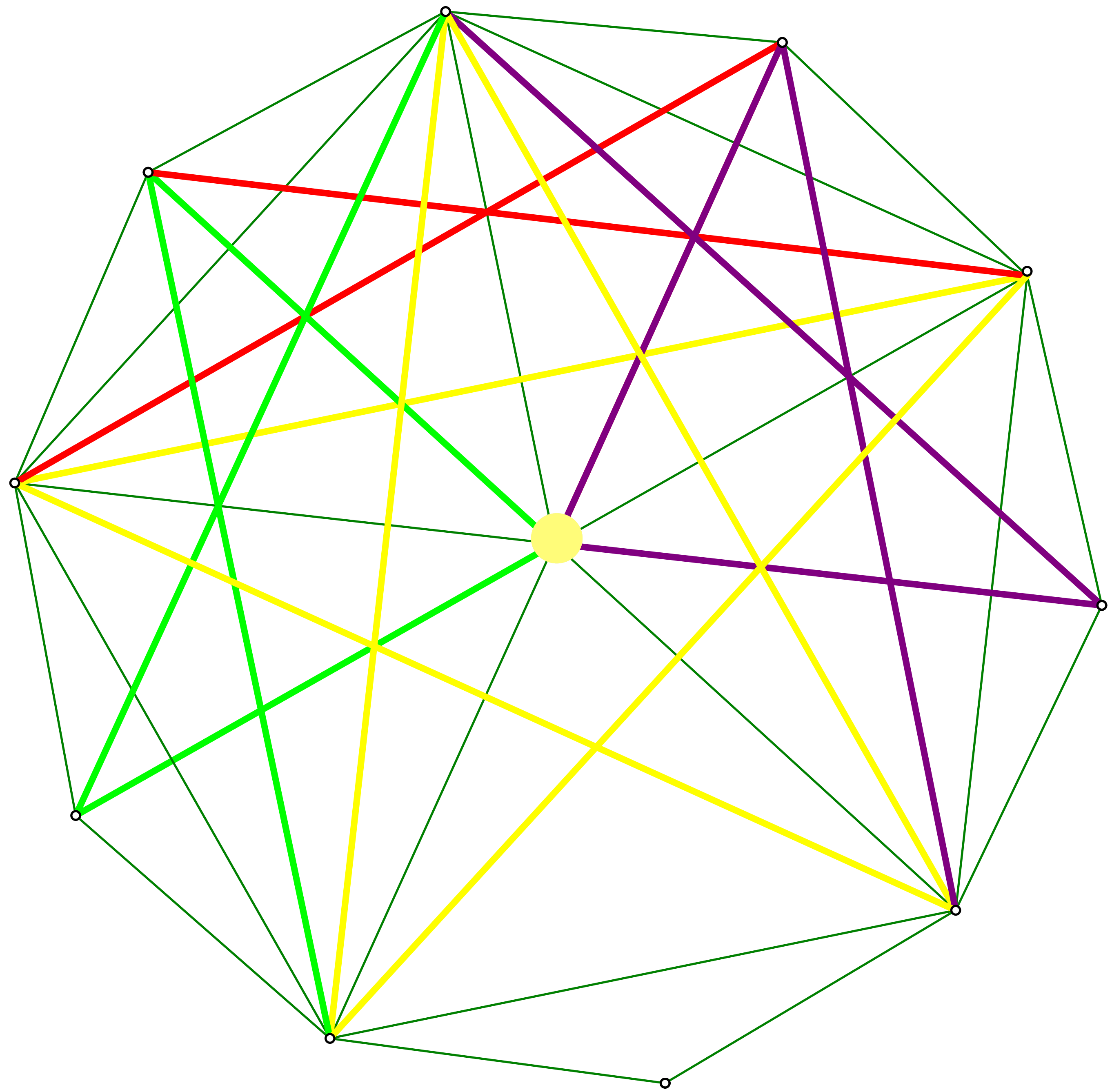

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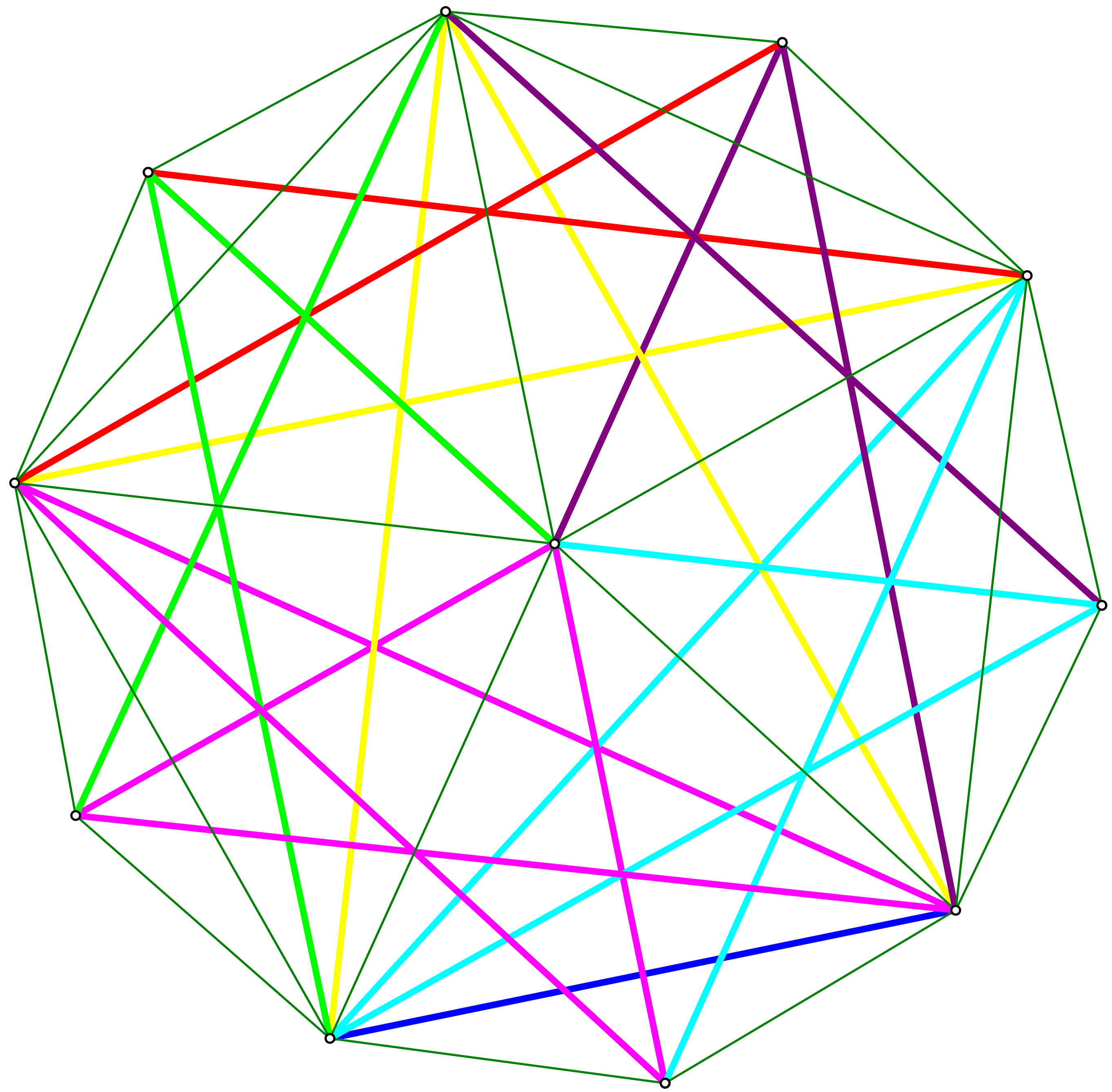

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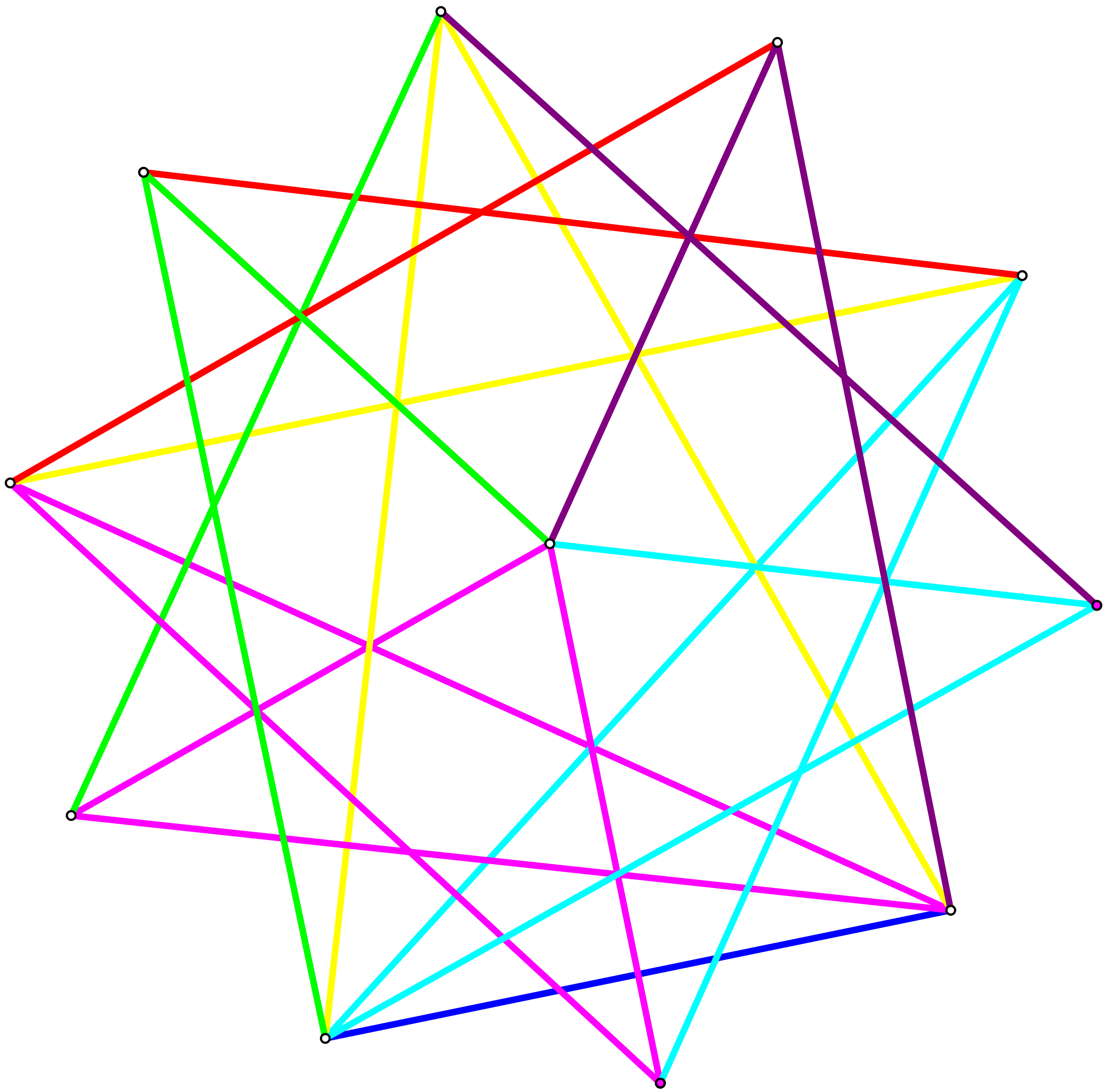
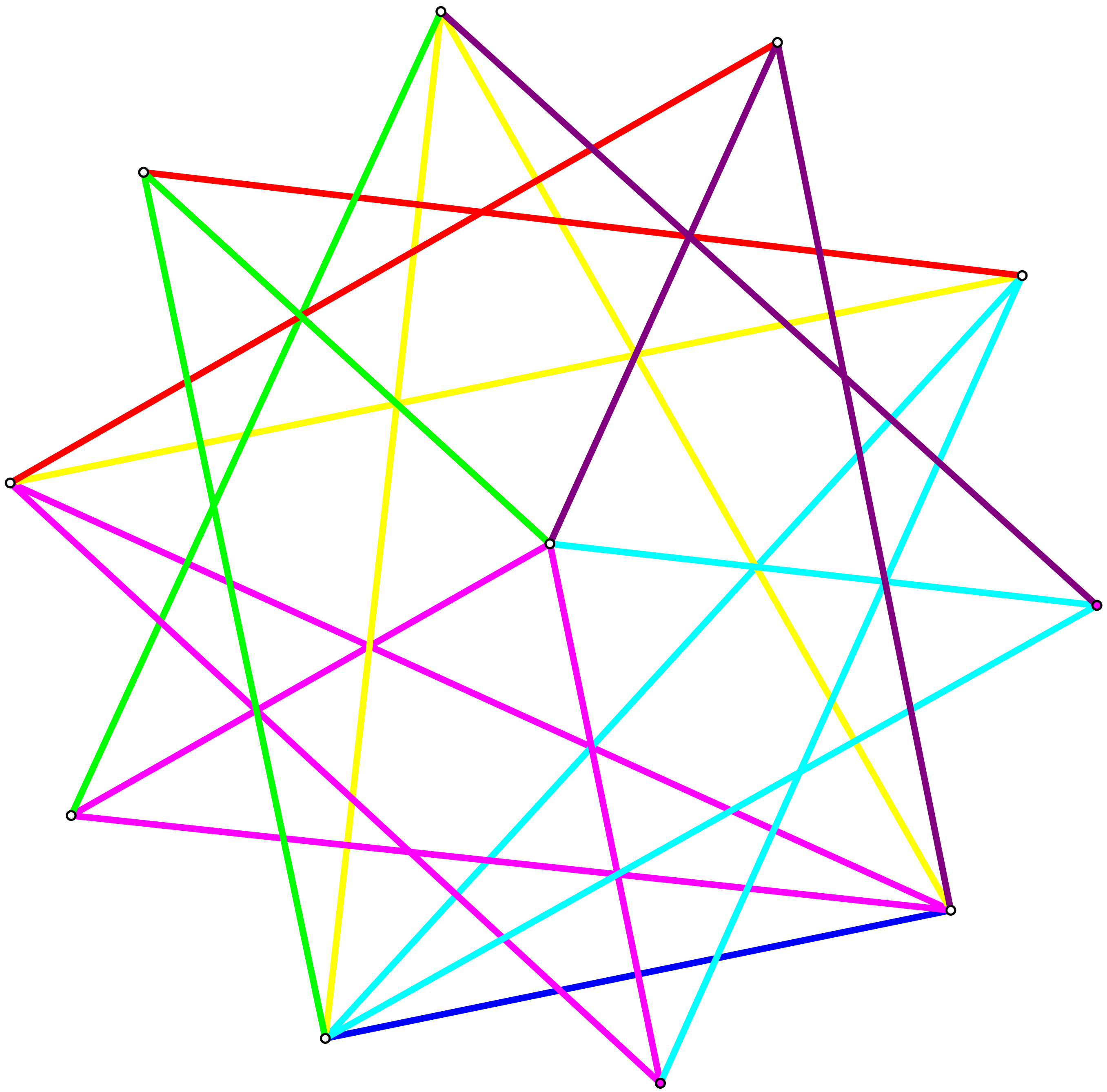


Regular stellated polyhedra



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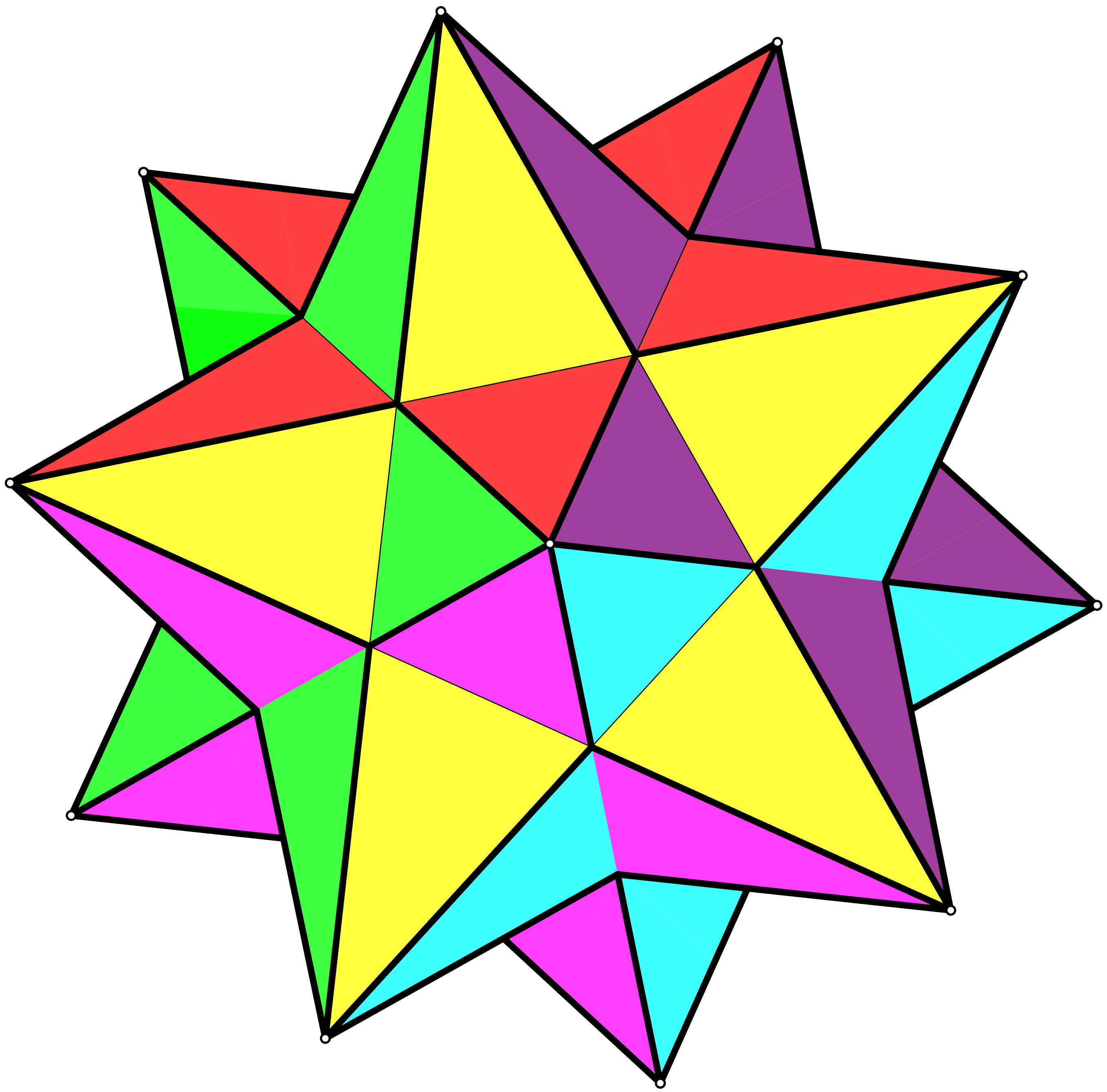

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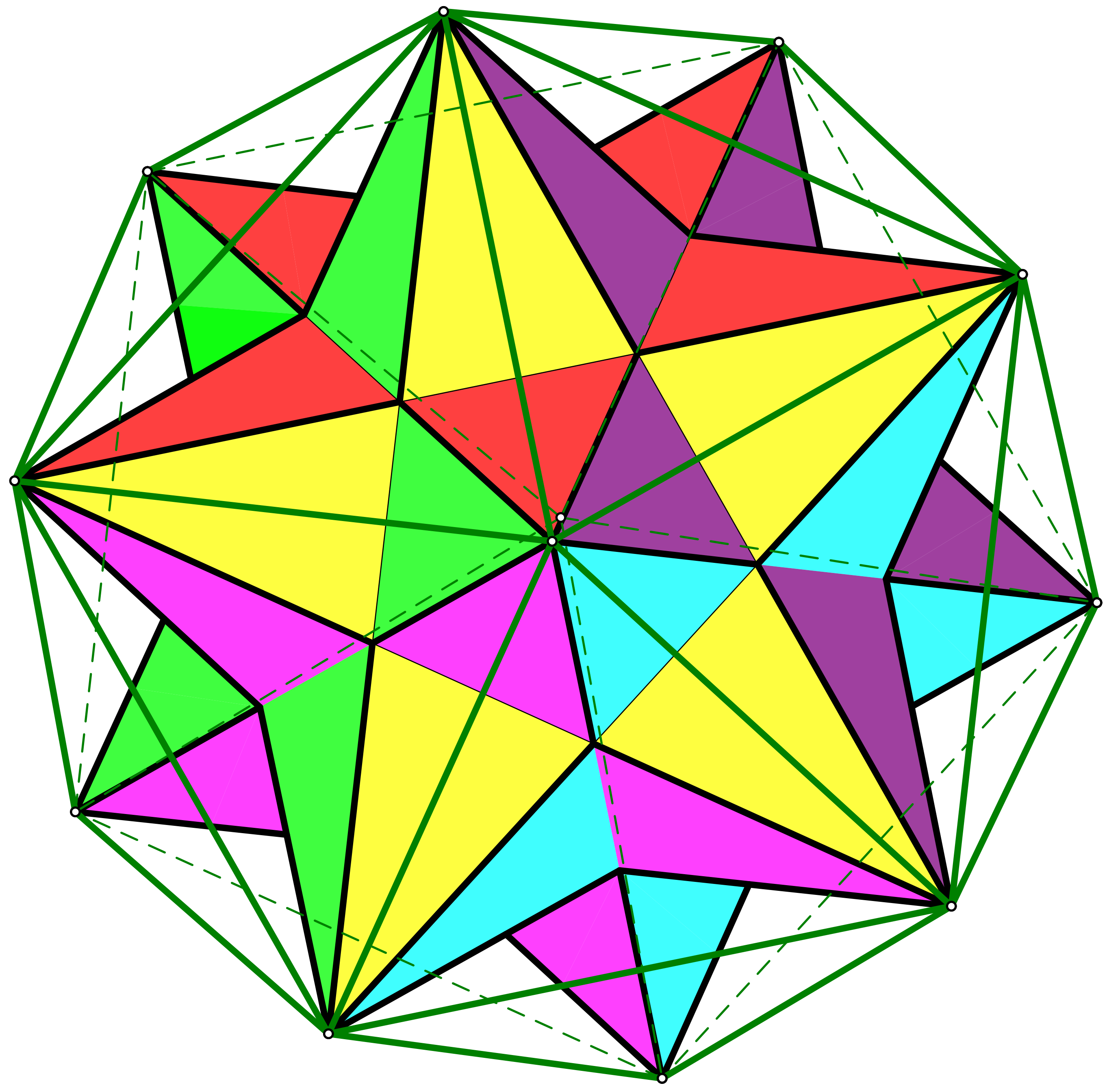

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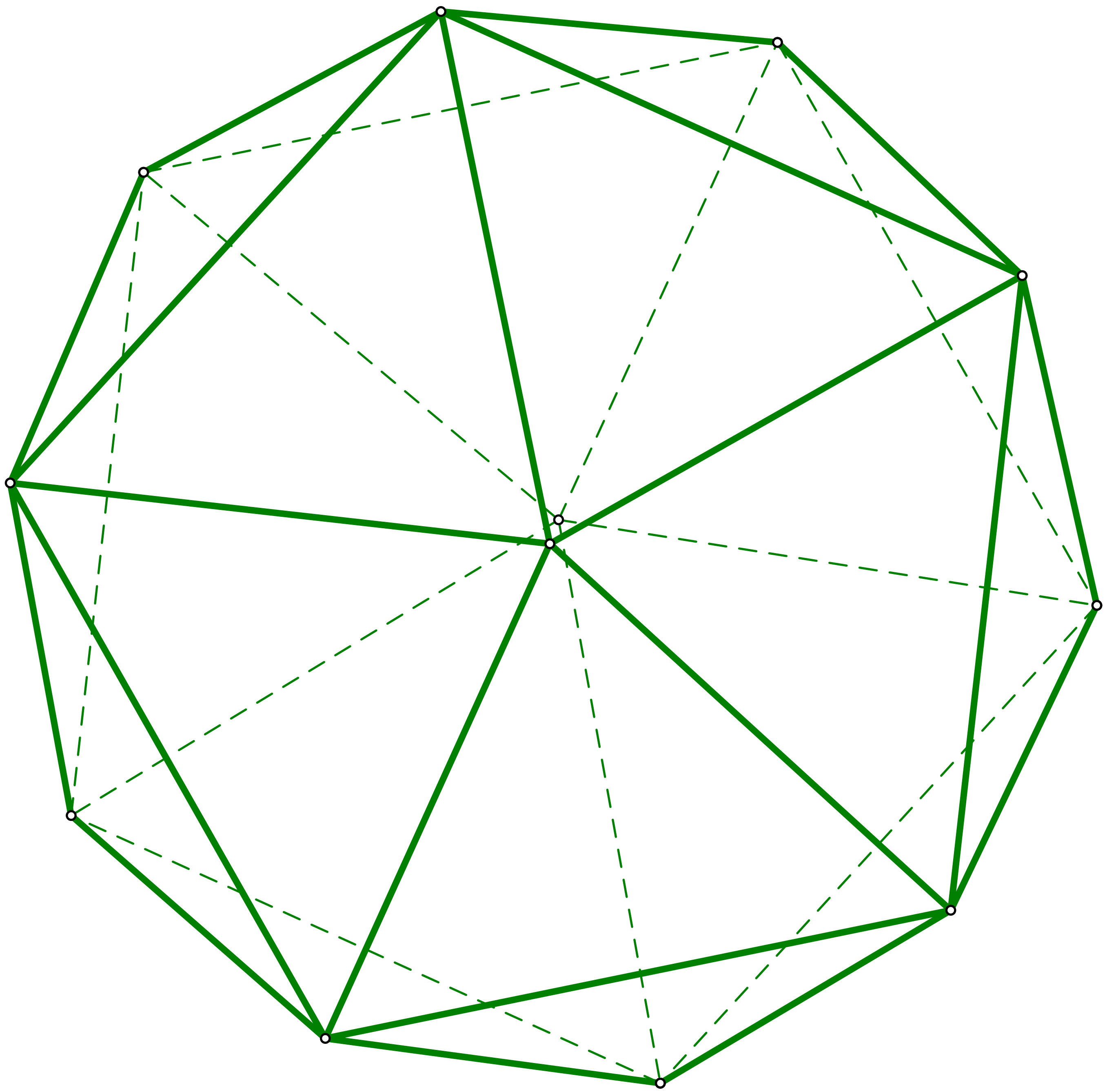

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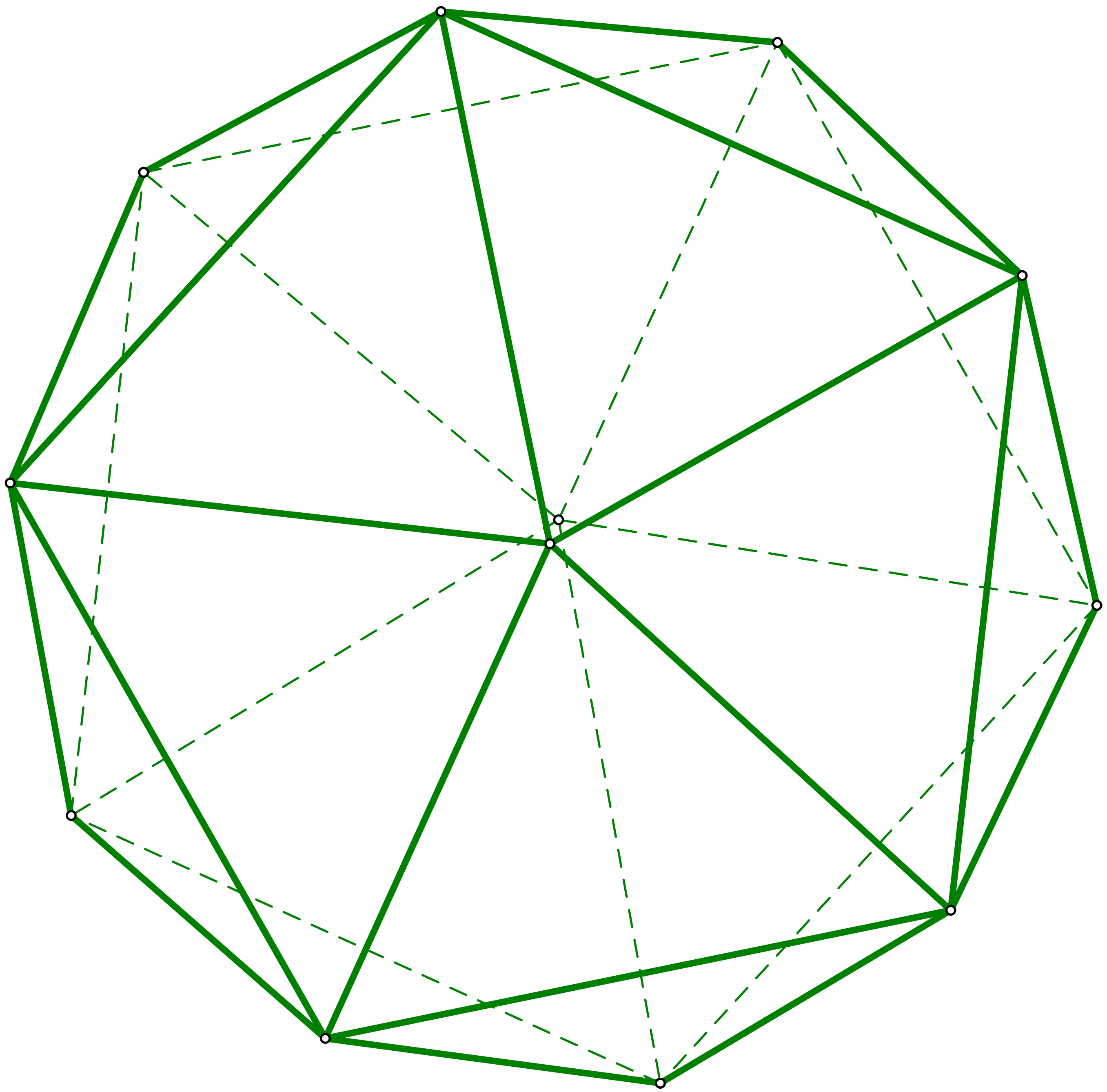

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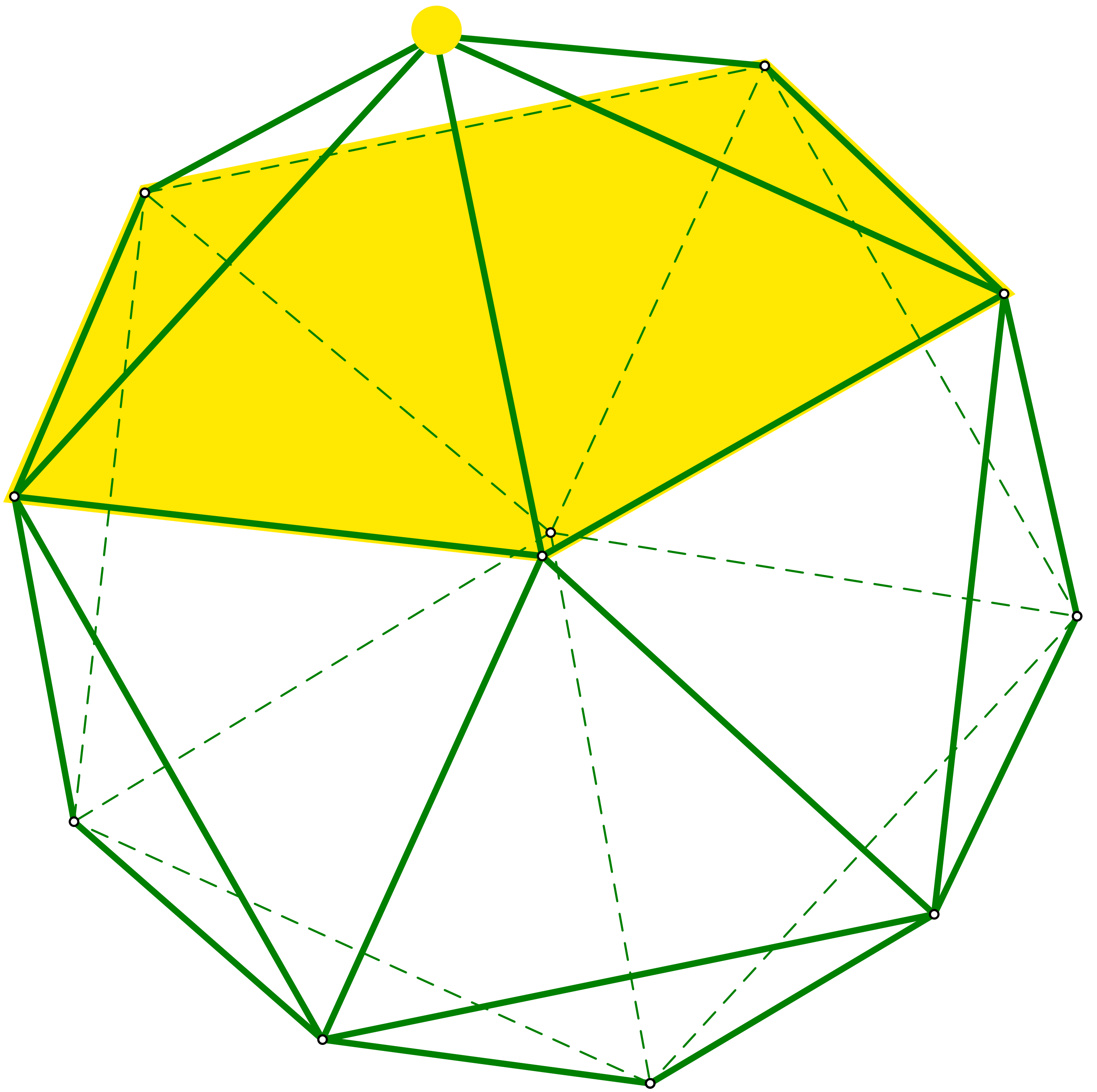

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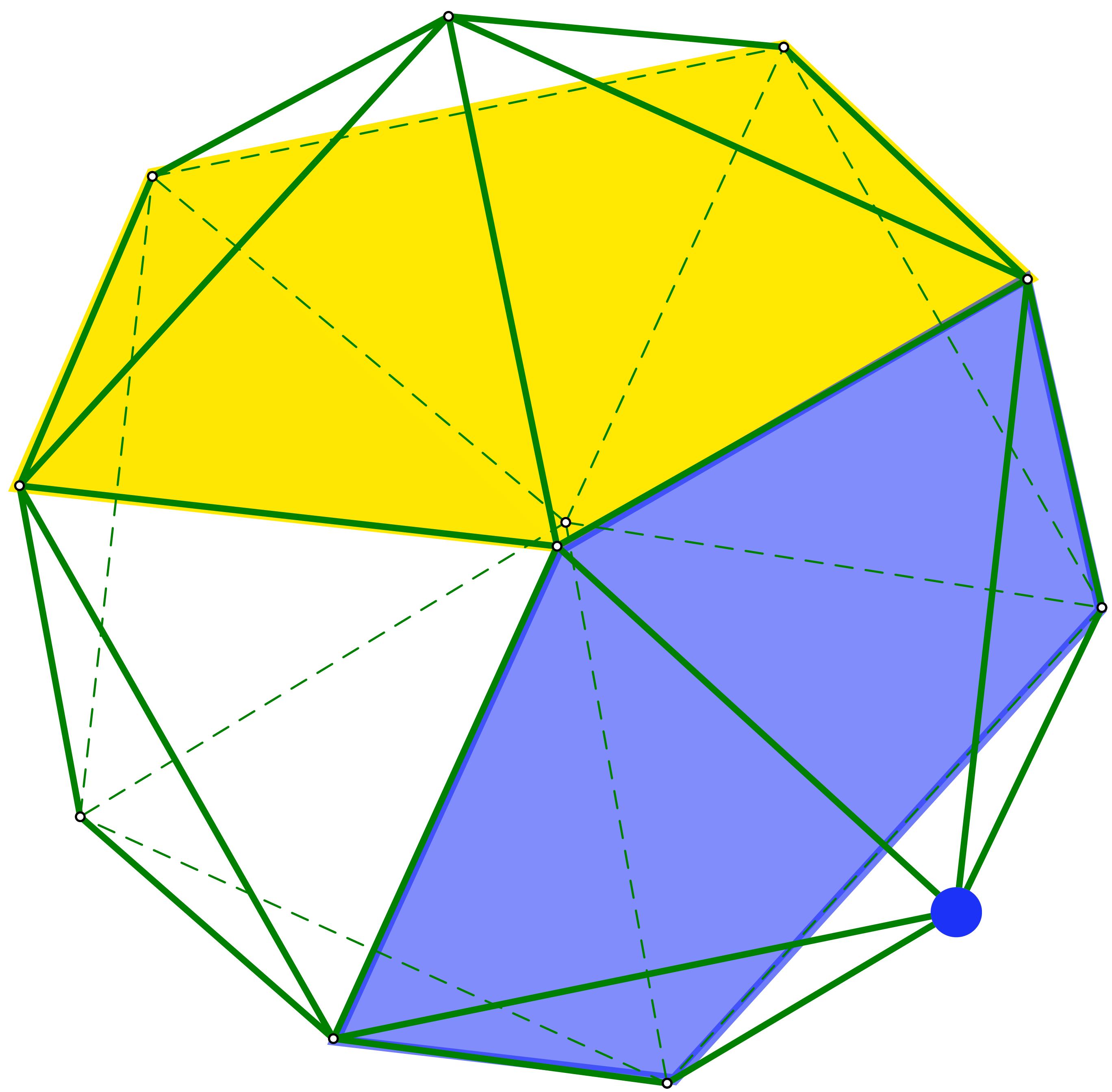

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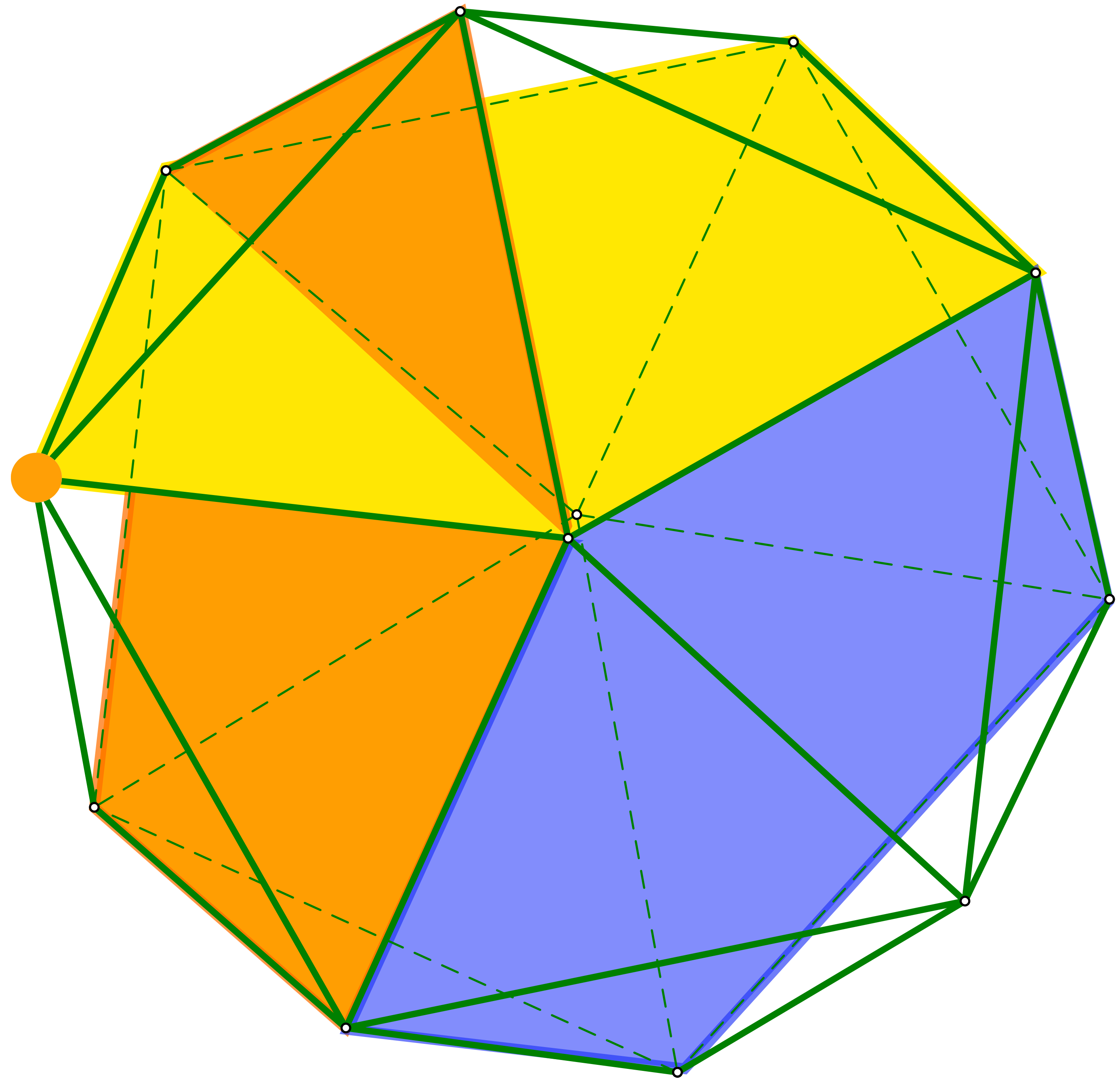

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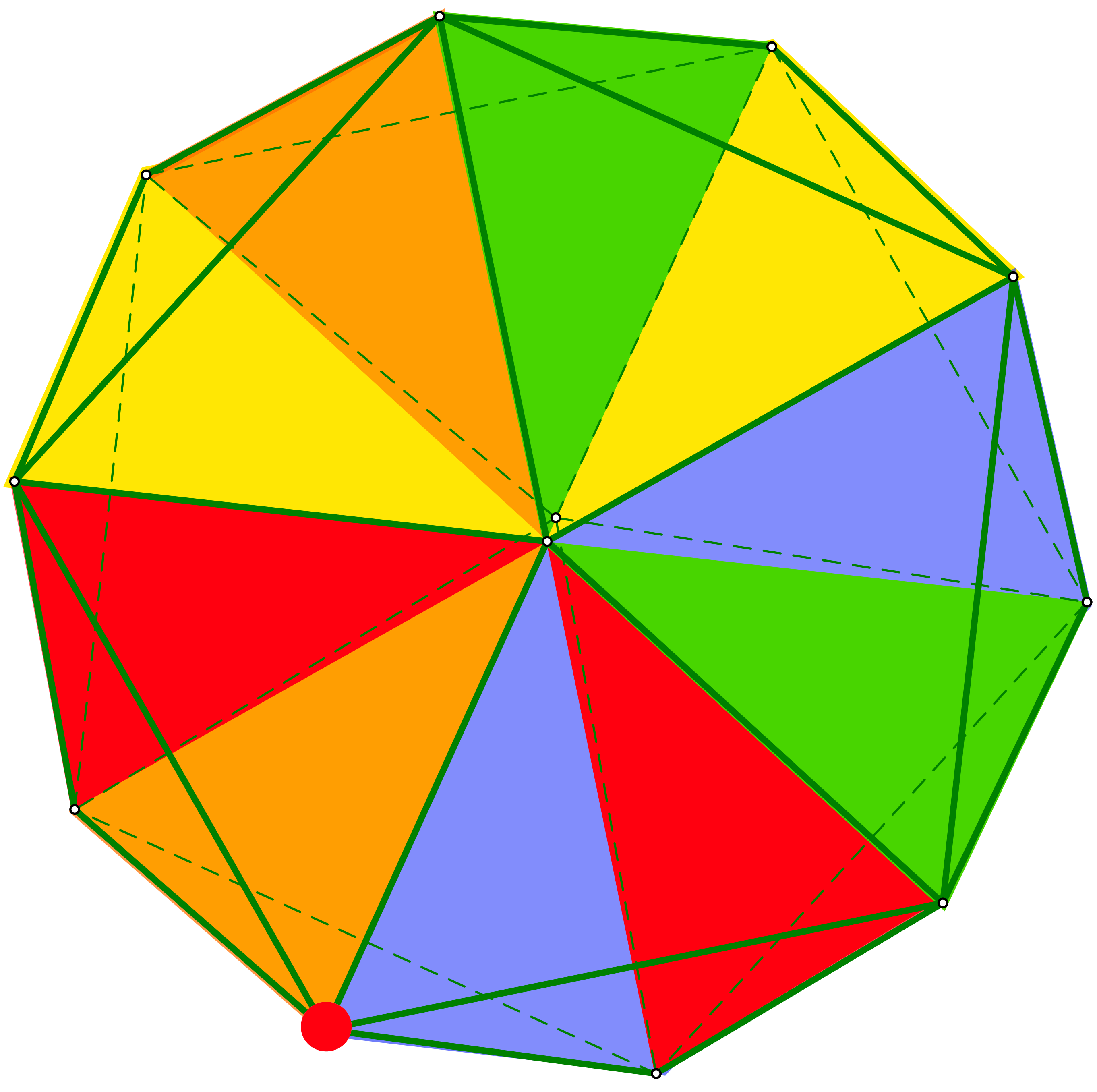

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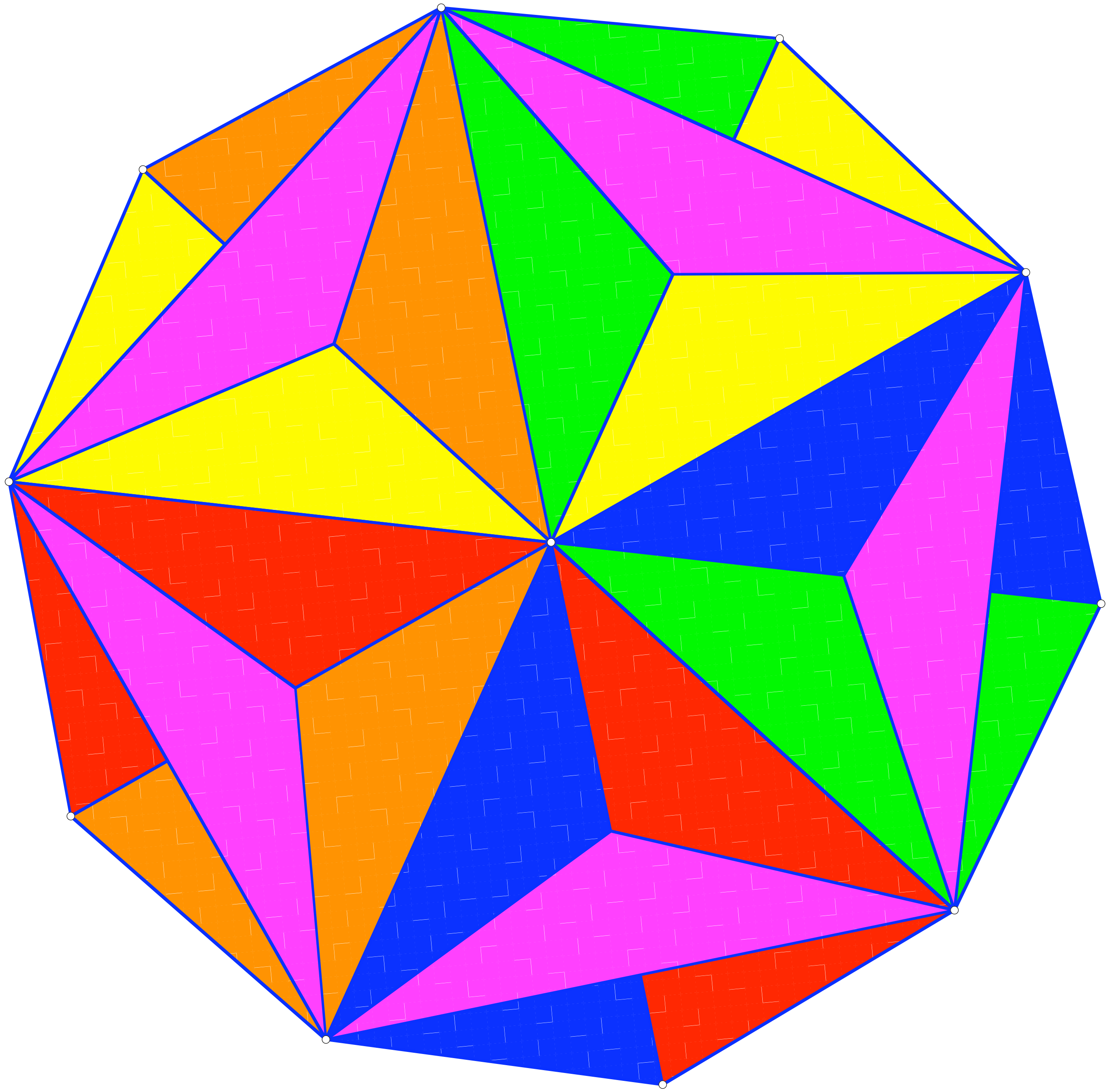

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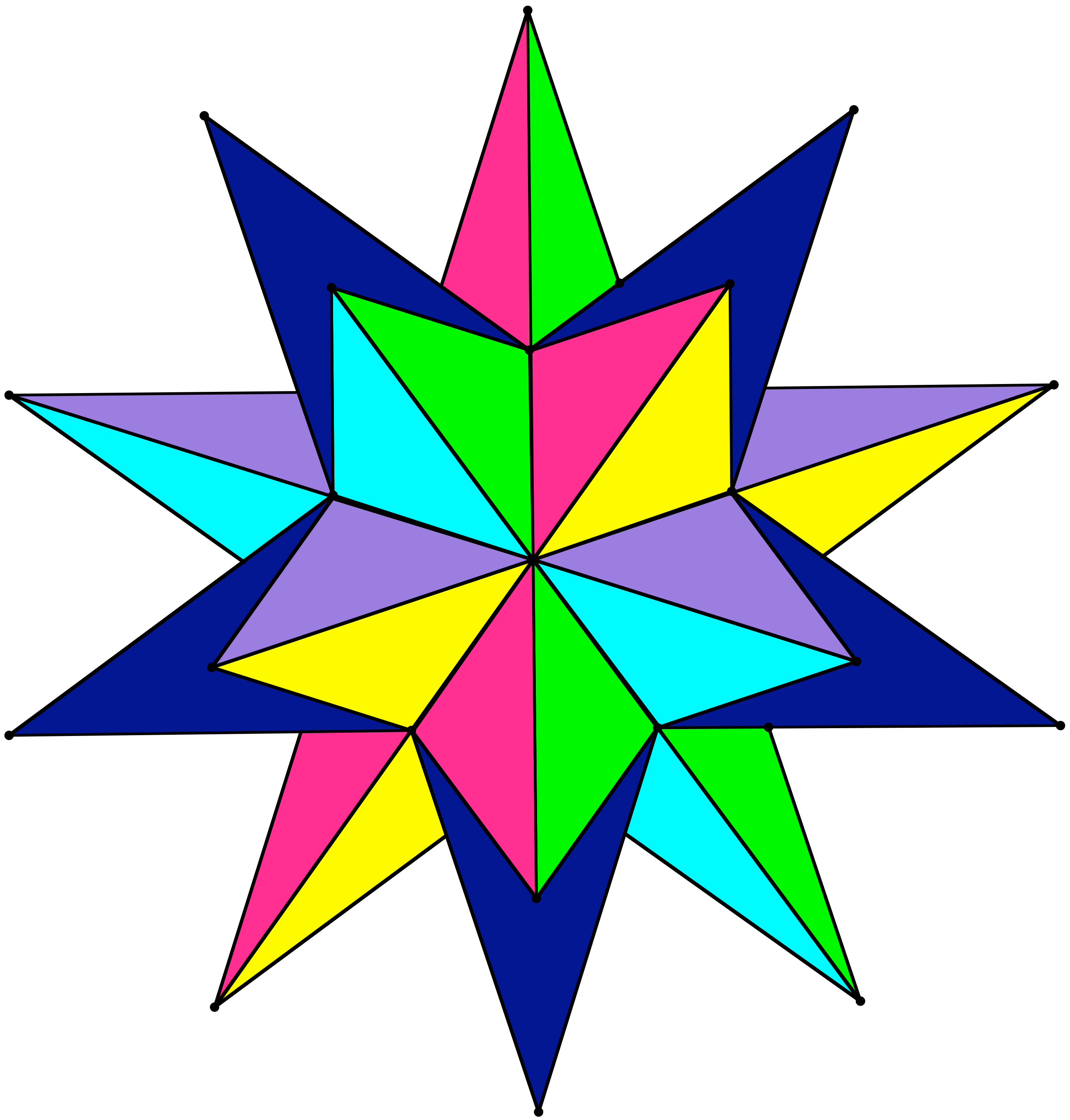

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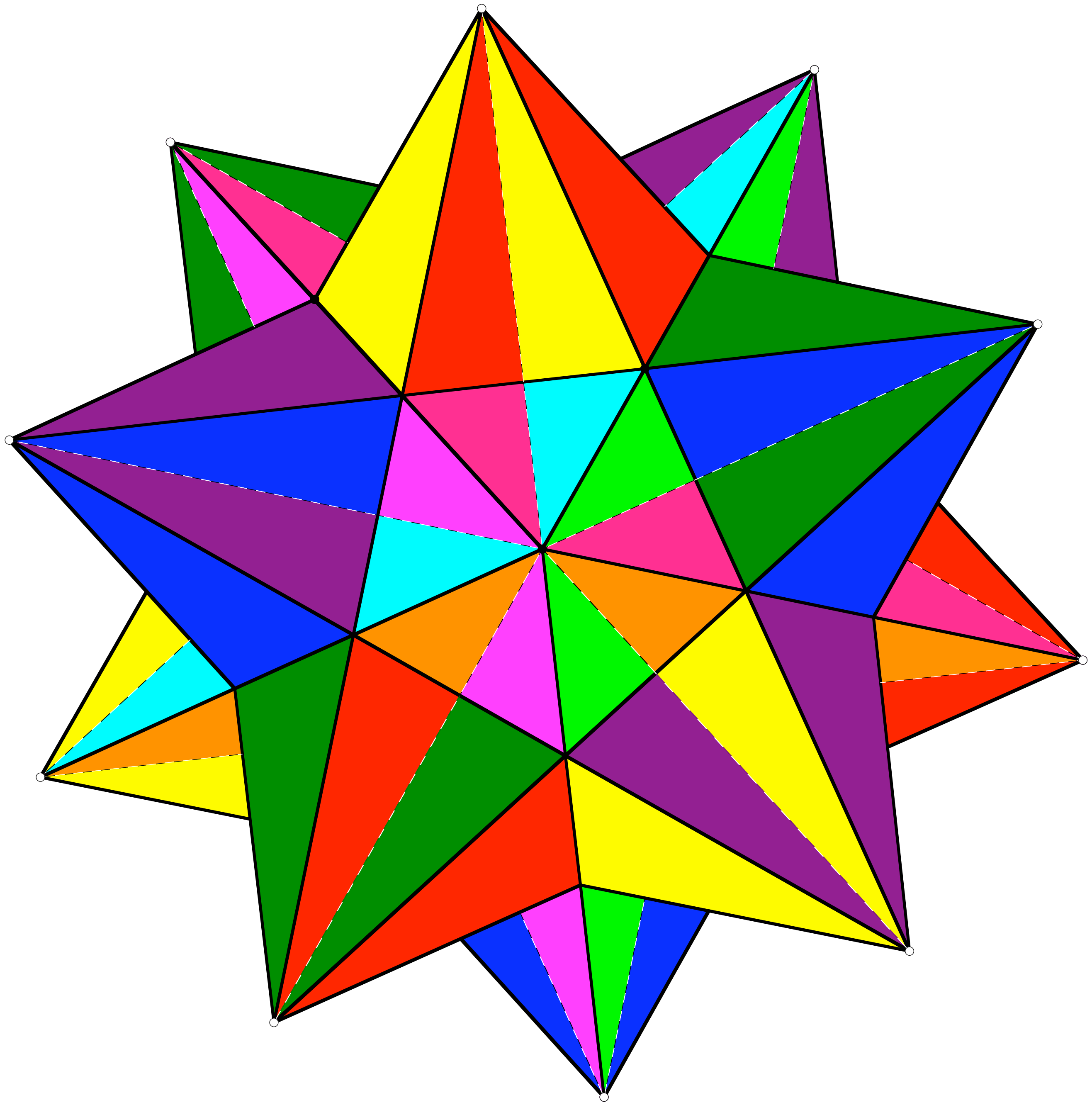

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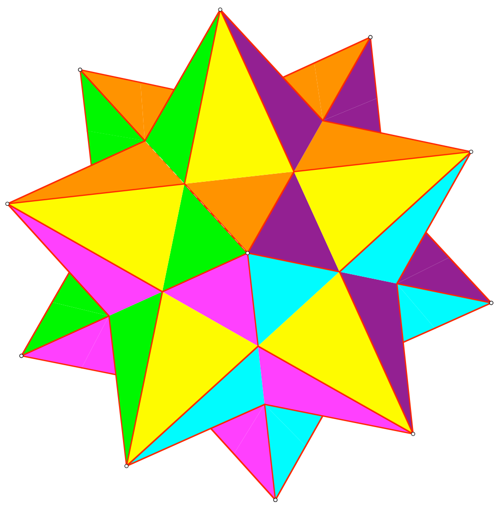
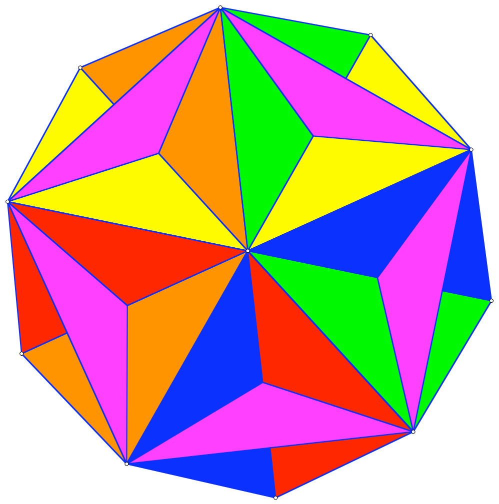
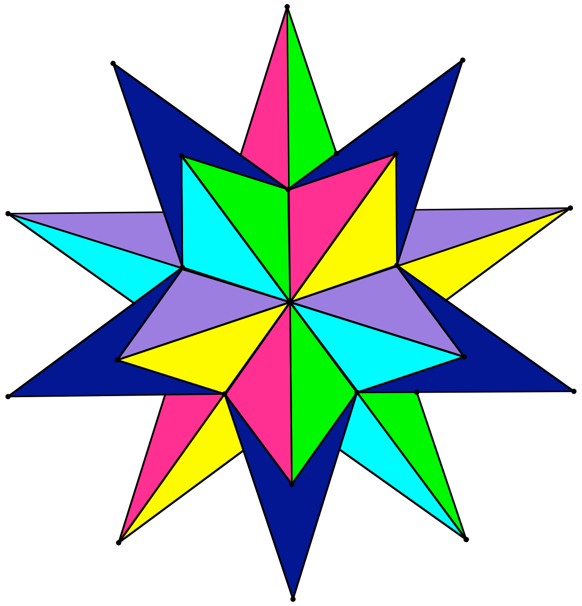


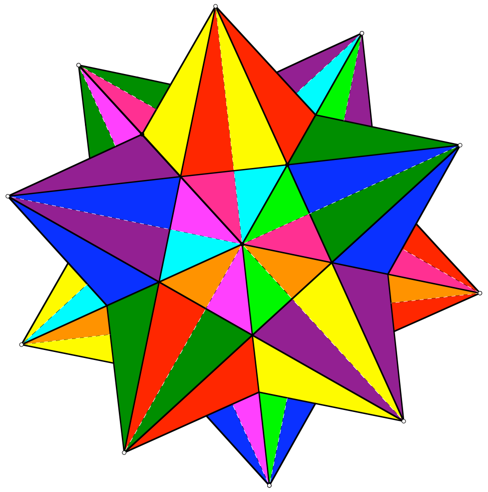
Small stellated dodecahedron



Great dodecahedron

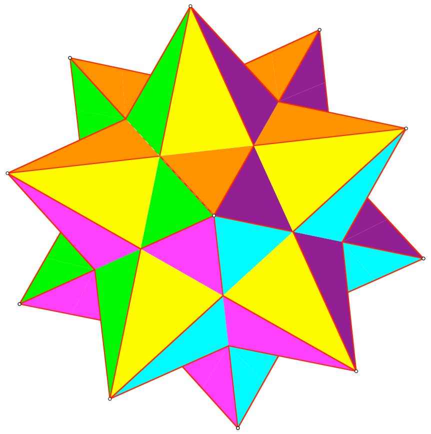


Great stellated dodecahedron

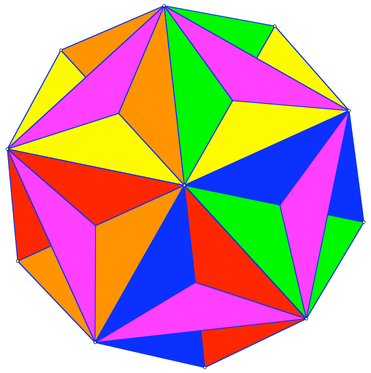


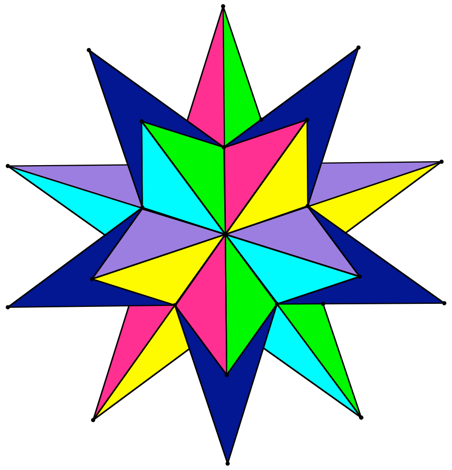
Great icosahedron

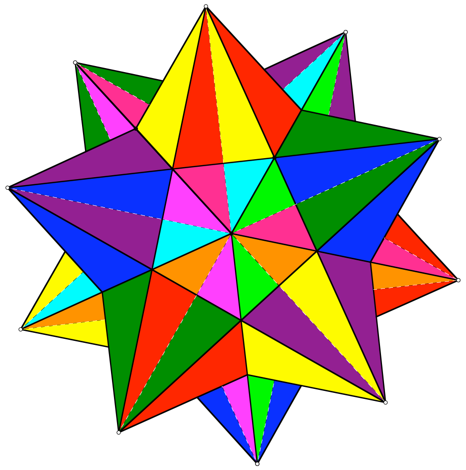
Regular stellated polyhedra

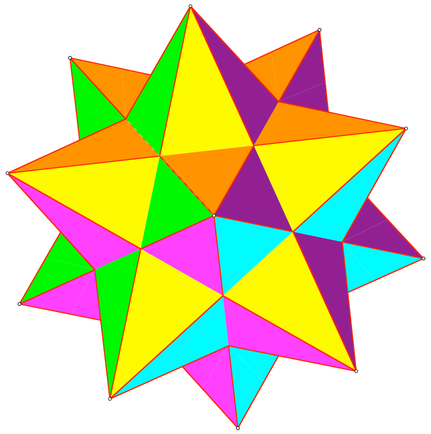


- Discovered by Kepler in 1619.

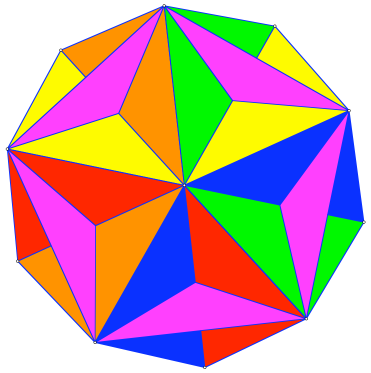




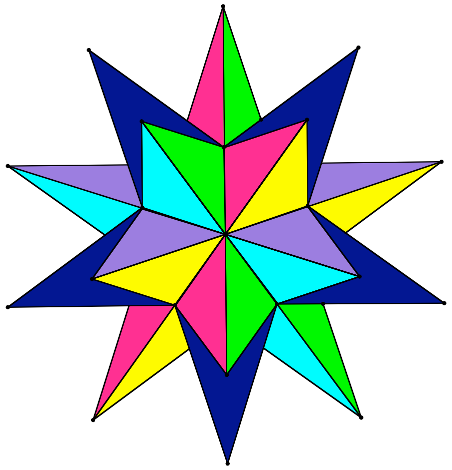


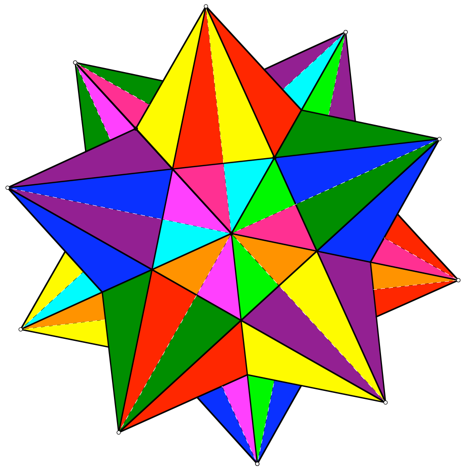


Regular stellated polyhedra

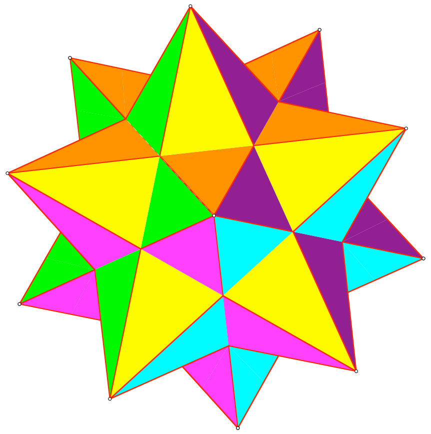


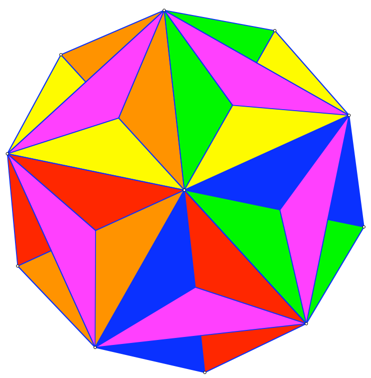
- Discovered by Kepler in 1619.
- Rediscovered by Poinsot in 1809.

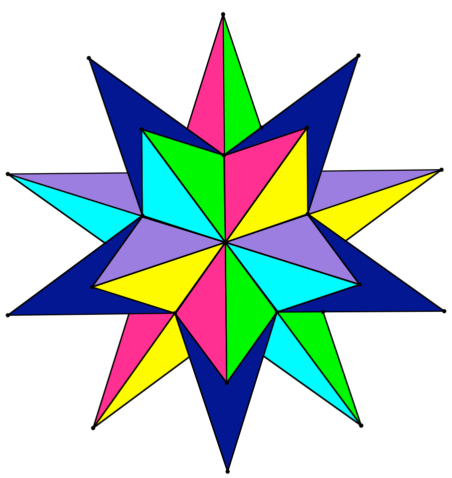


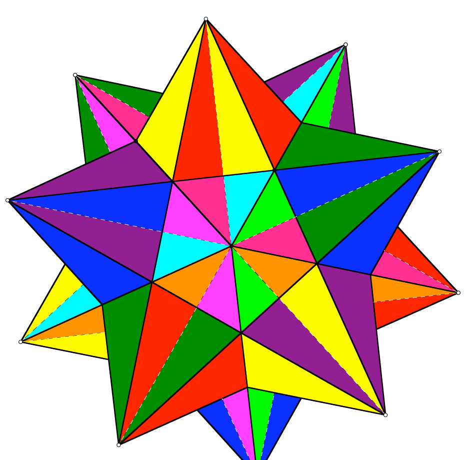


Regular stellated polyhedra

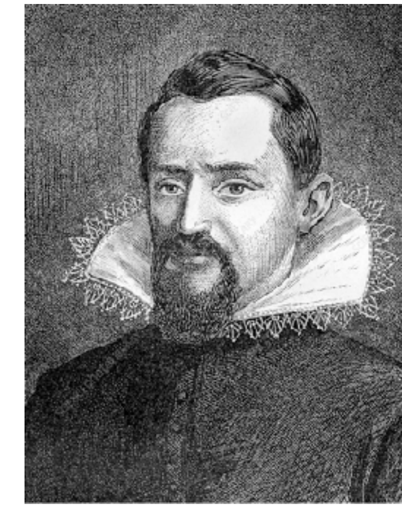




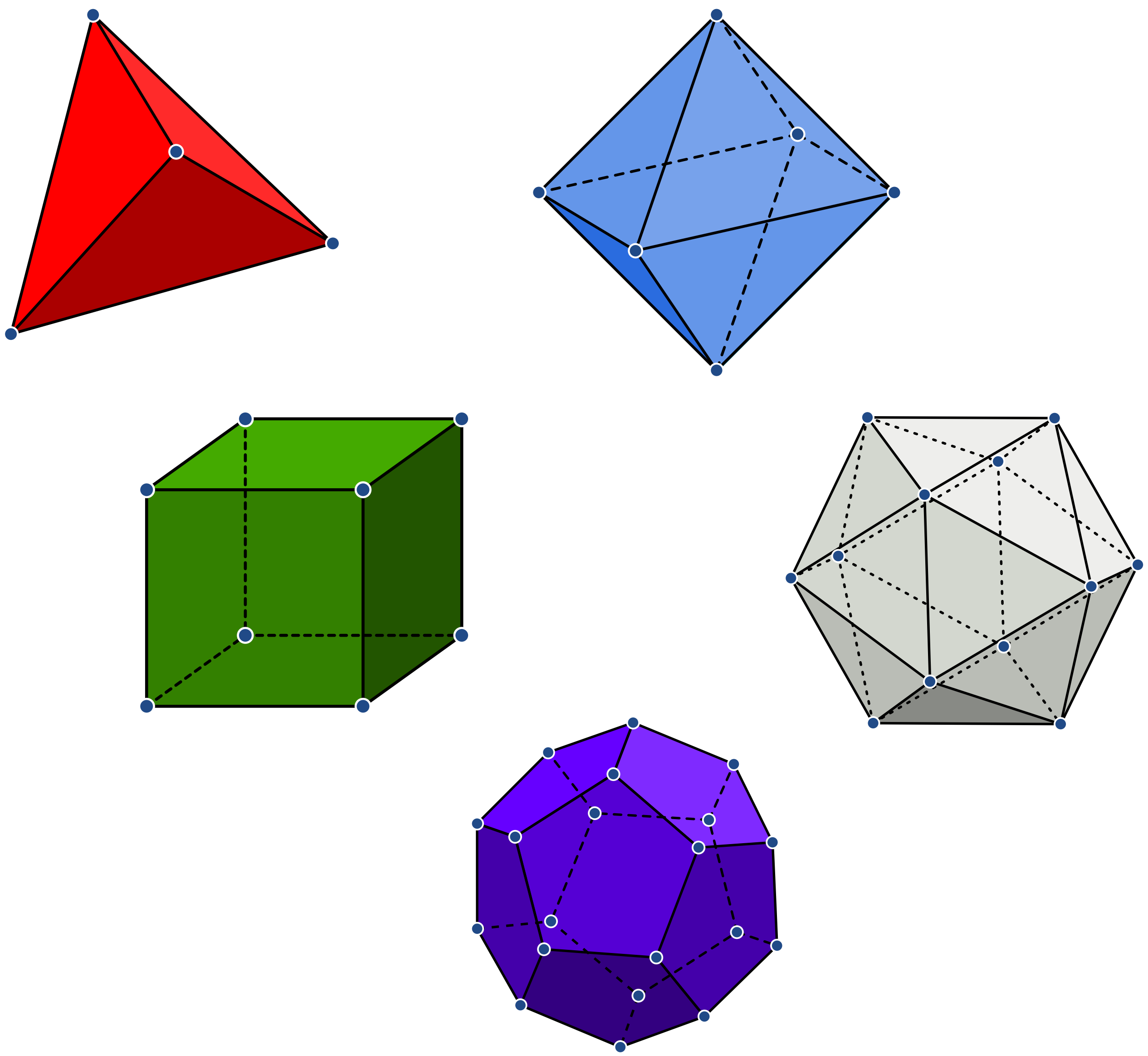




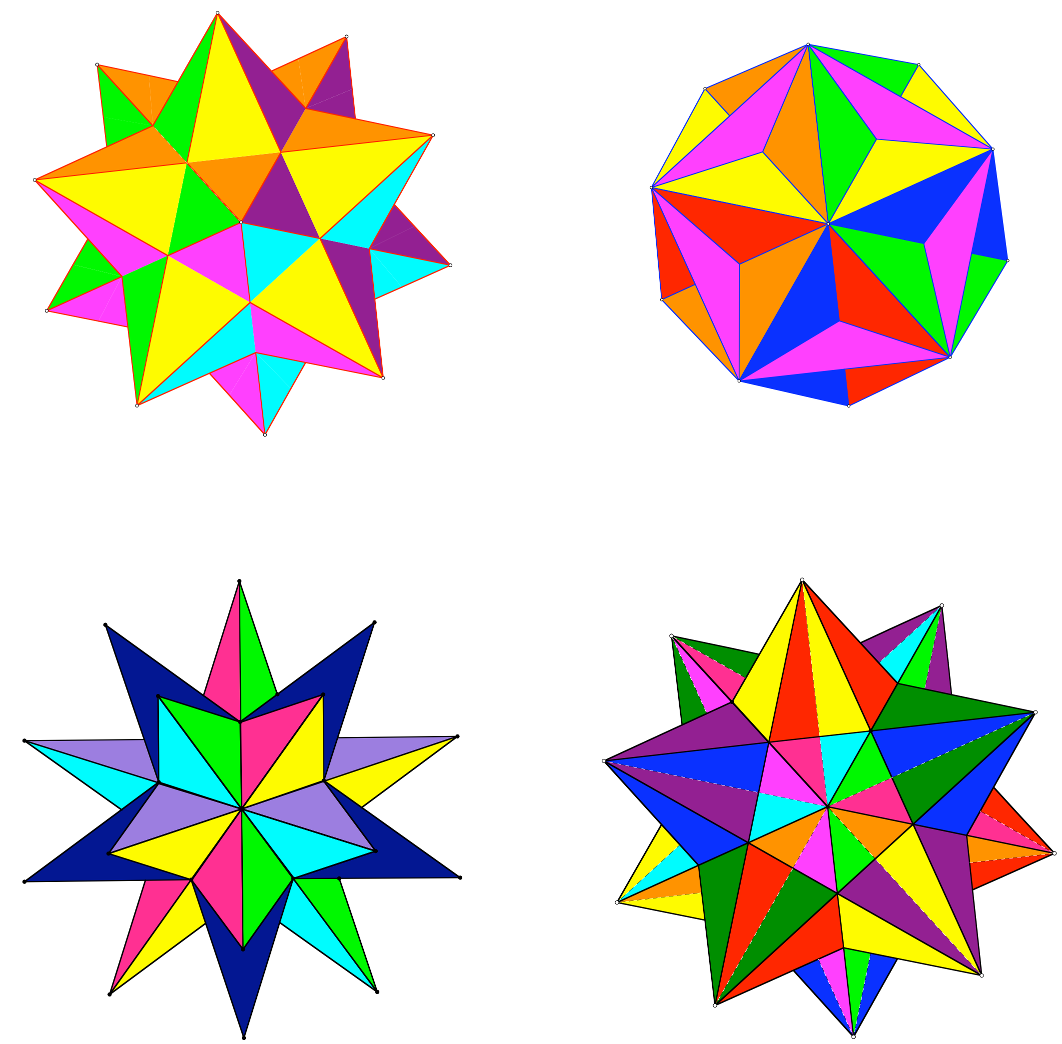
- Discovered by Kepler in 1619.
- Rediscovered by Poinsot in 1809.
- Cauchy proved that they are all the stellated polyhedra (1812).



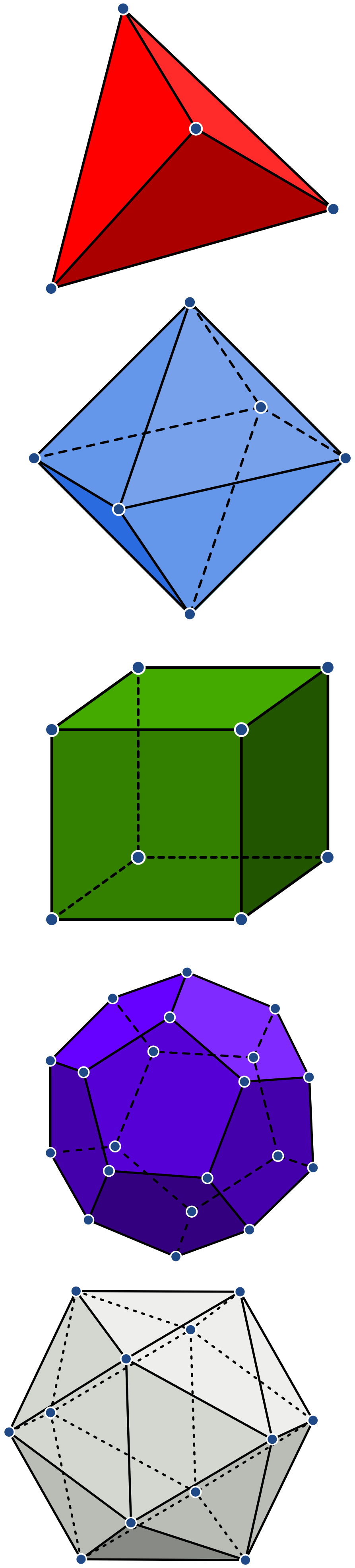
Regular polyhedra



Platonic solids



Kepler - Poinsot polyhedra



Regular polyhedra

They are built by glueing polygons (faces) along their edges (two per edge)

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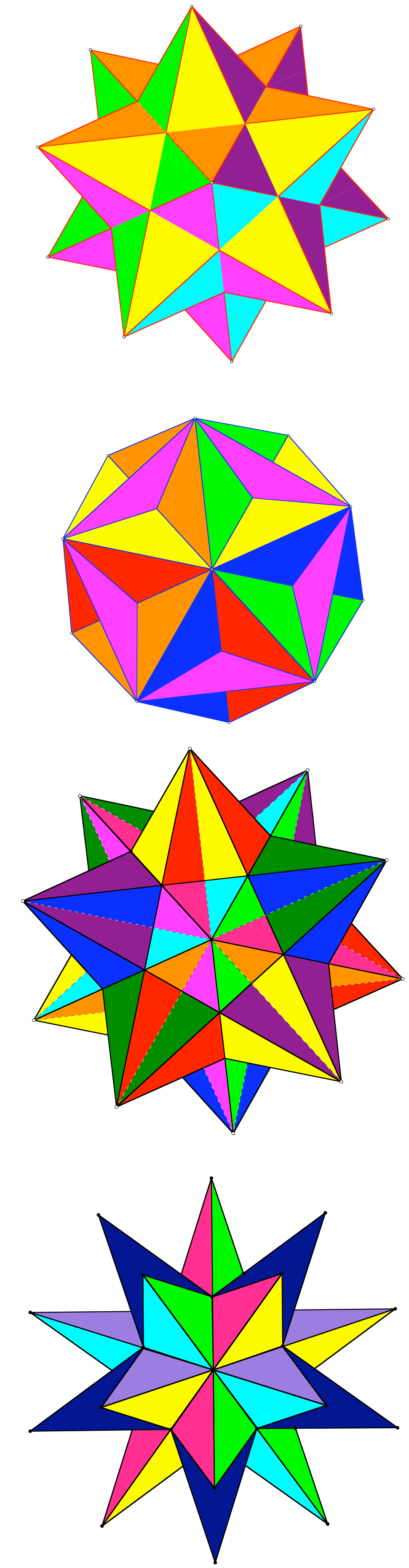


Europe | 1600s - 1800s

Europe | 1600s - 1800s

London | 1926

H. S. M. Coxeter
1907 - 2003

John F. Petrie
1907 - 1972

H. S. M. Coxeter
1907 - 2003

John F. Petrie
1907 - 1972

H. S. M. Coxeter
1907 - 2003

Dear Donald,
I found two regular polyhedra without fake vertices.

Dear Donald,
I found two regular polyhedra without fake vertices.

…

One of them has six squares around each vertex, the other one has four hexagons.

One of them has six squares around each vertex, the other one has four hexagons.

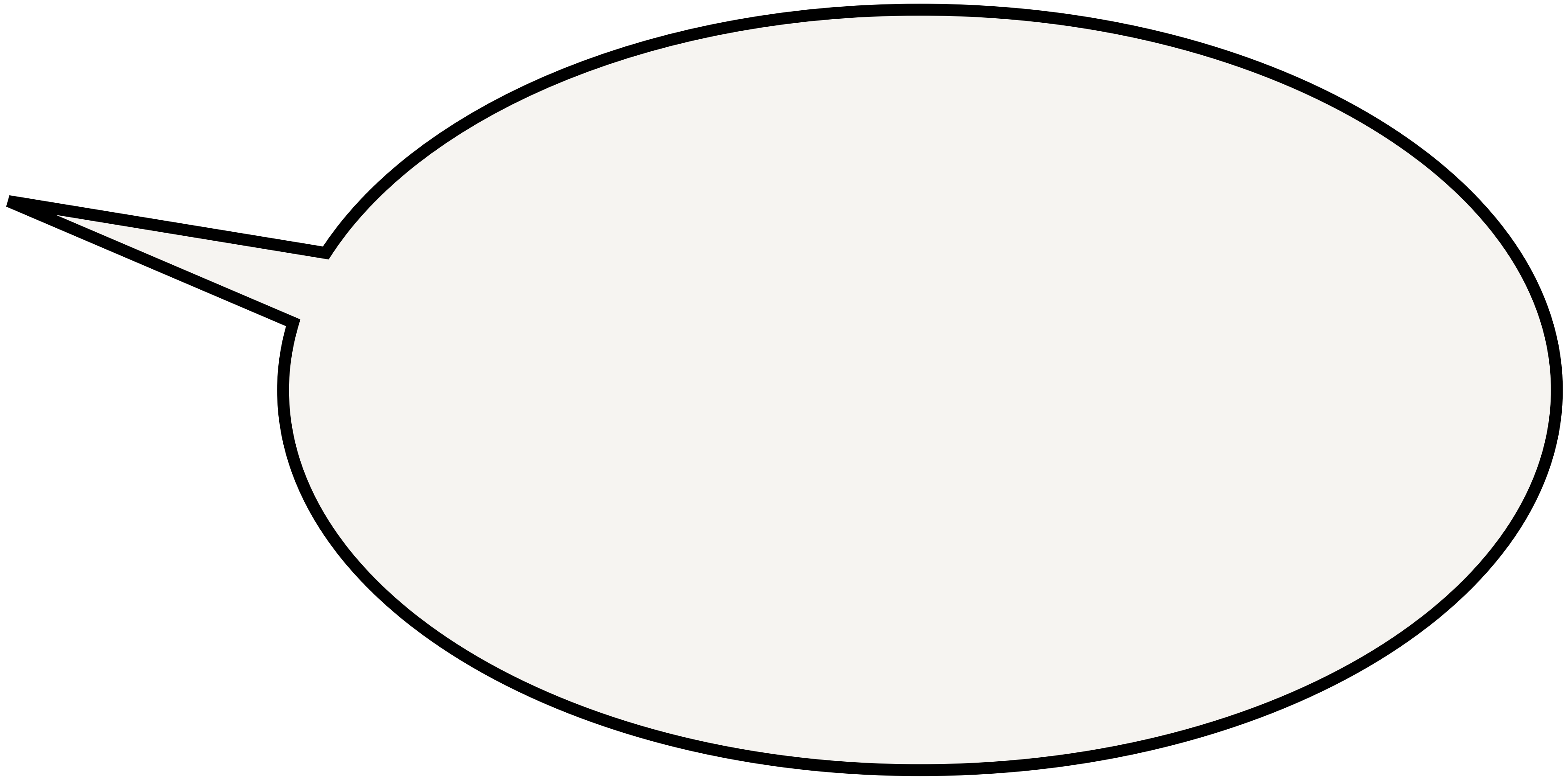
But...

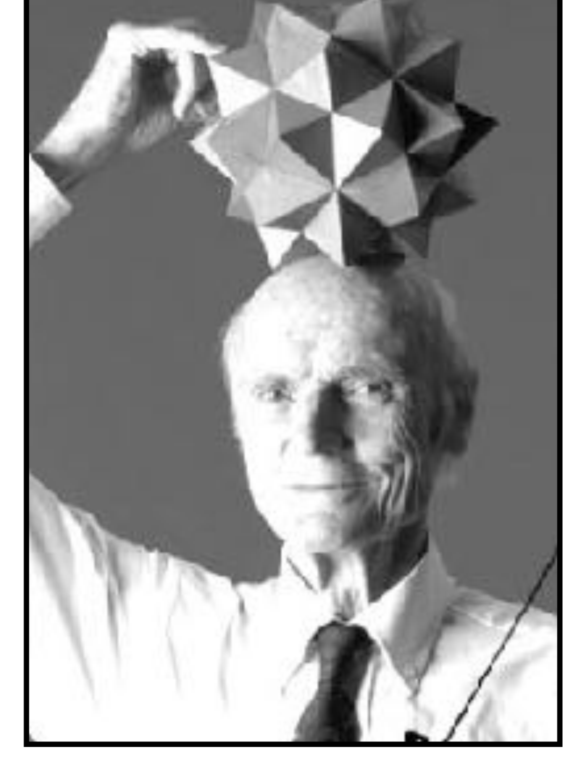
Yes, I know; they do not fit.

Unless…

Yes, I know; they do not fit.
Unless…

But then… they have to…

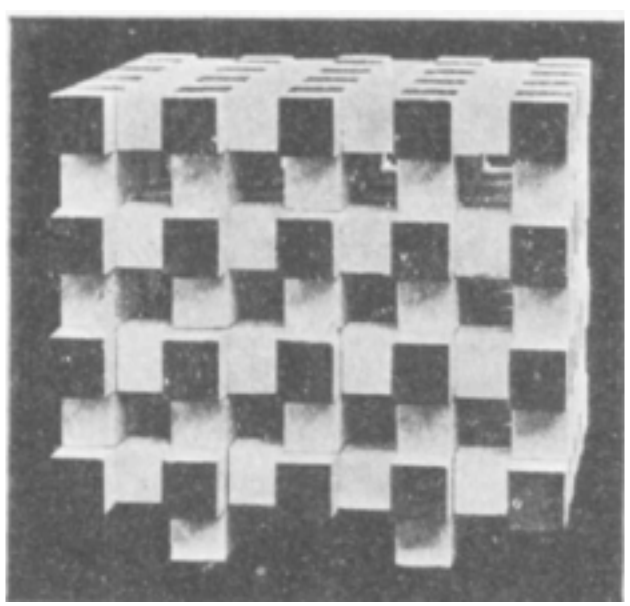


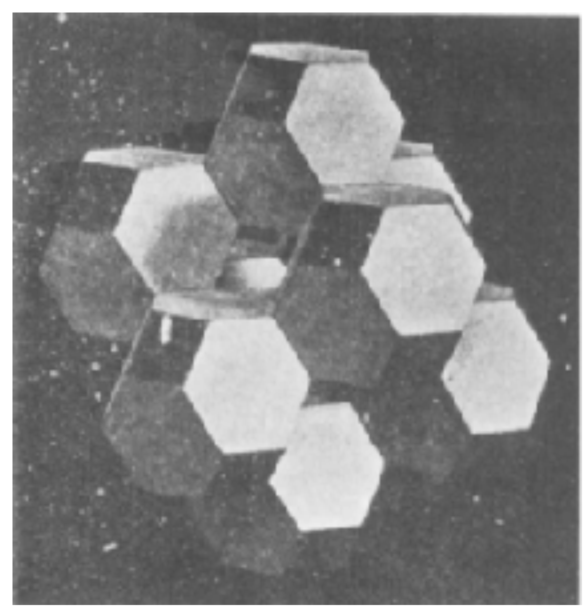


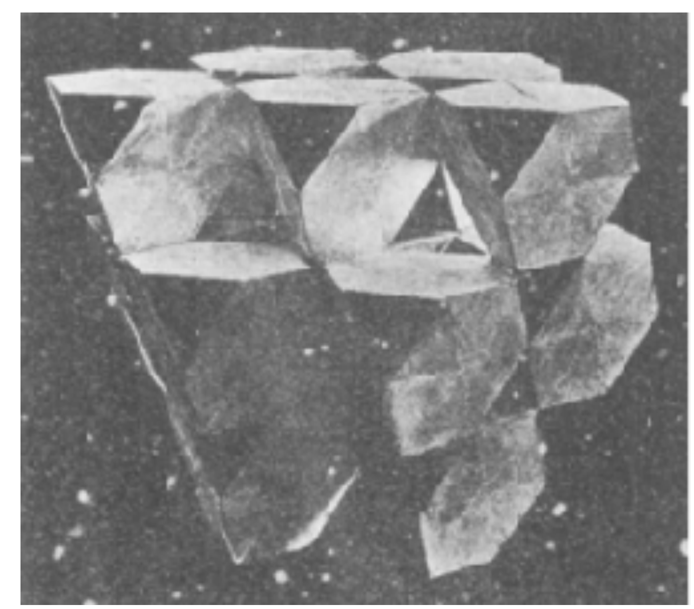
YES!!
They are infinite!!

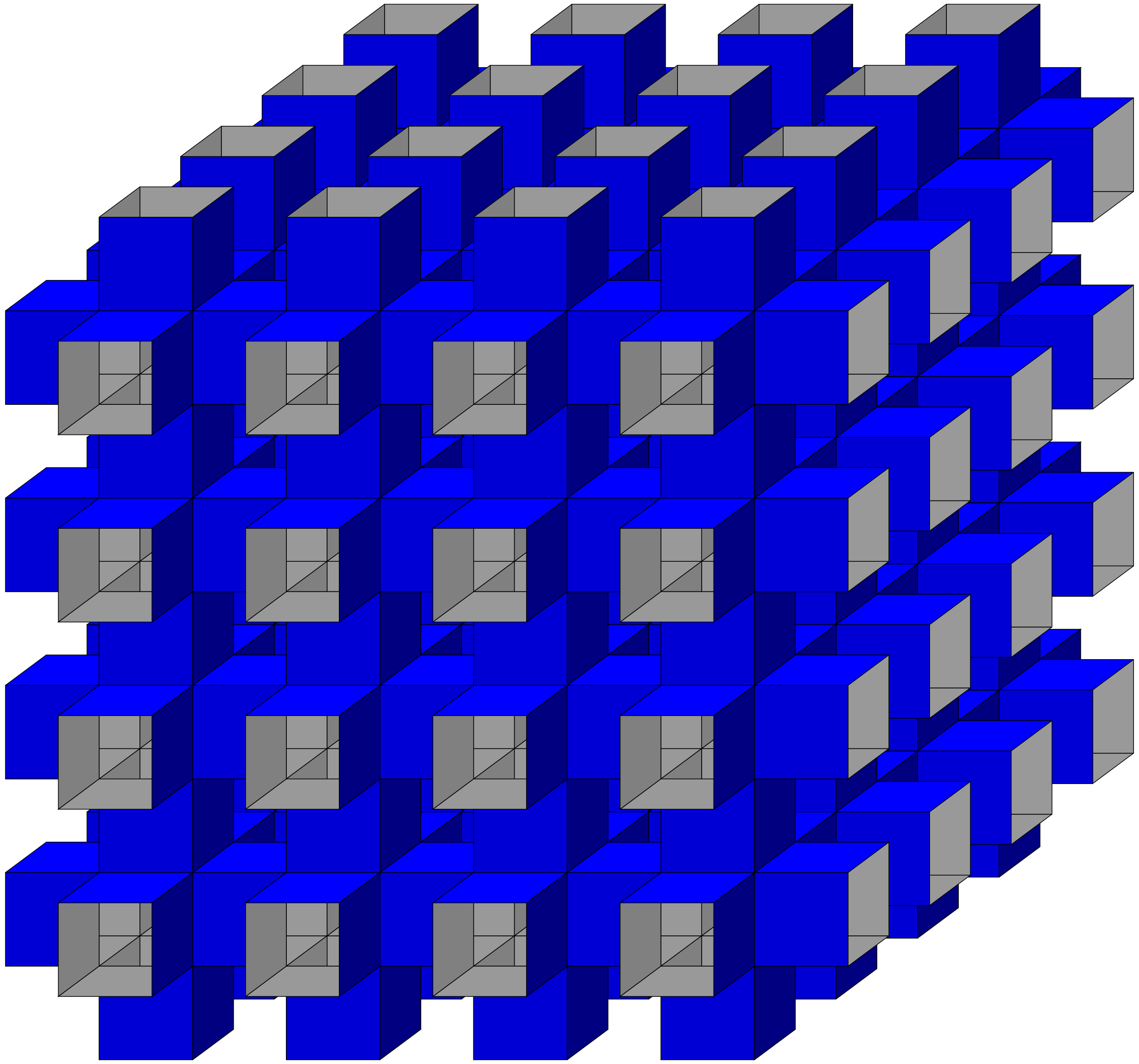
YES!!
They are infinite!!

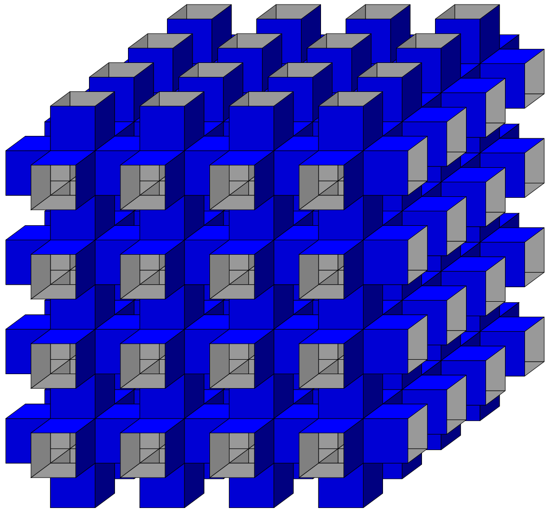
Aha! There is another possibility: six hexagons at each vertex!

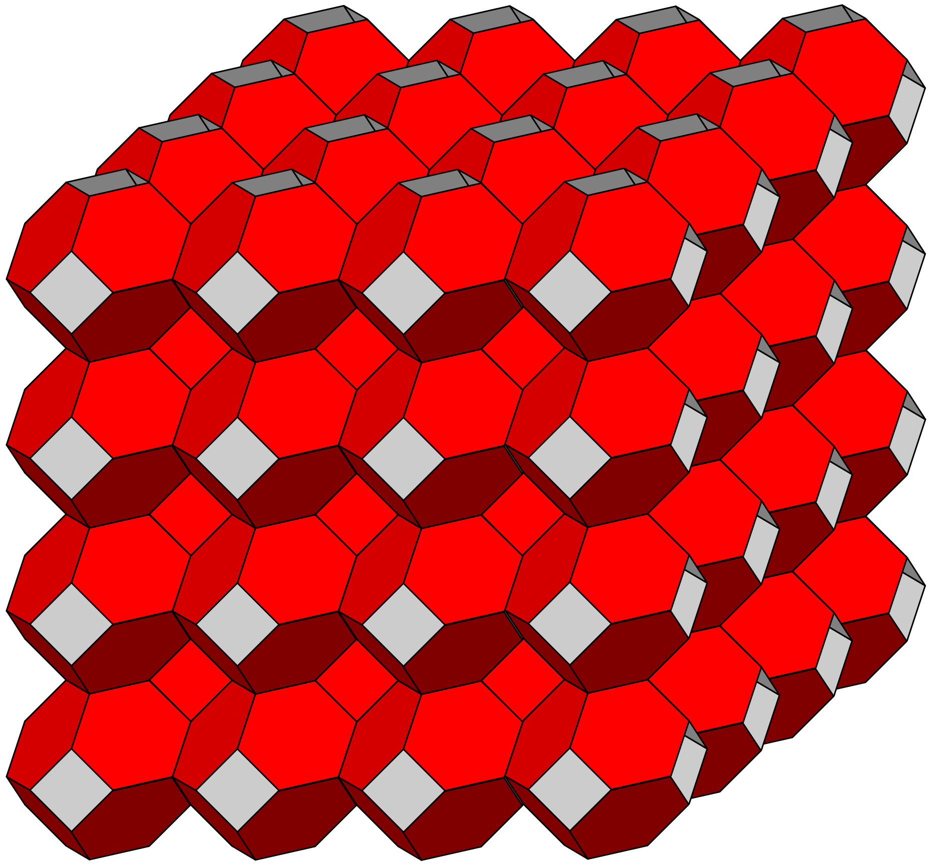





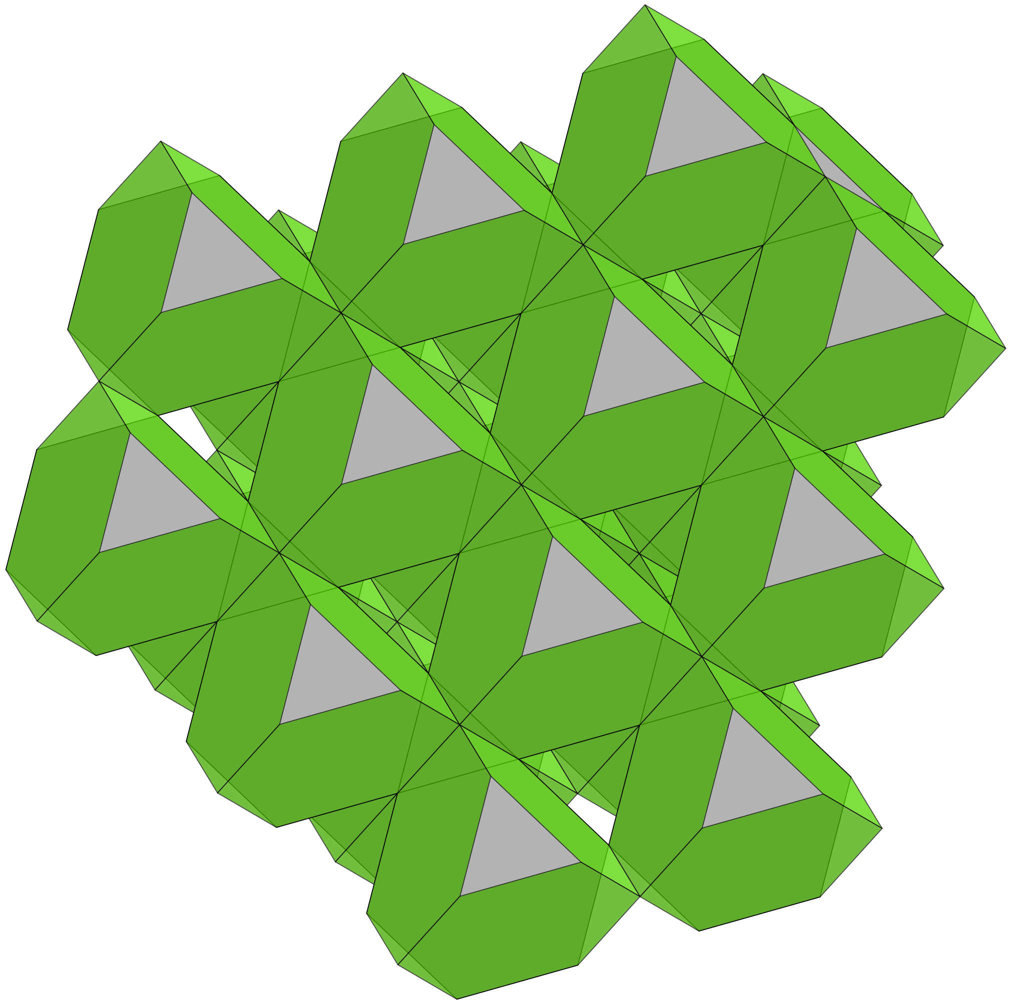
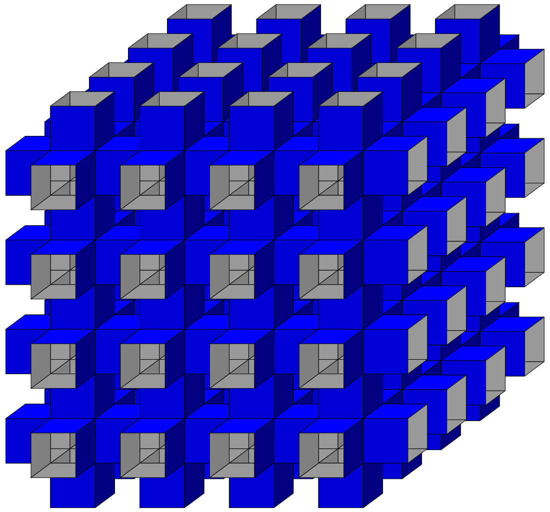
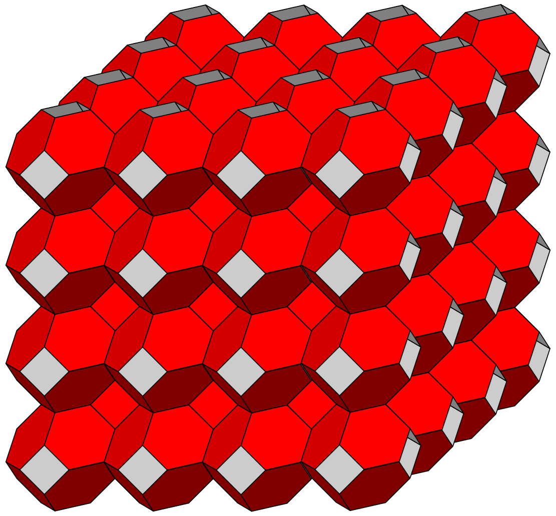


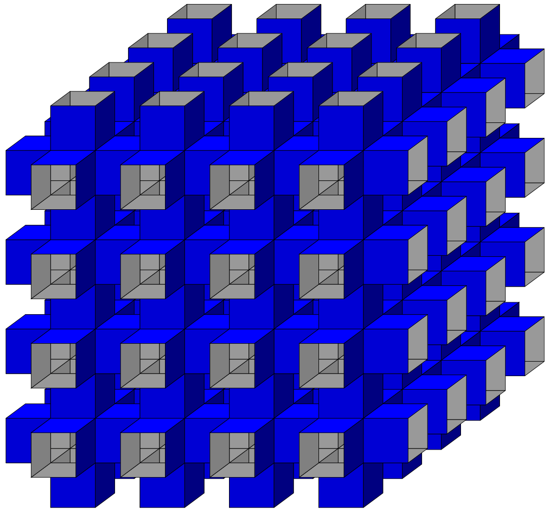
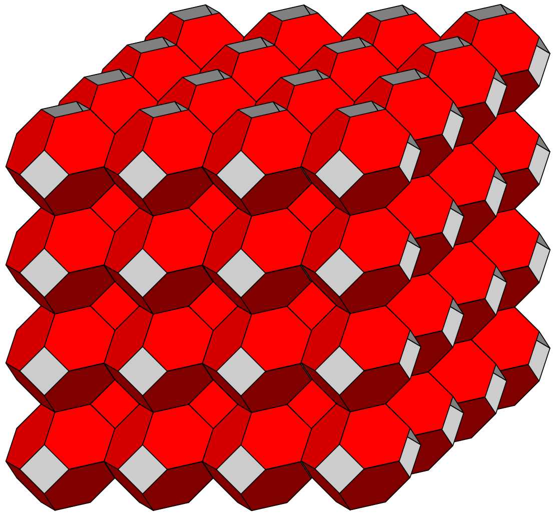
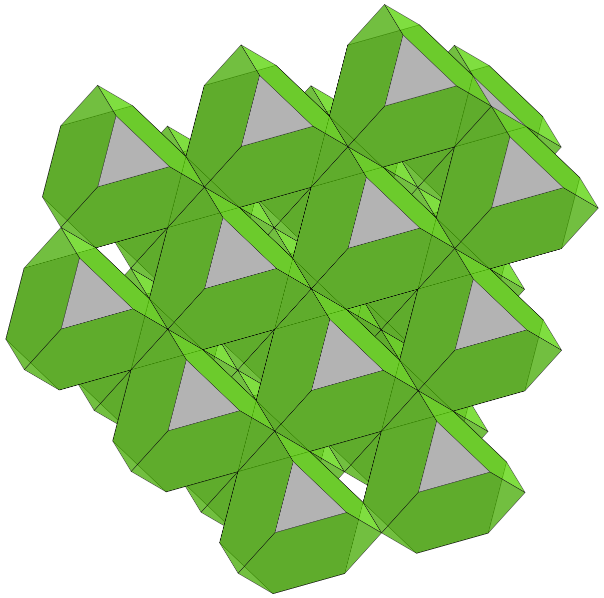


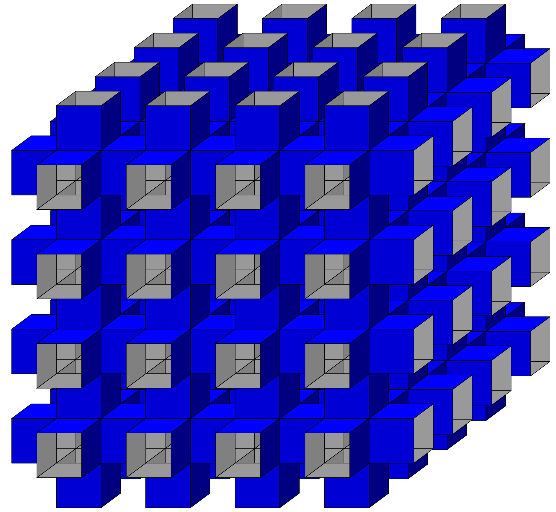


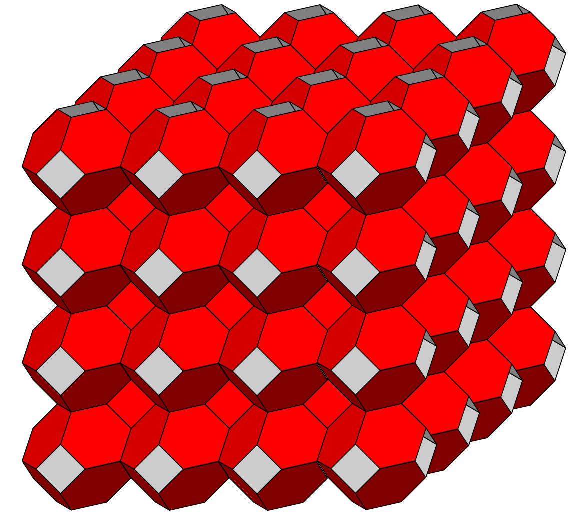


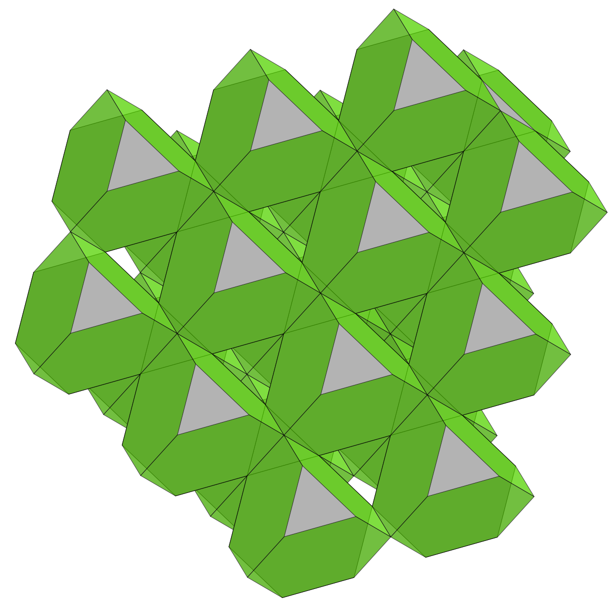






I also
proved that the list is
complete

London | 1926

London | 1926

Osijek, Croatia | 1929

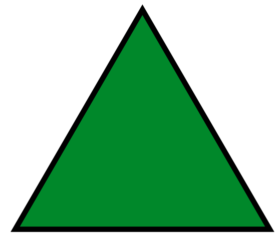
Branko Grünbaum
1929 - 2018

Regular polyhedra

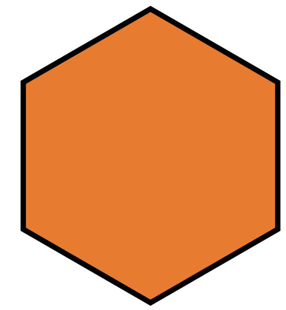
They are built by glueing polygons (faces) along their edges (two per edge)

I. All the faces must be equal.
II. Every face is a regular polygon.
III. The number of faces at every vertex must be the same.

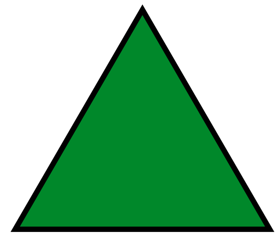
Faces?



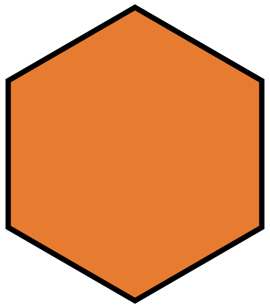
How many around each vertex?

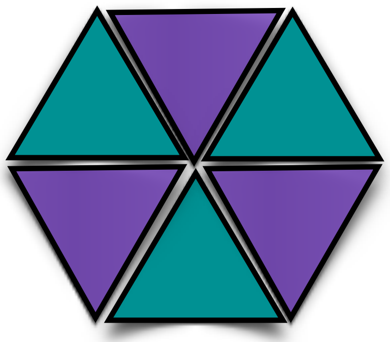


Faces?



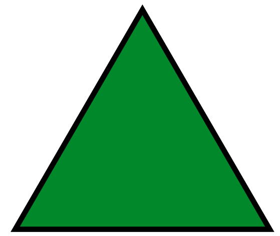
How many around each vertex?

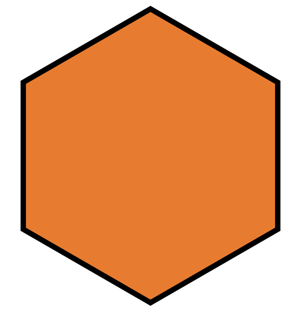


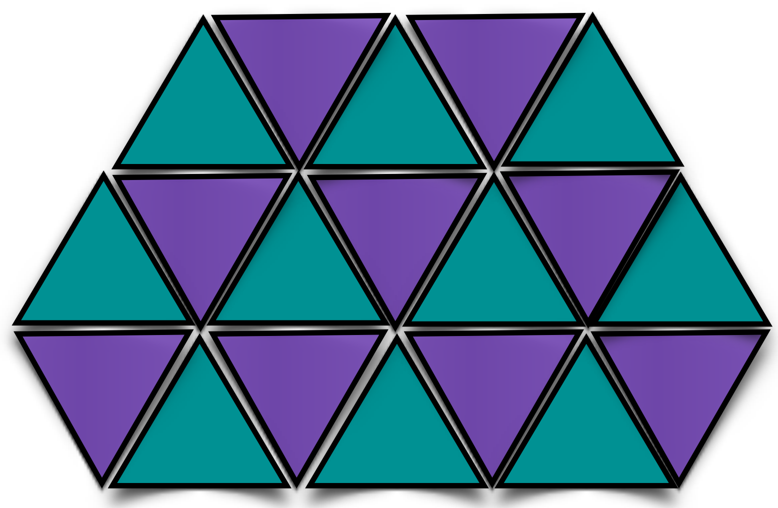


Faces?

How many around each vertex?

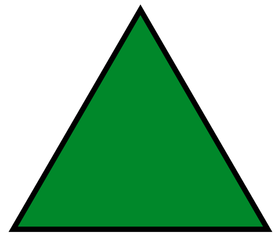


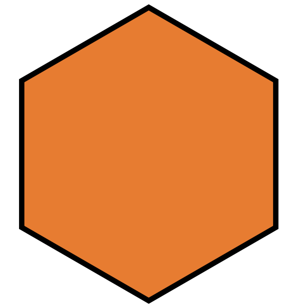


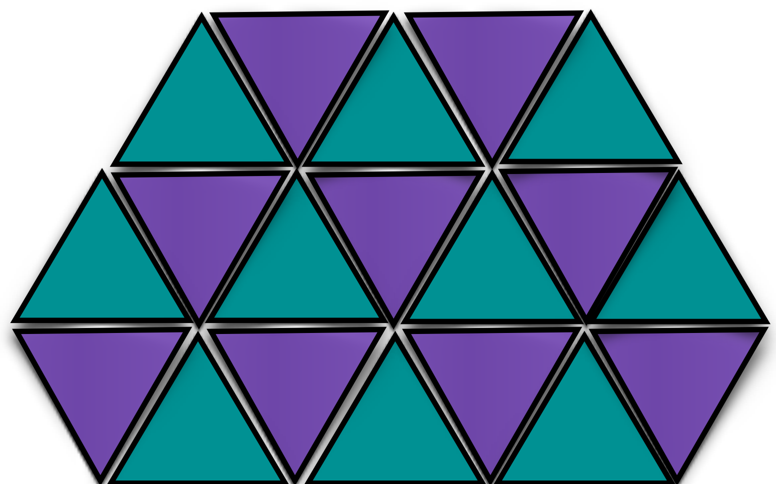


Faces?

How many around each vertex?

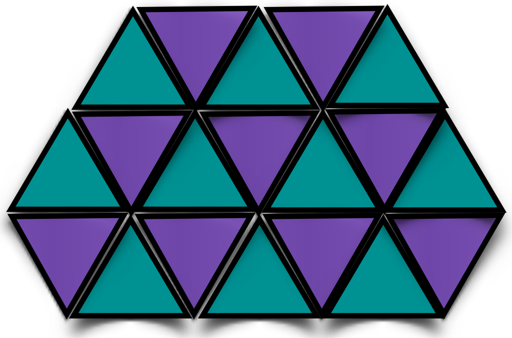






Faces?

How many around each vertex?



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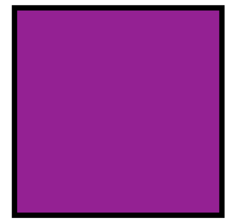
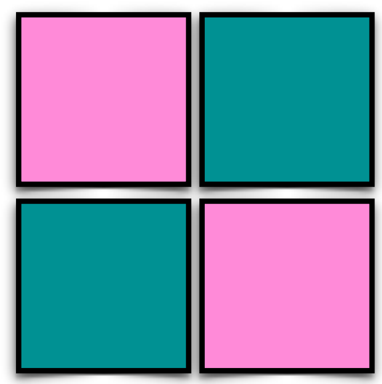
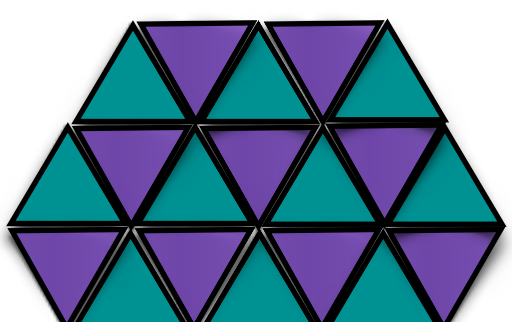
How many around each vertex?

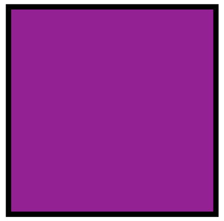


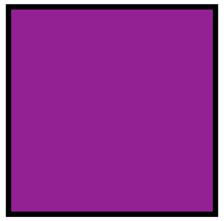


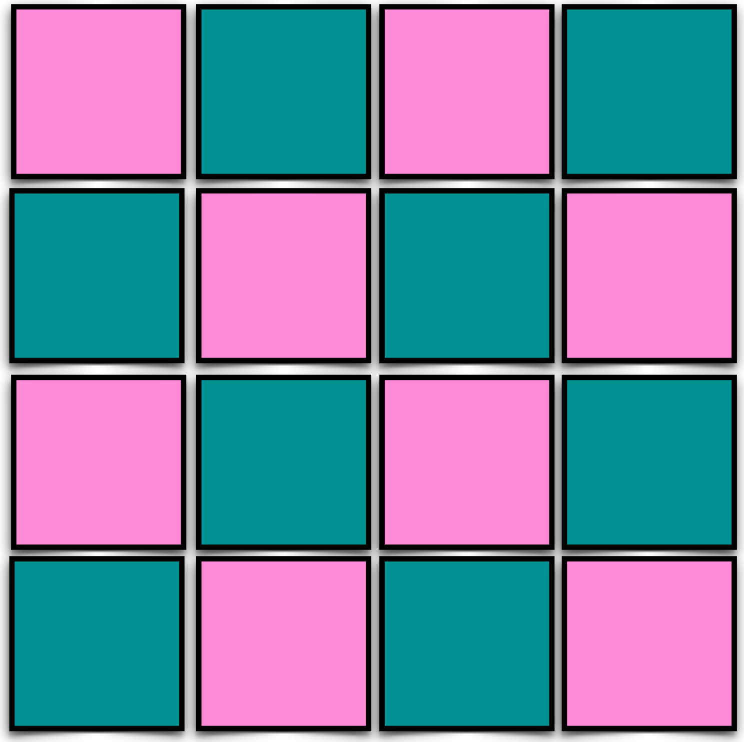


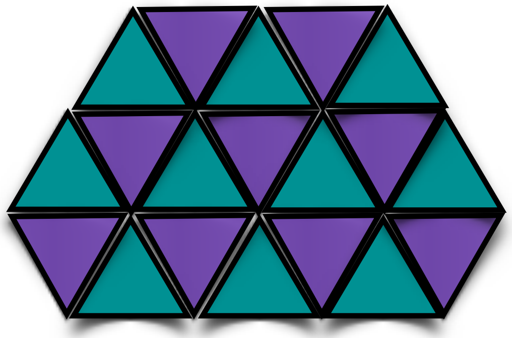
Faces?

How many around each vertex?



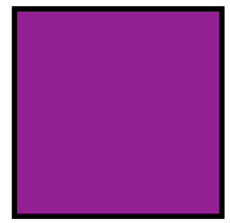


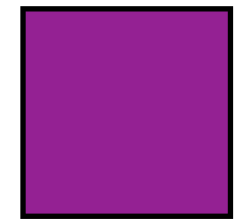


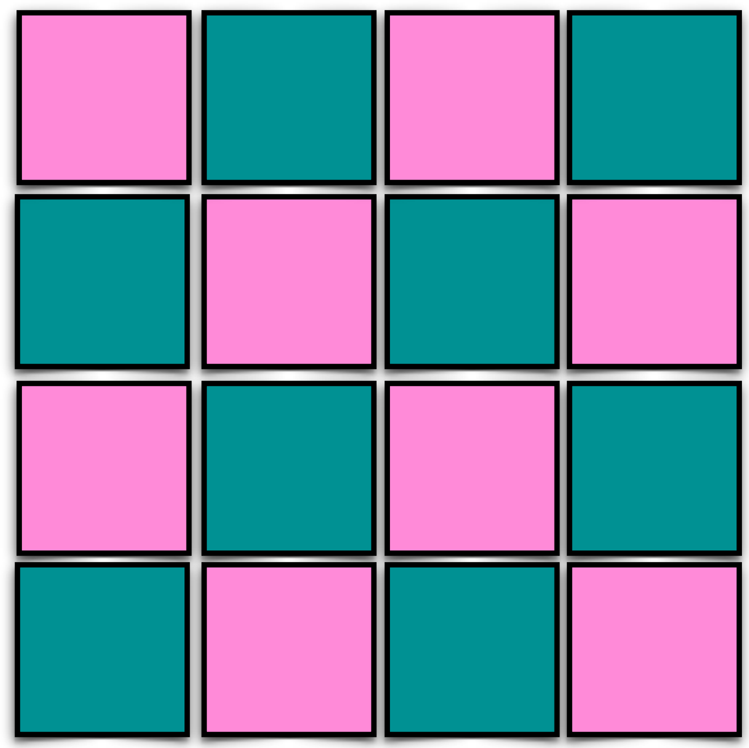


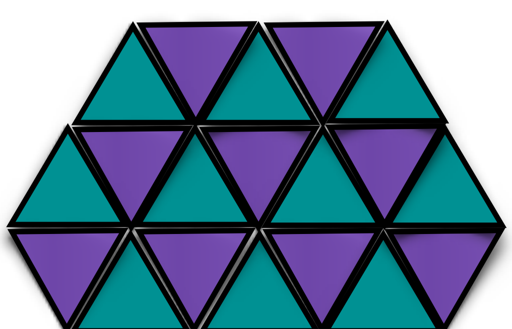
Faces?

How many around each vertex?









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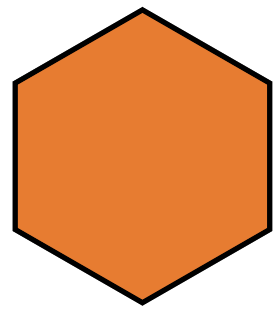
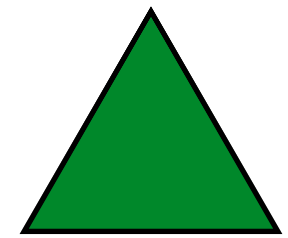
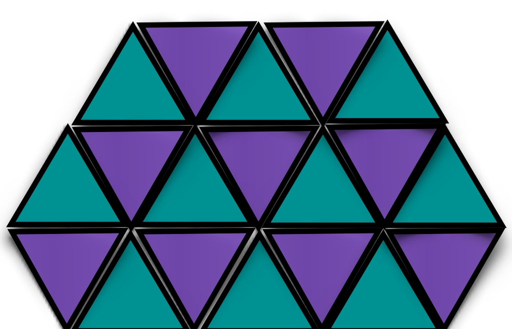
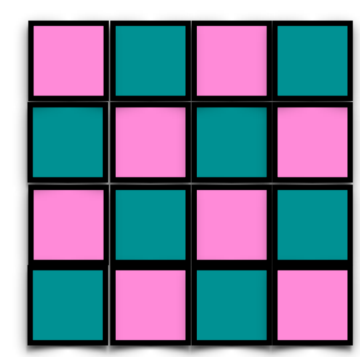
How many around each vertex?









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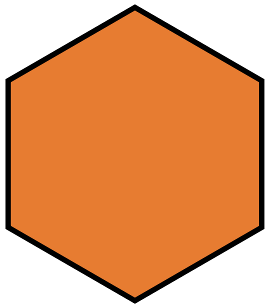

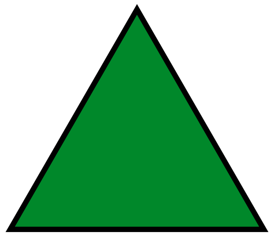

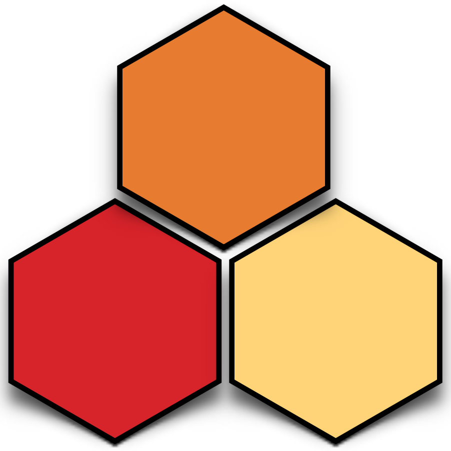

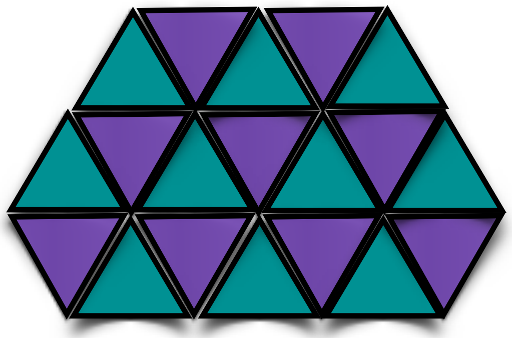

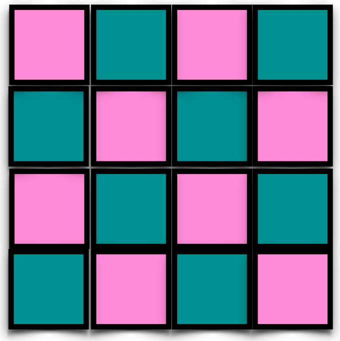

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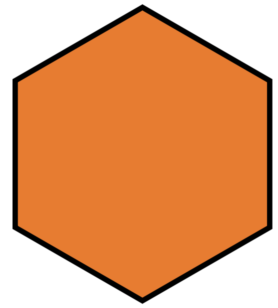

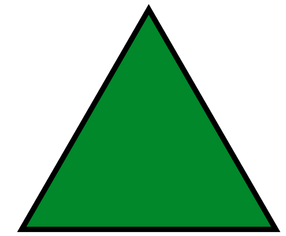

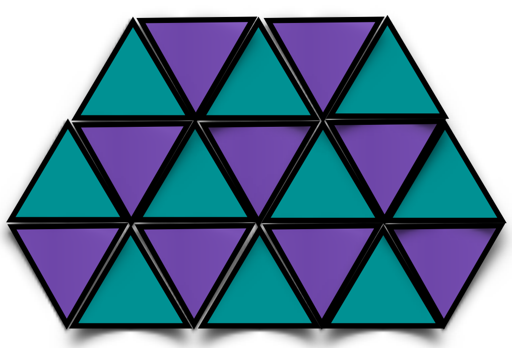

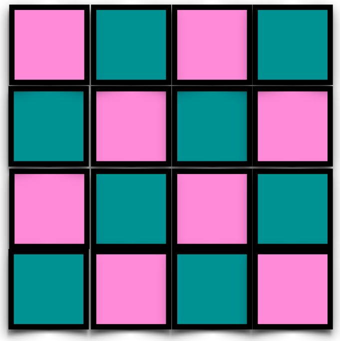

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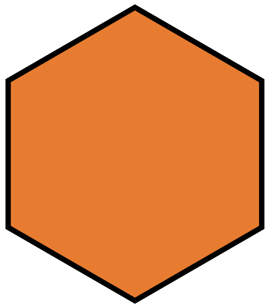
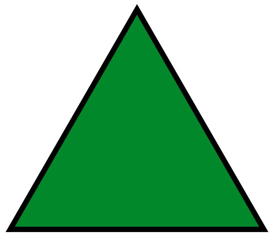
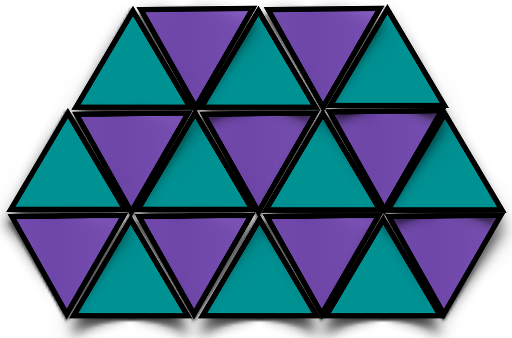
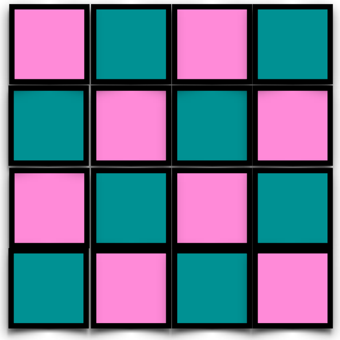
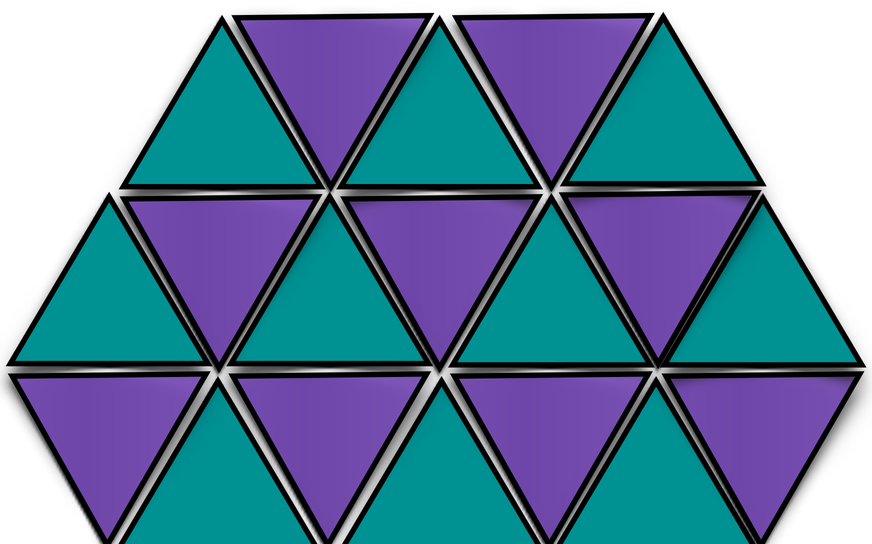
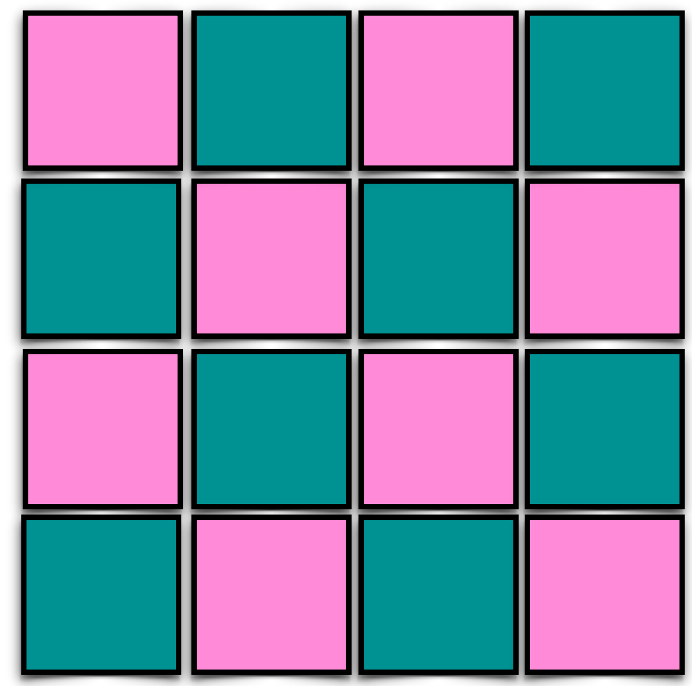




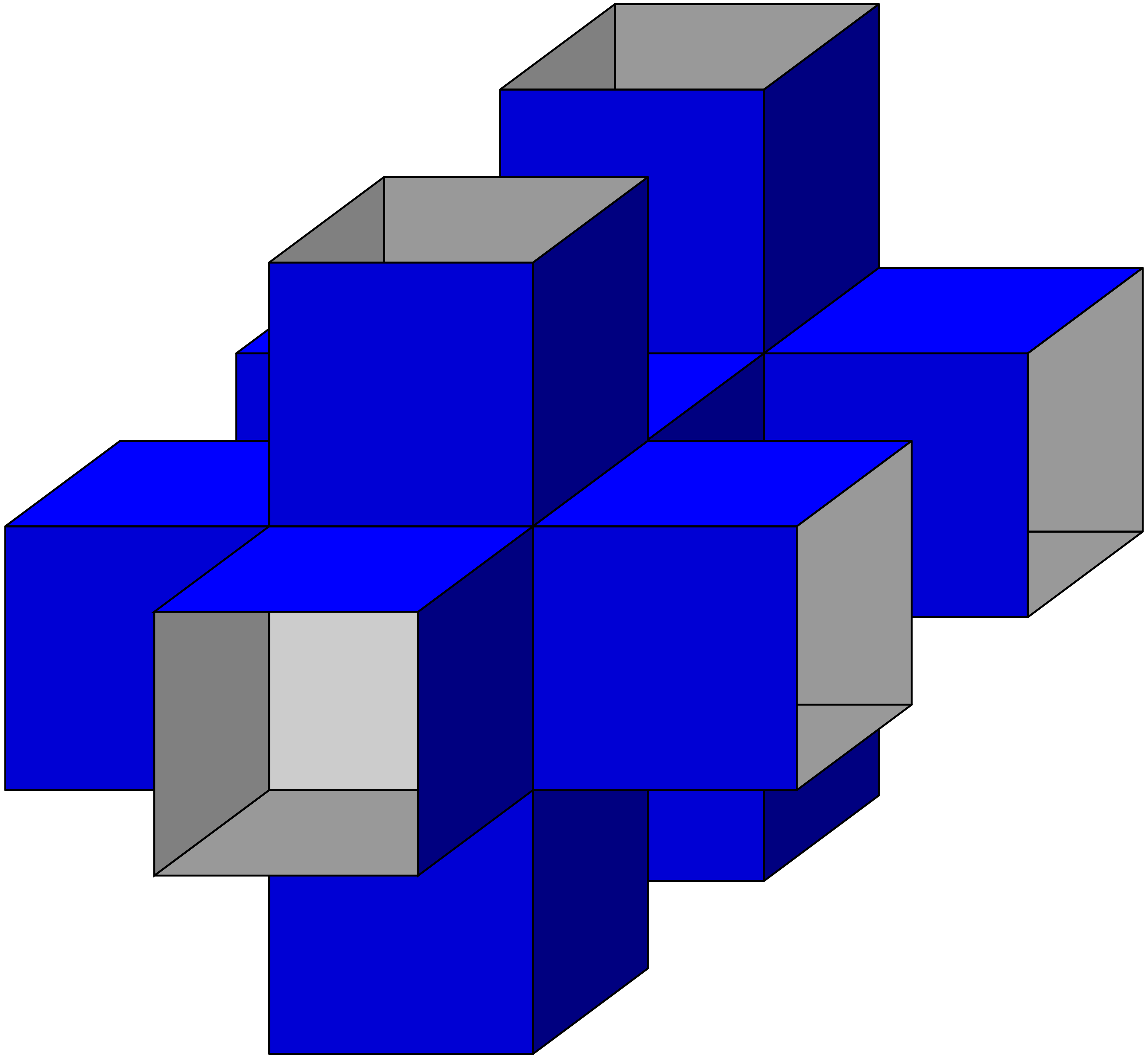


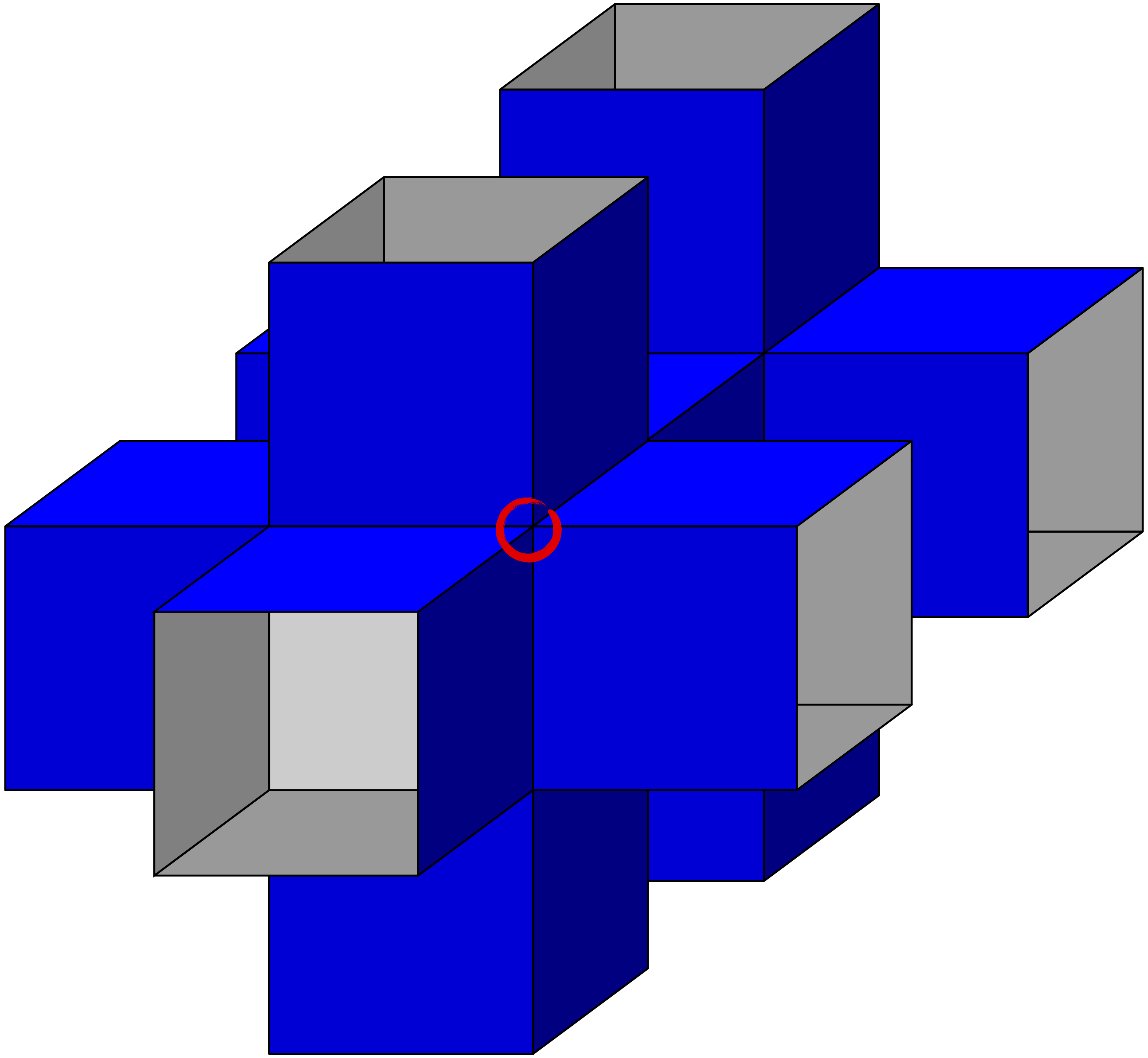


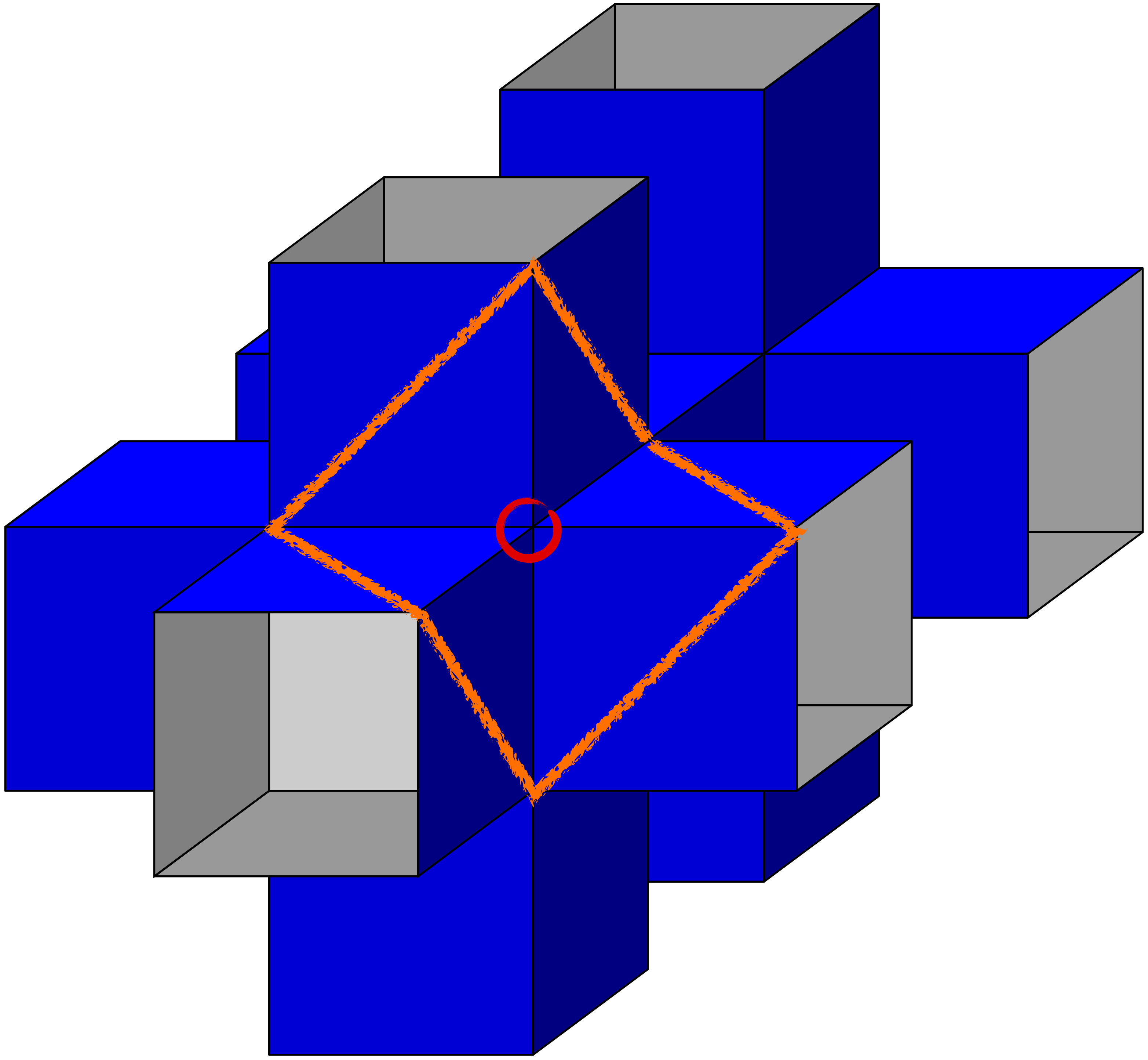




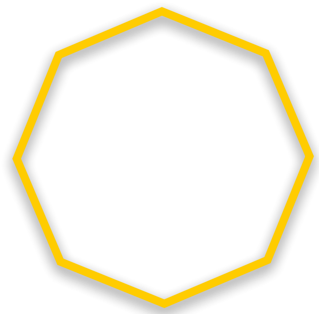
Infinite planar regular polyhedra

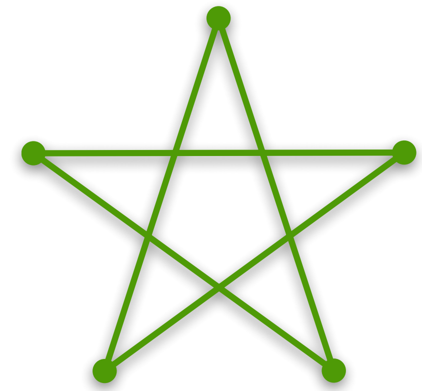




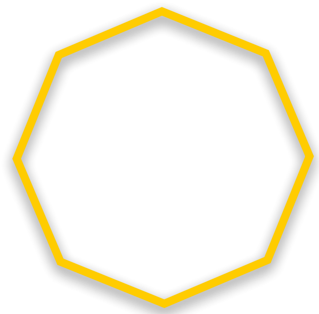


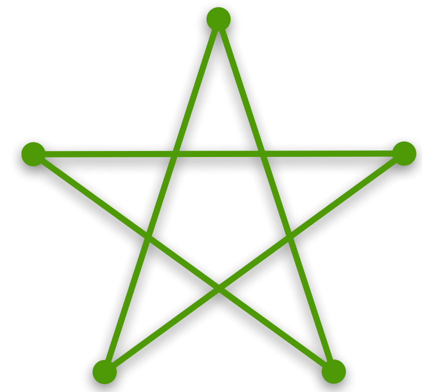
Regular polygons

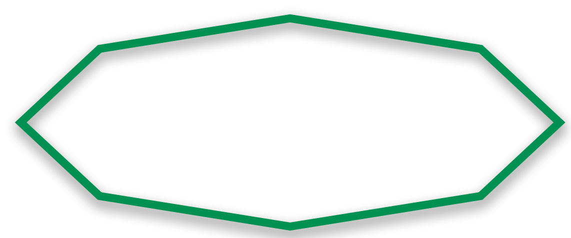




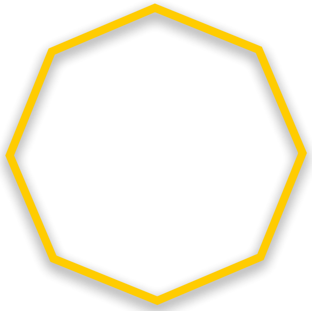
Regular polygons

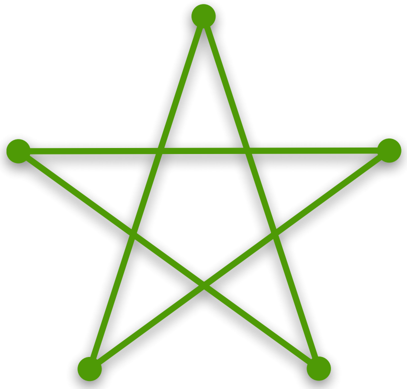


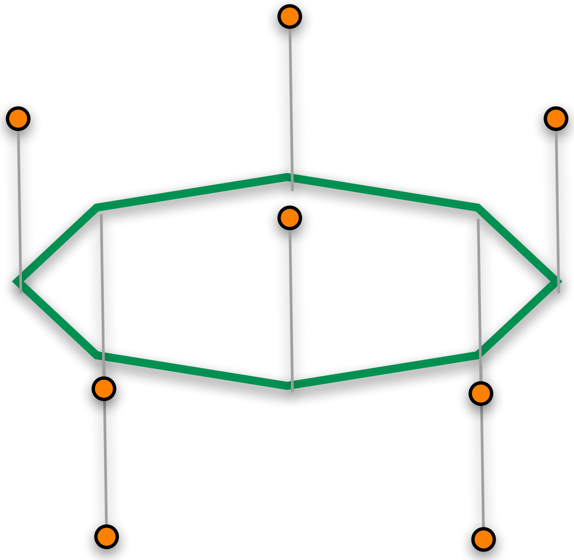




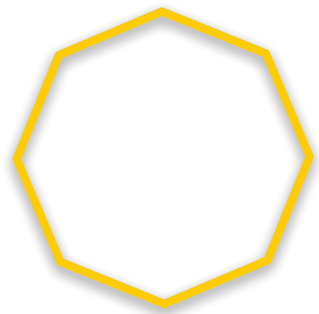
Regular polygons

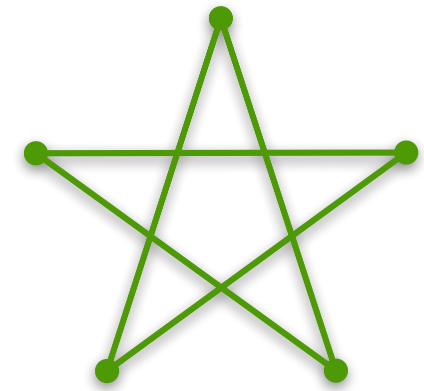


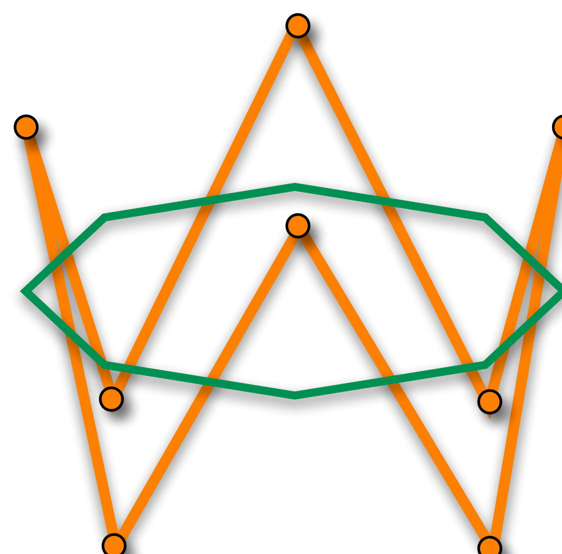




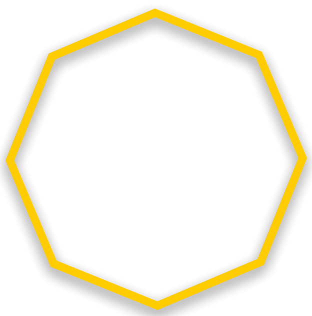
Regular polygons

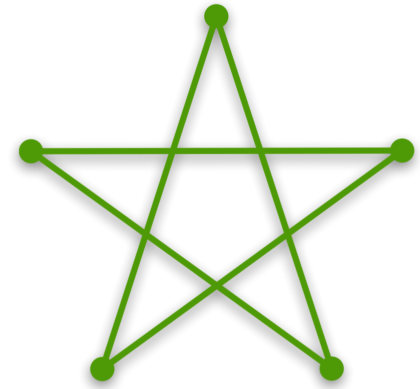


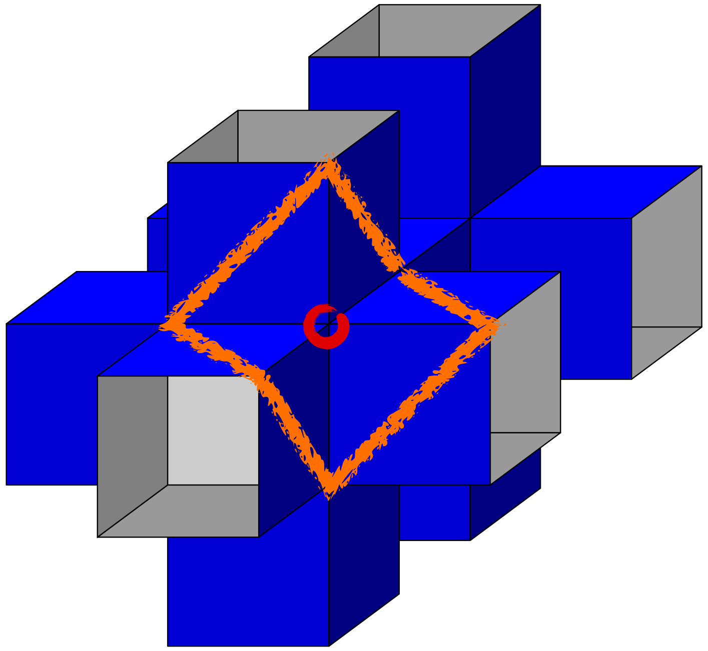


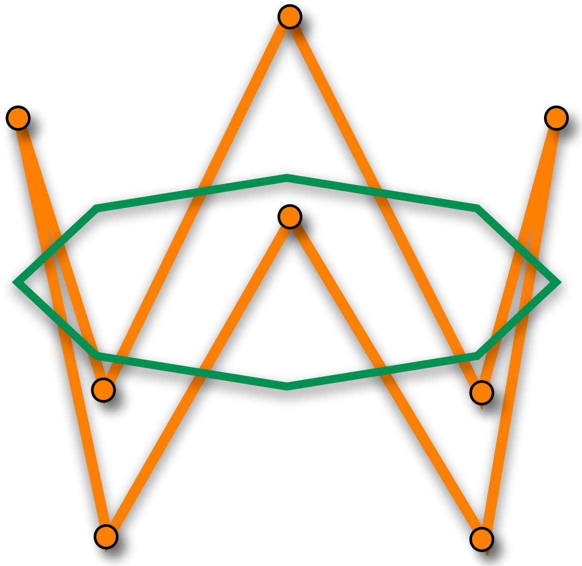


Regular polygons

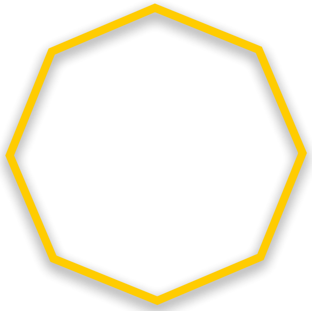


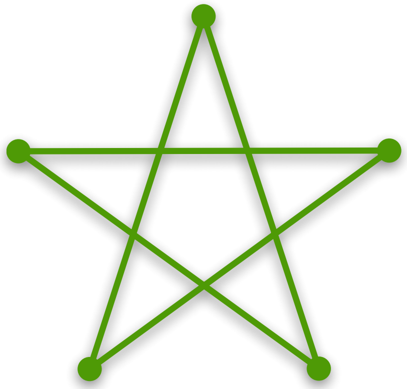


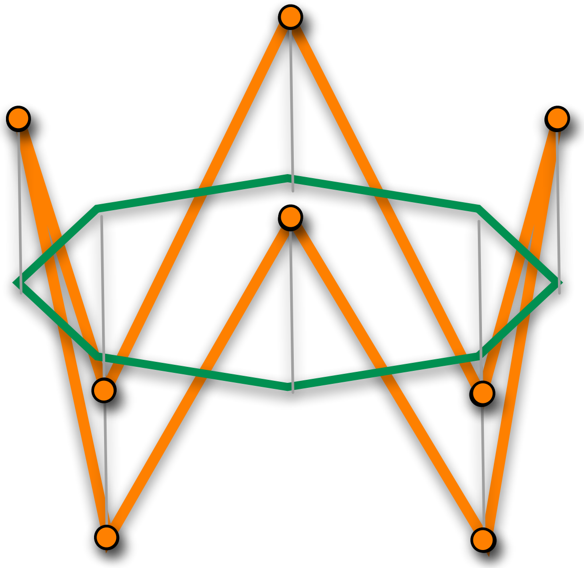




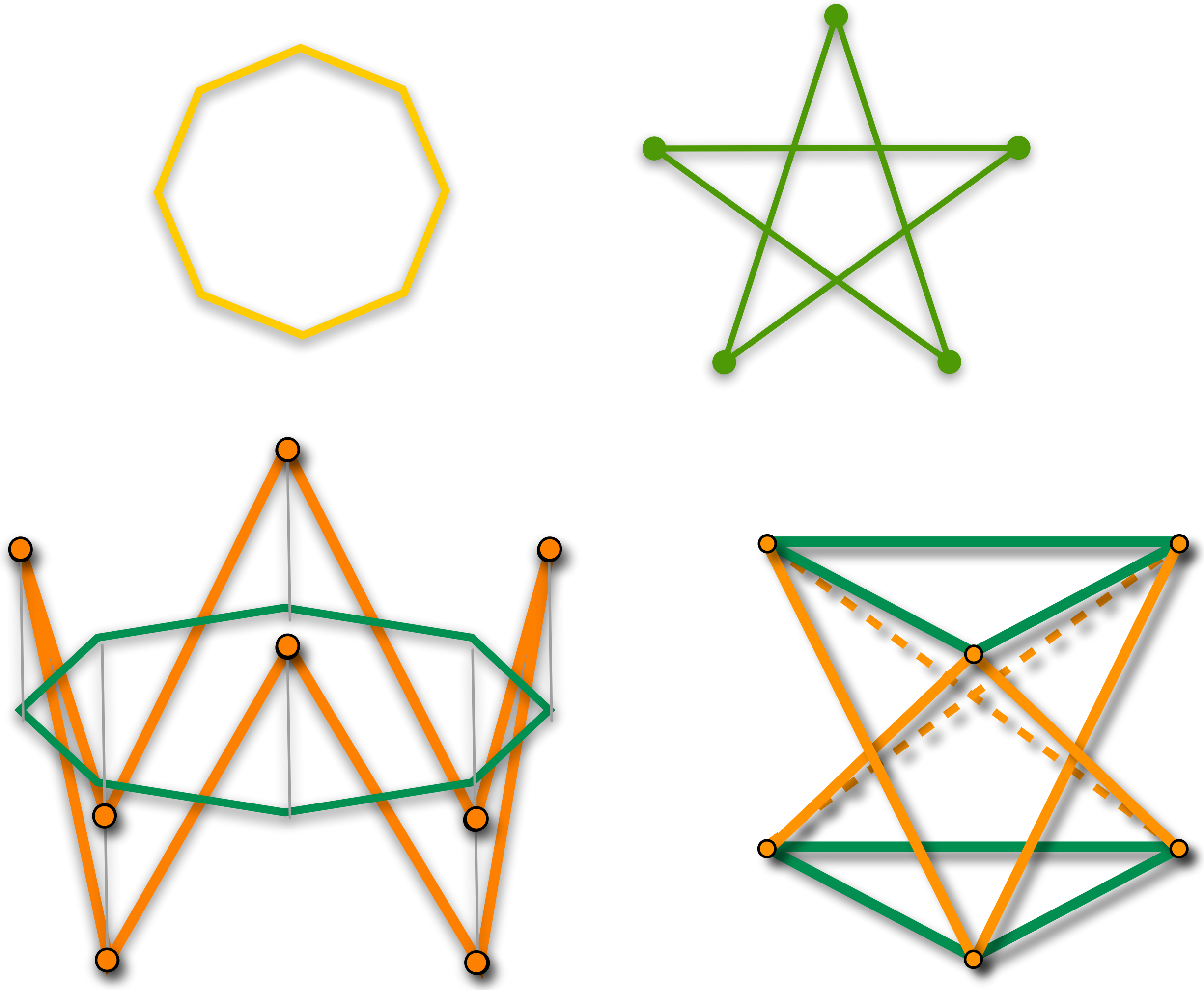
Regular polygons







Regular polygons



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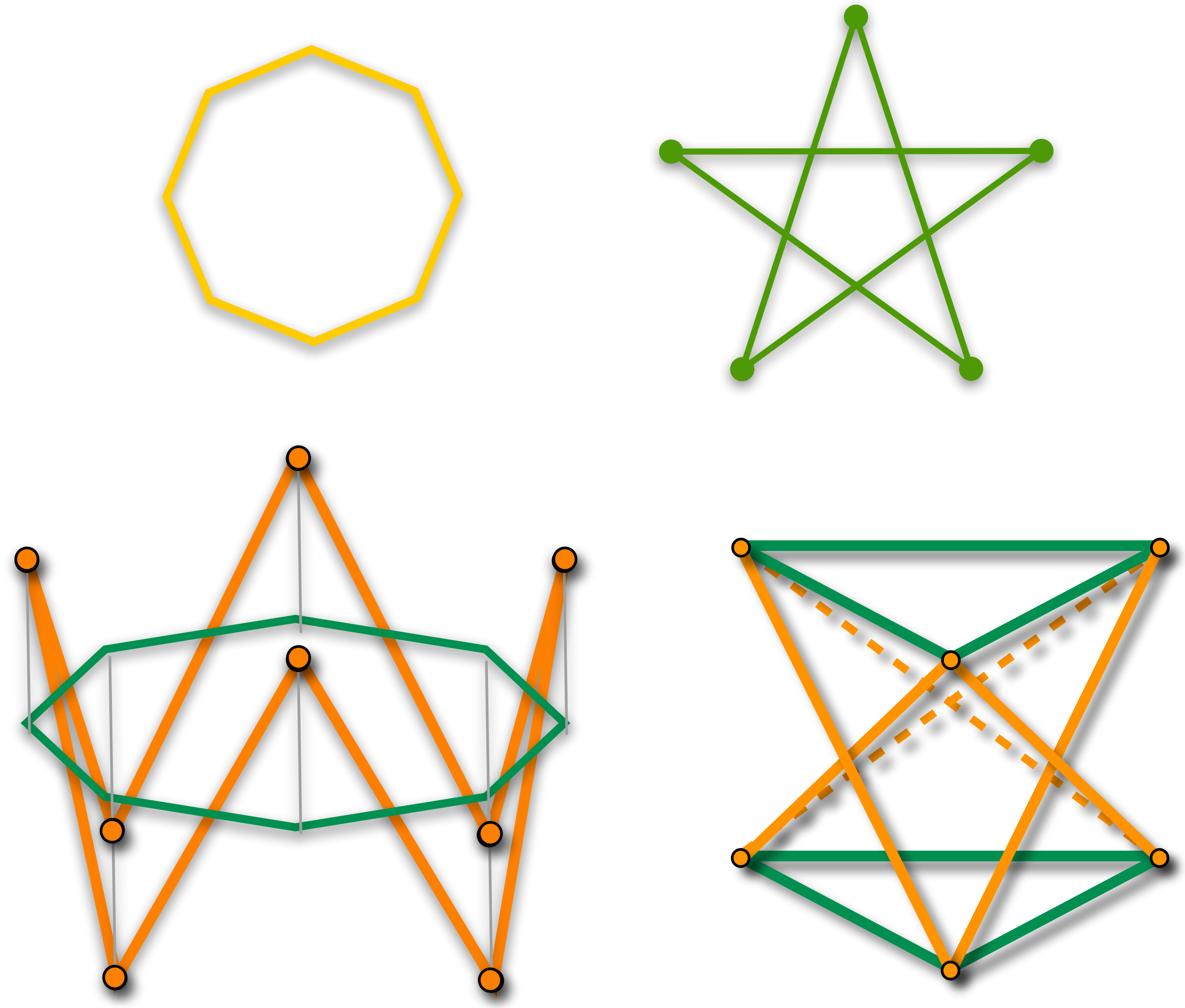
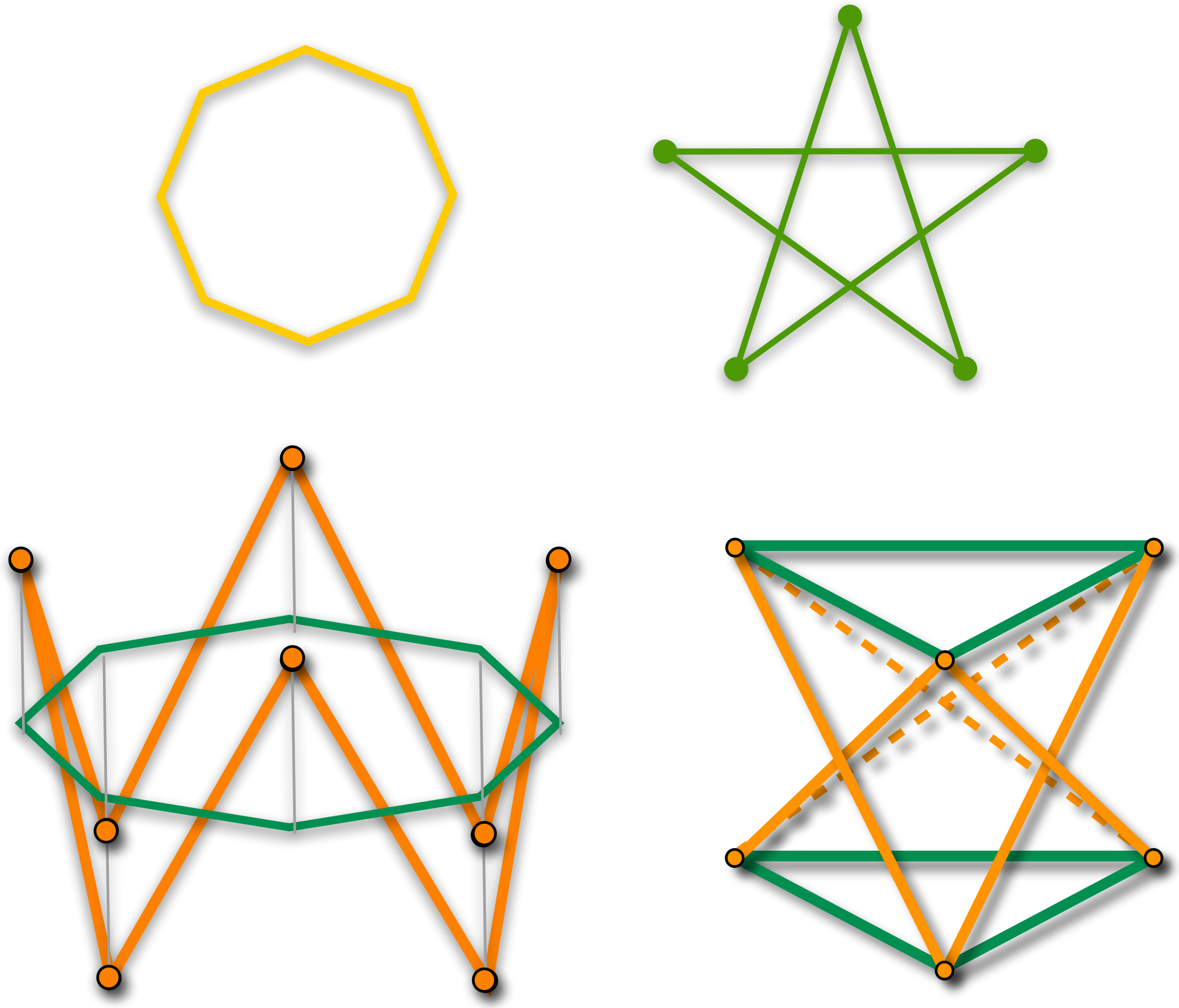
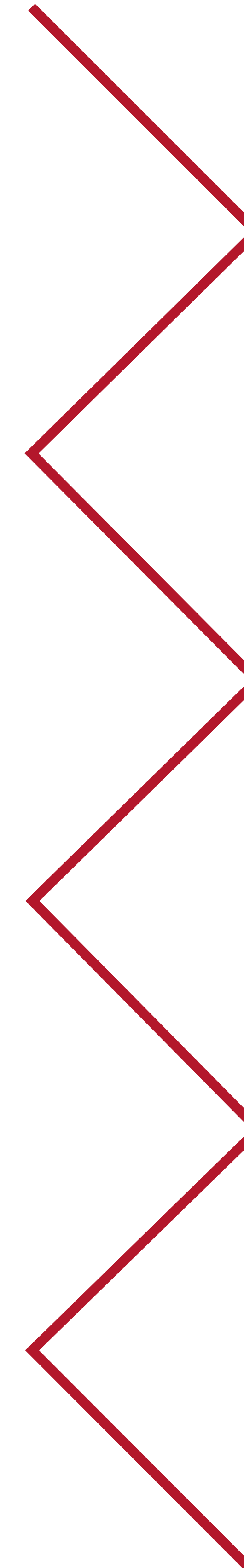


Regular polygons





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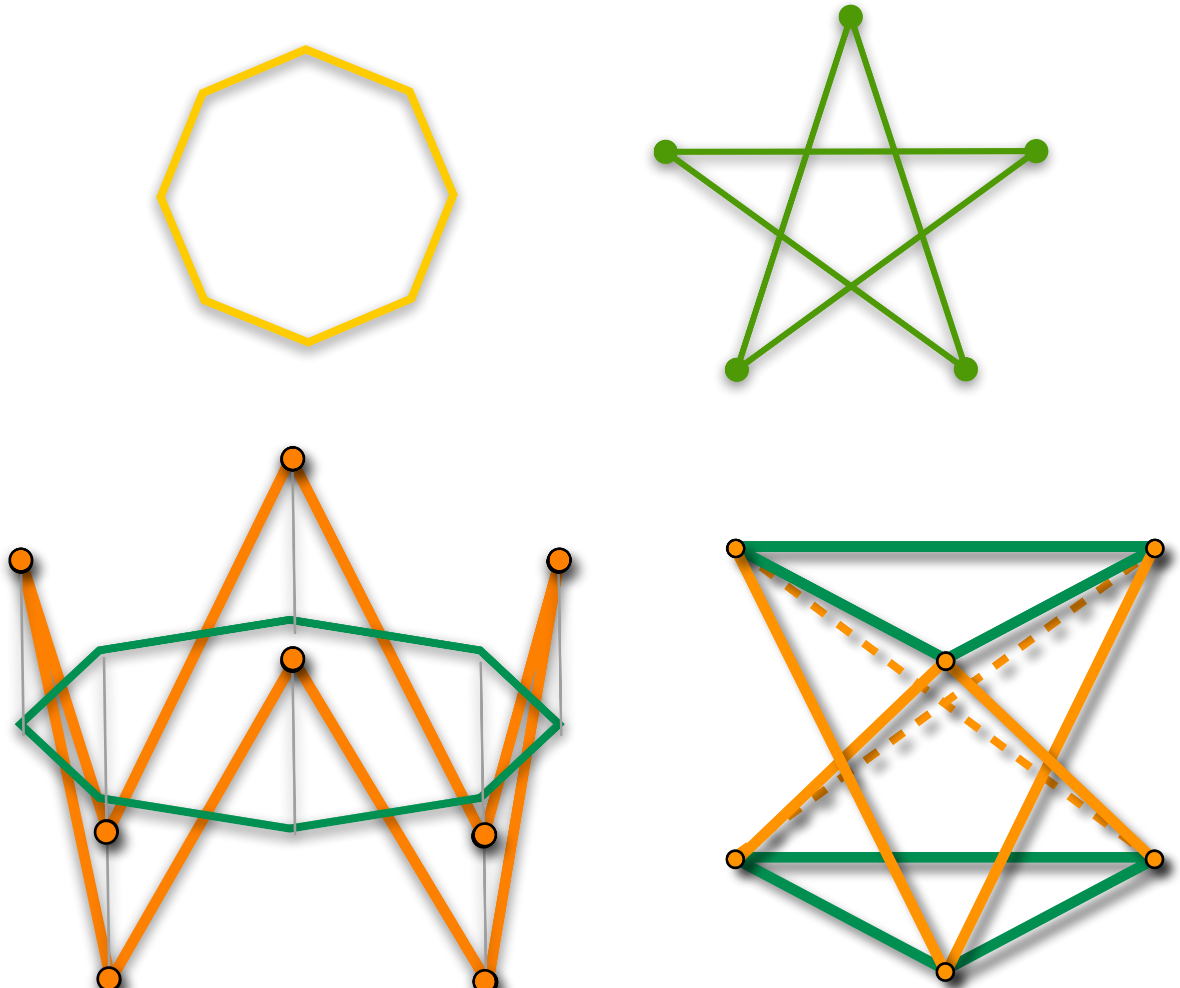

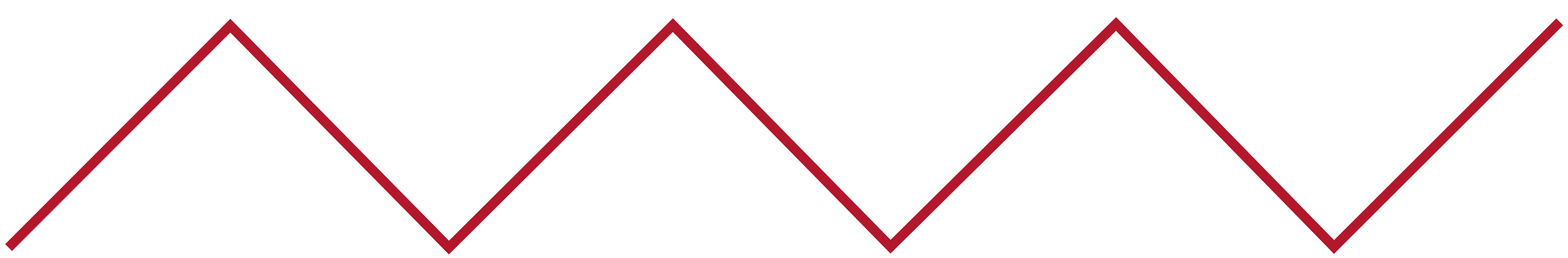

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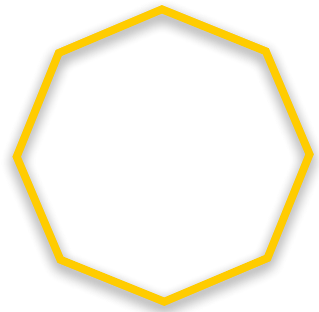

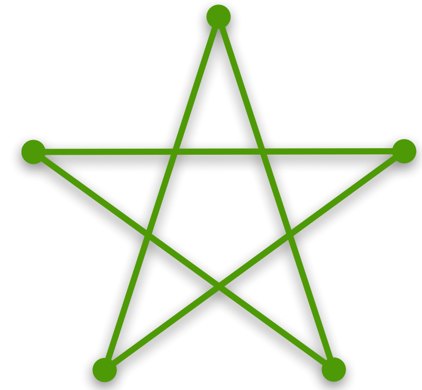

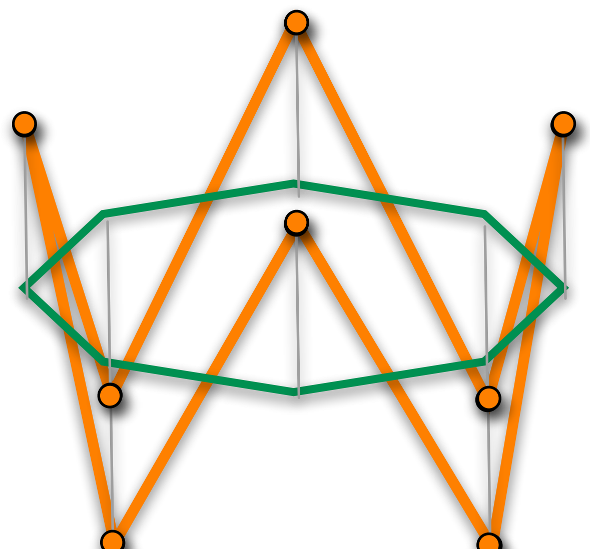

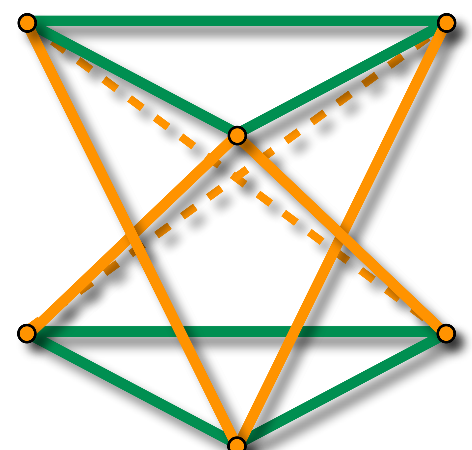

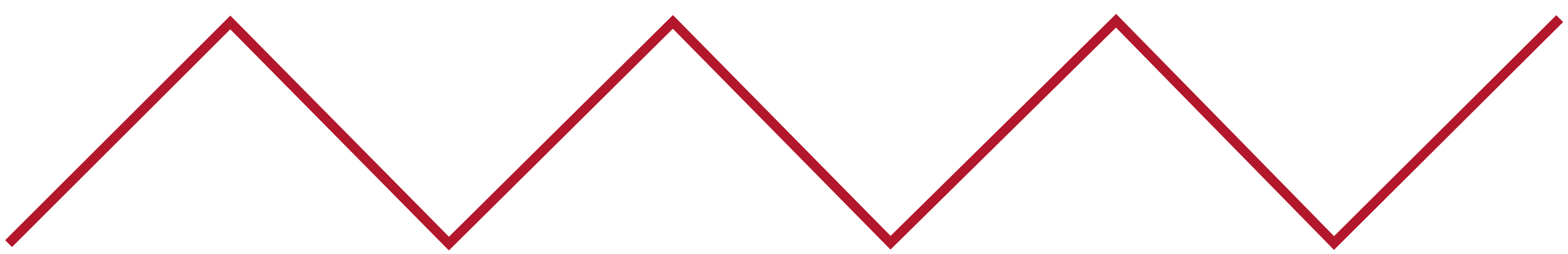

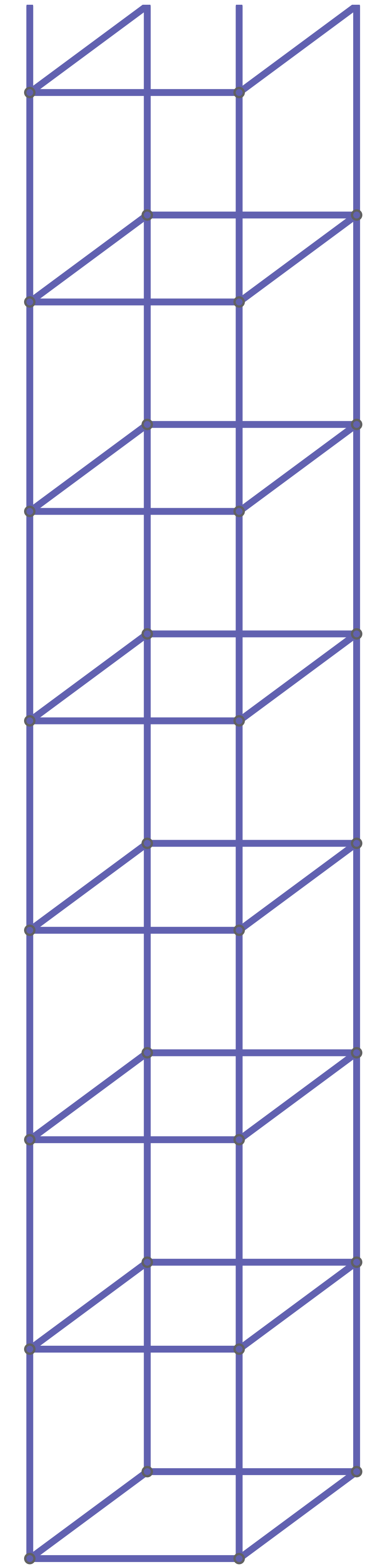

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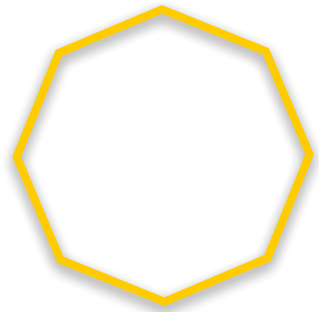
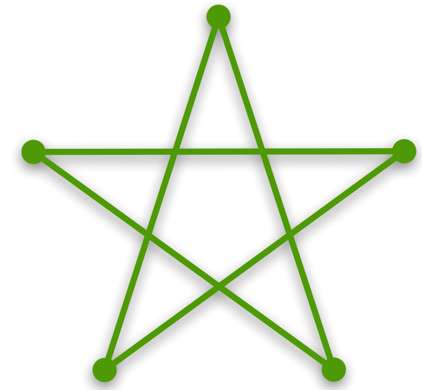
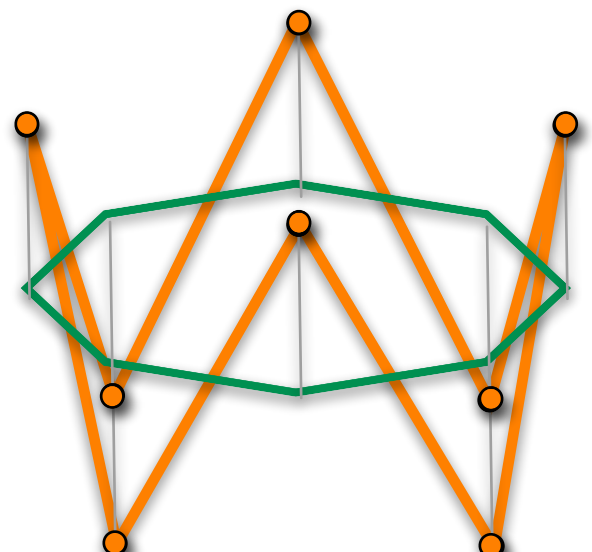
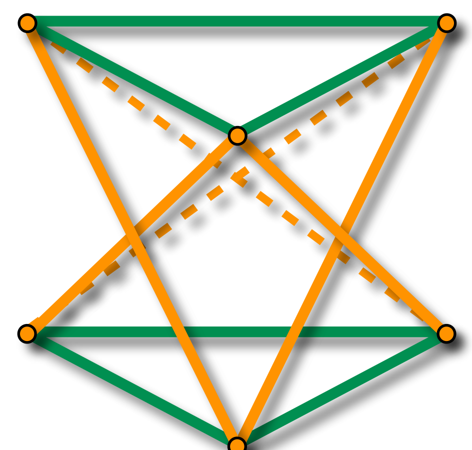
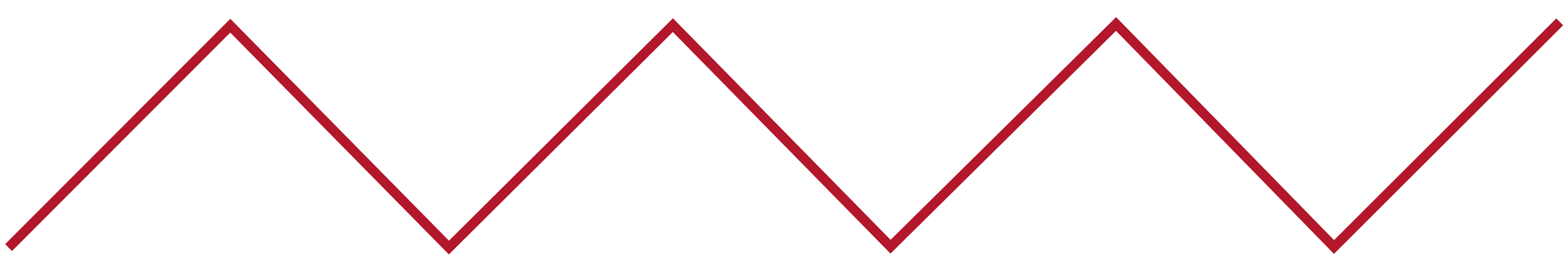
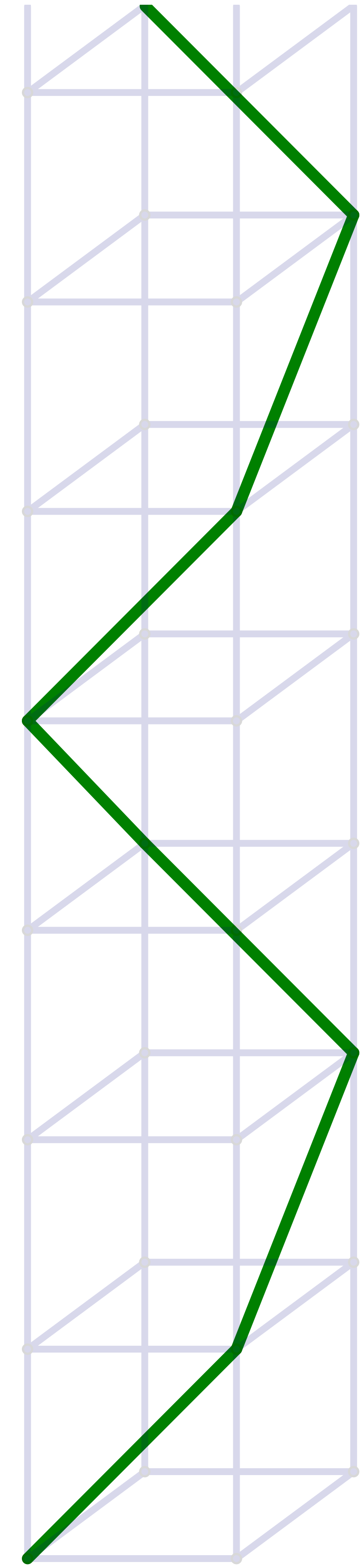






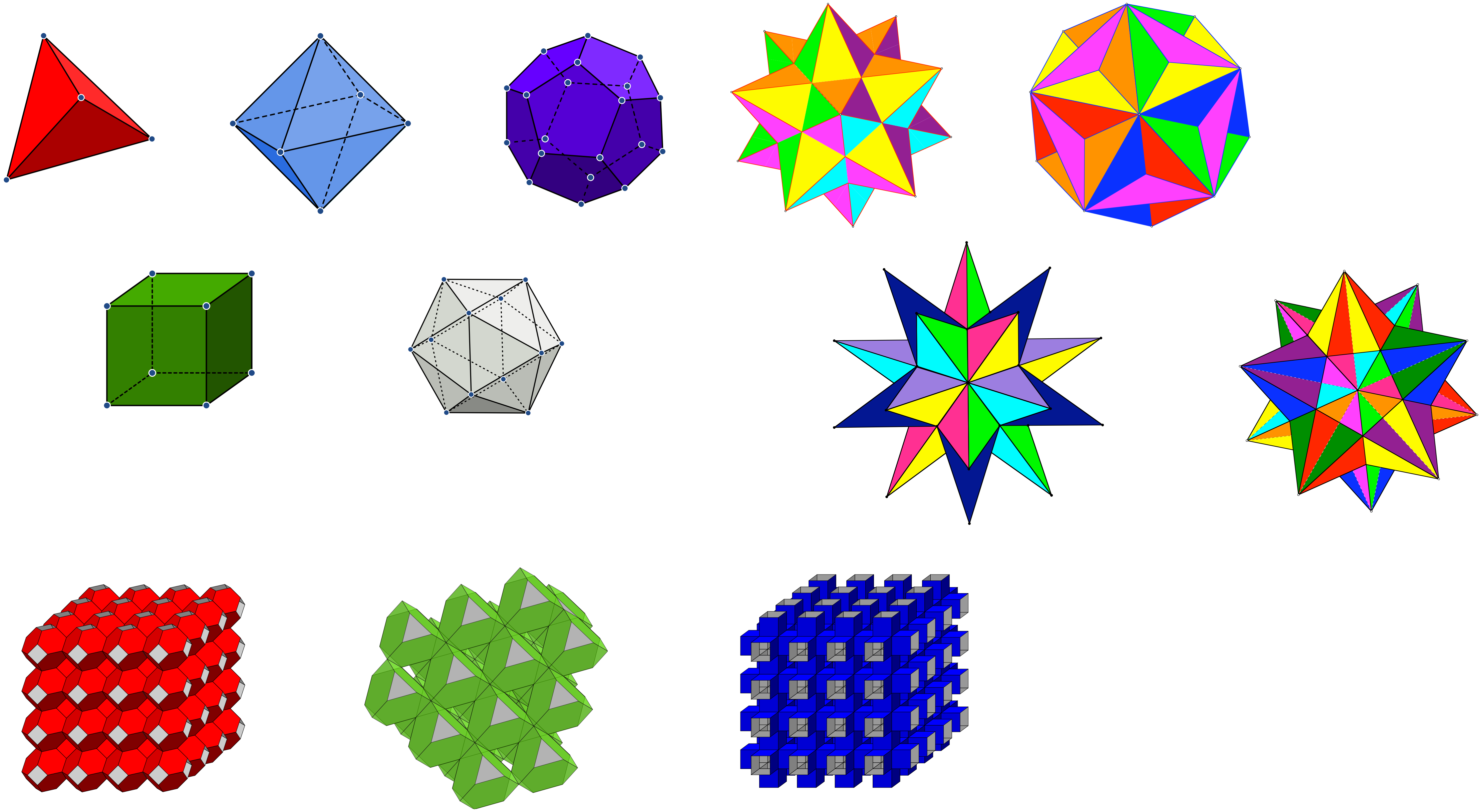






Grünbaum 1977

Finds 47 regular polyhedra in $\mathbb{E}^3$



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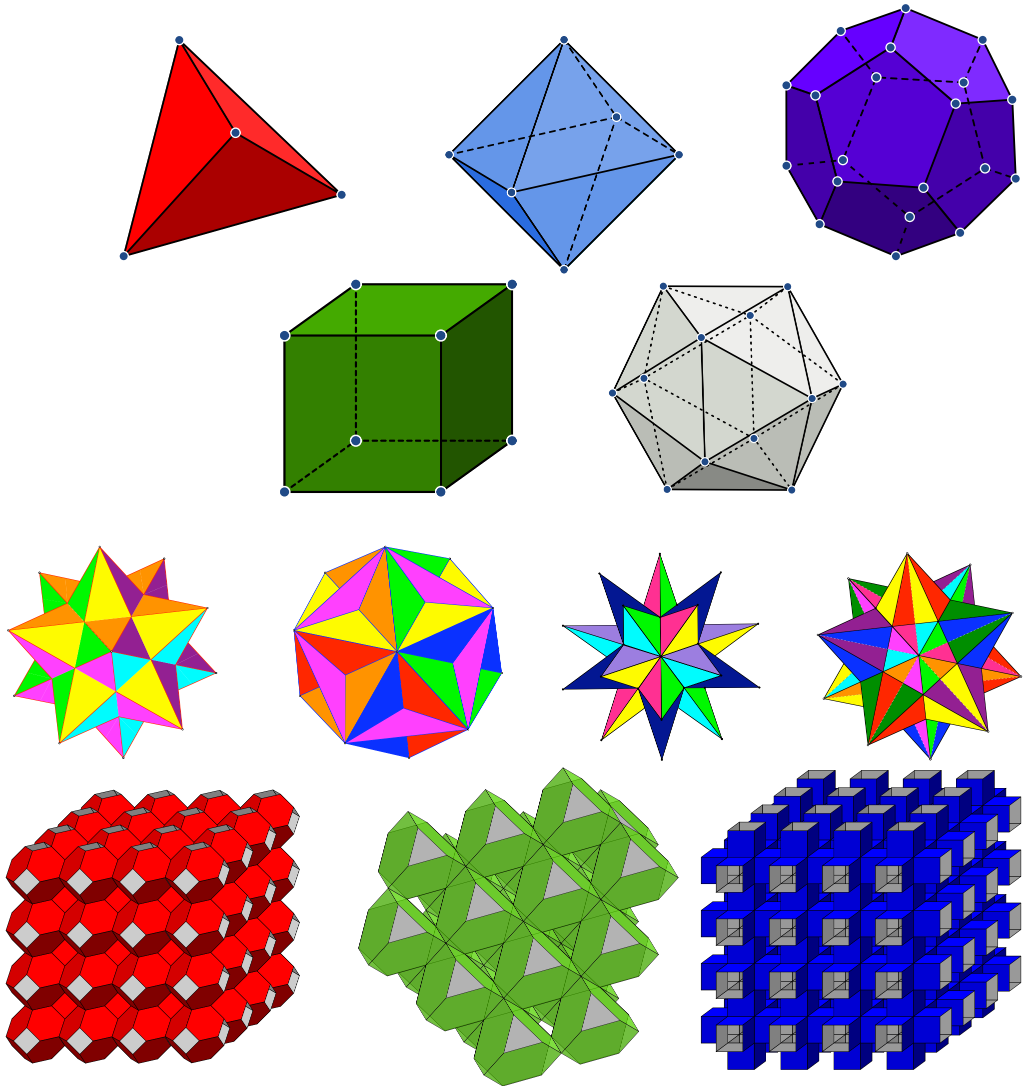

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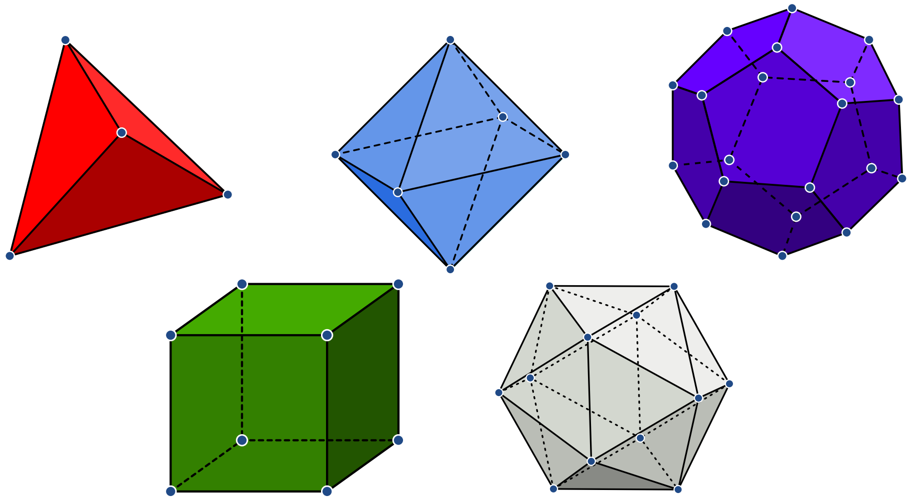
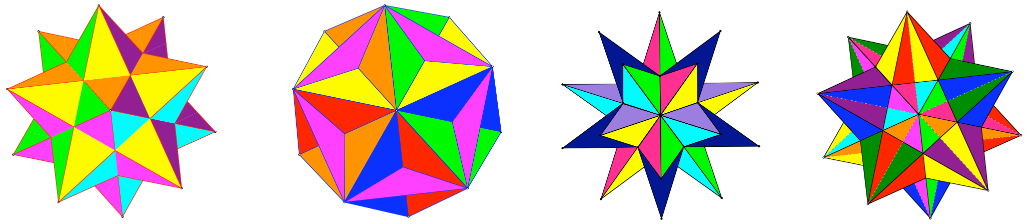
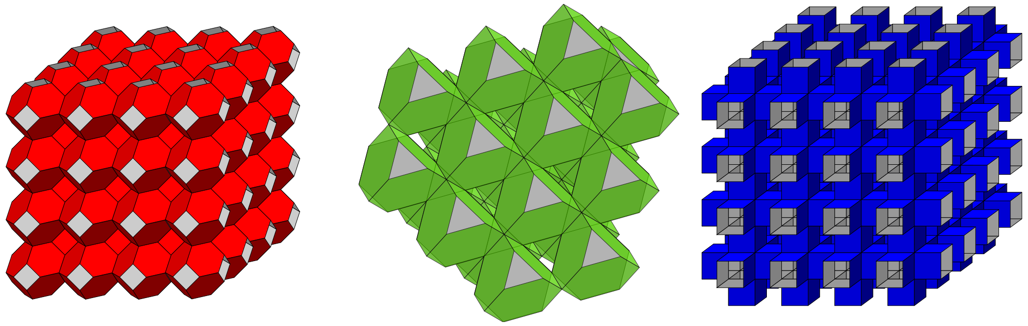
- Gives a definition
- Never claims his list is complete







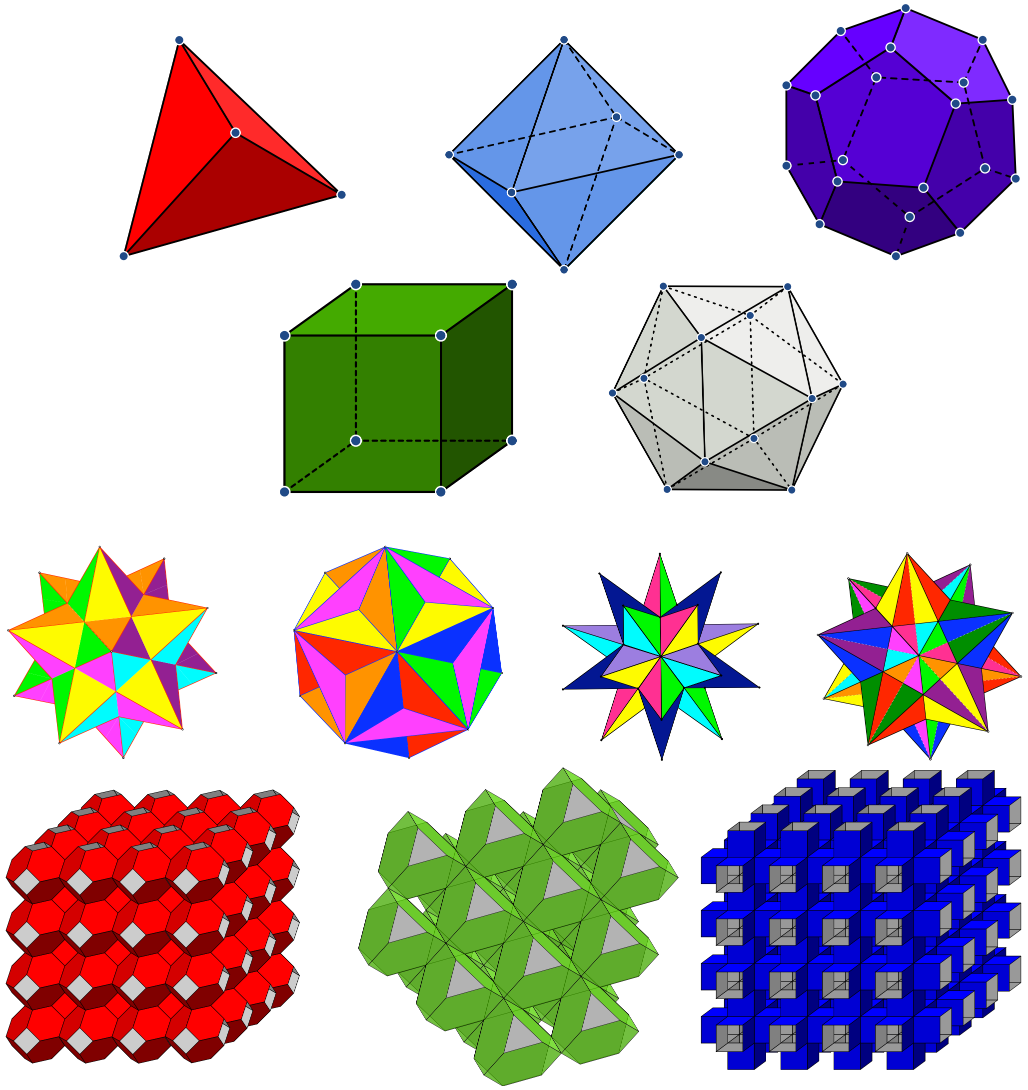
Grünbaum 1977

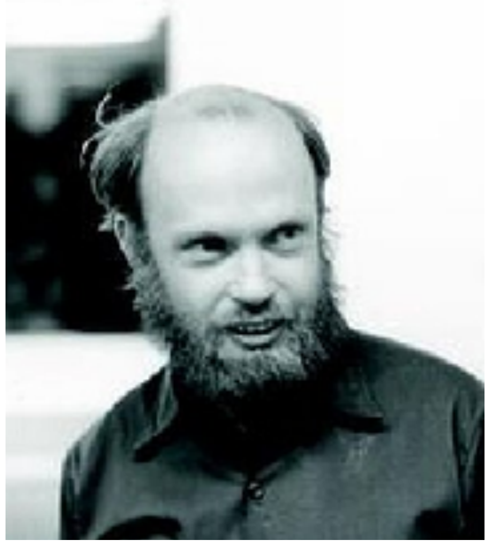
Finds 47 regular polyhedra in $\mathbb{E}^3$

- Gives a definition
- Never claims his list is complete

A. Dress 1981, 1985

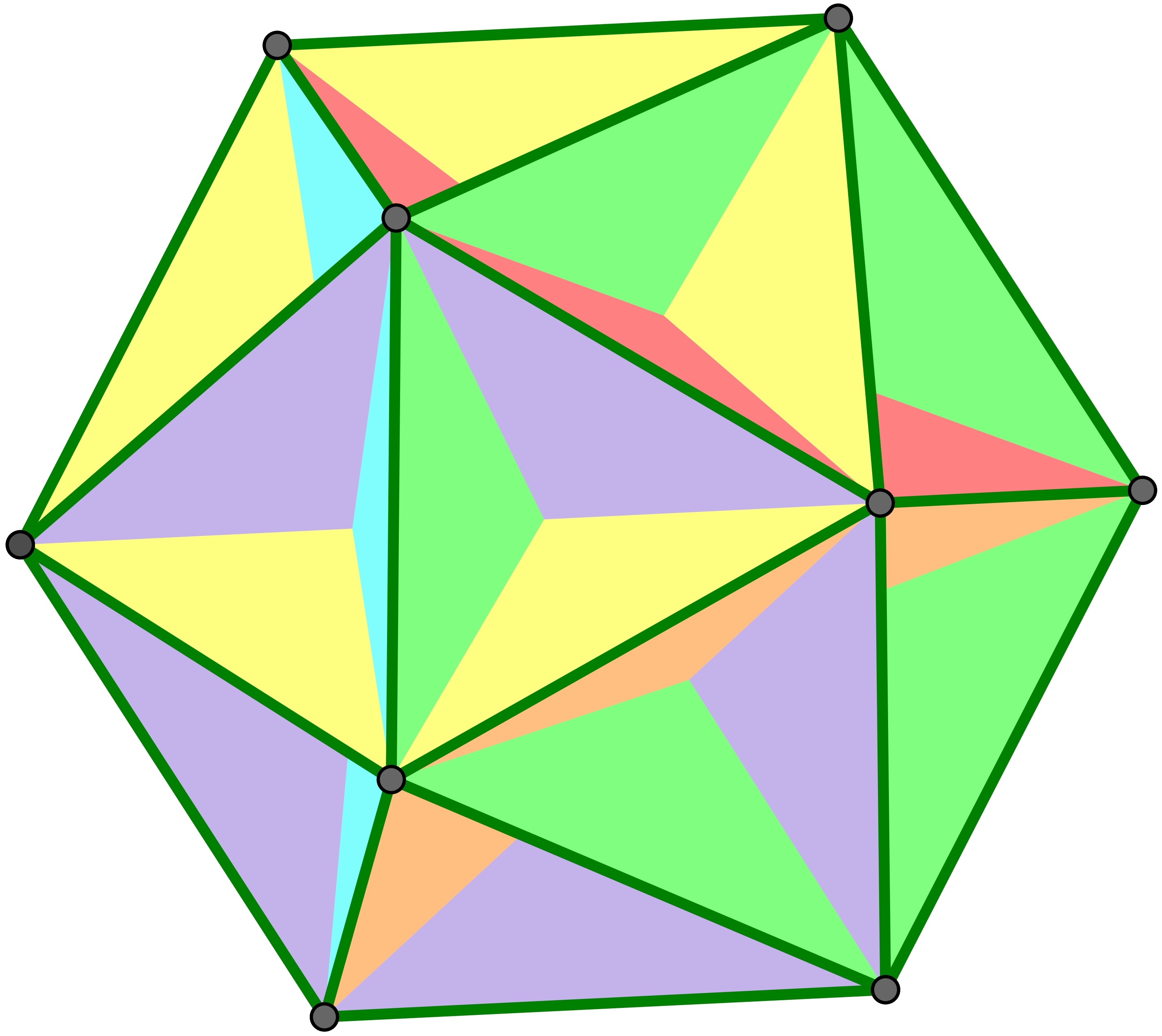
- Finds another one
- Proves that the list is complete





Andreas Dress
1938 - 2024

One of the many...



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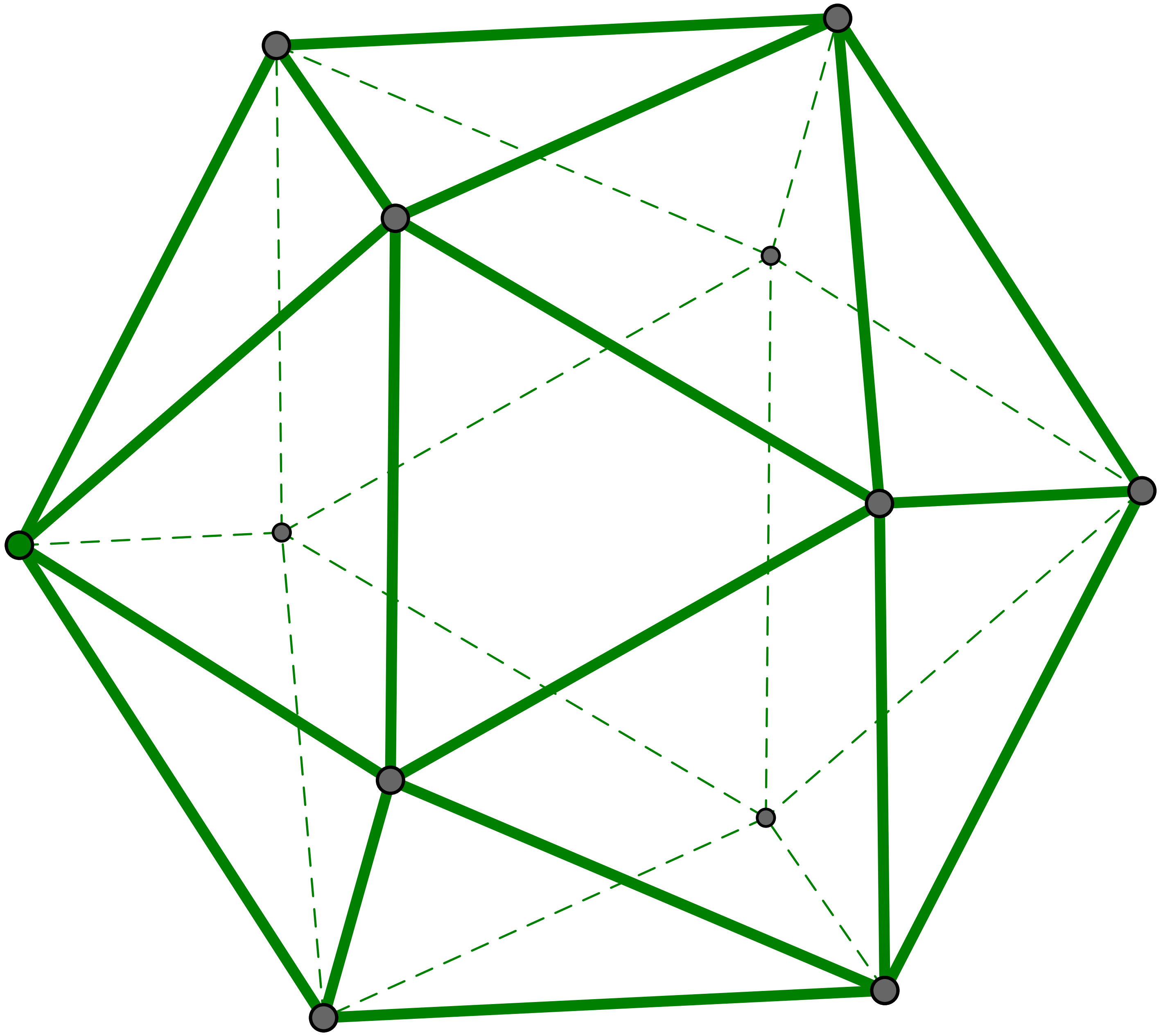

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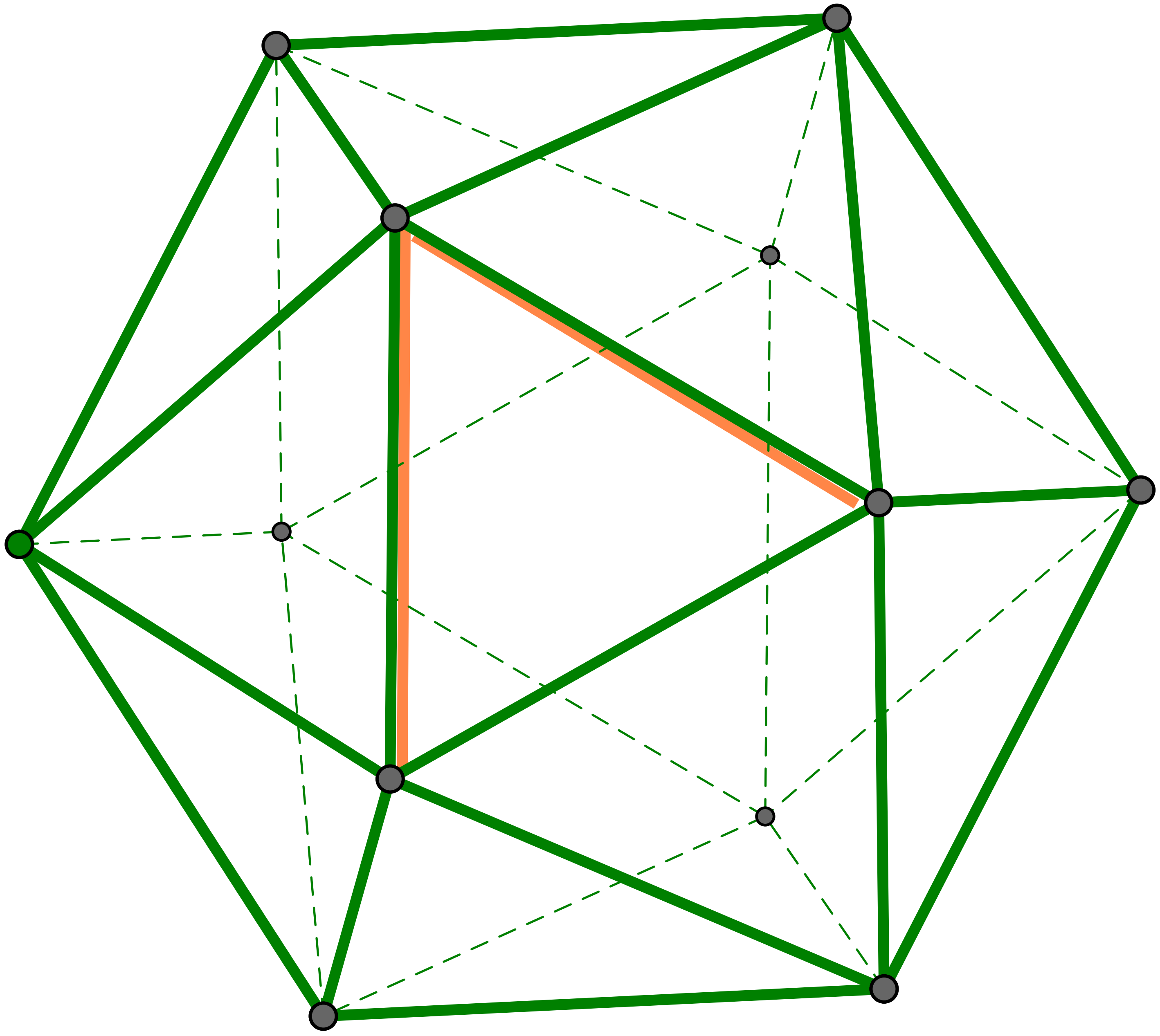

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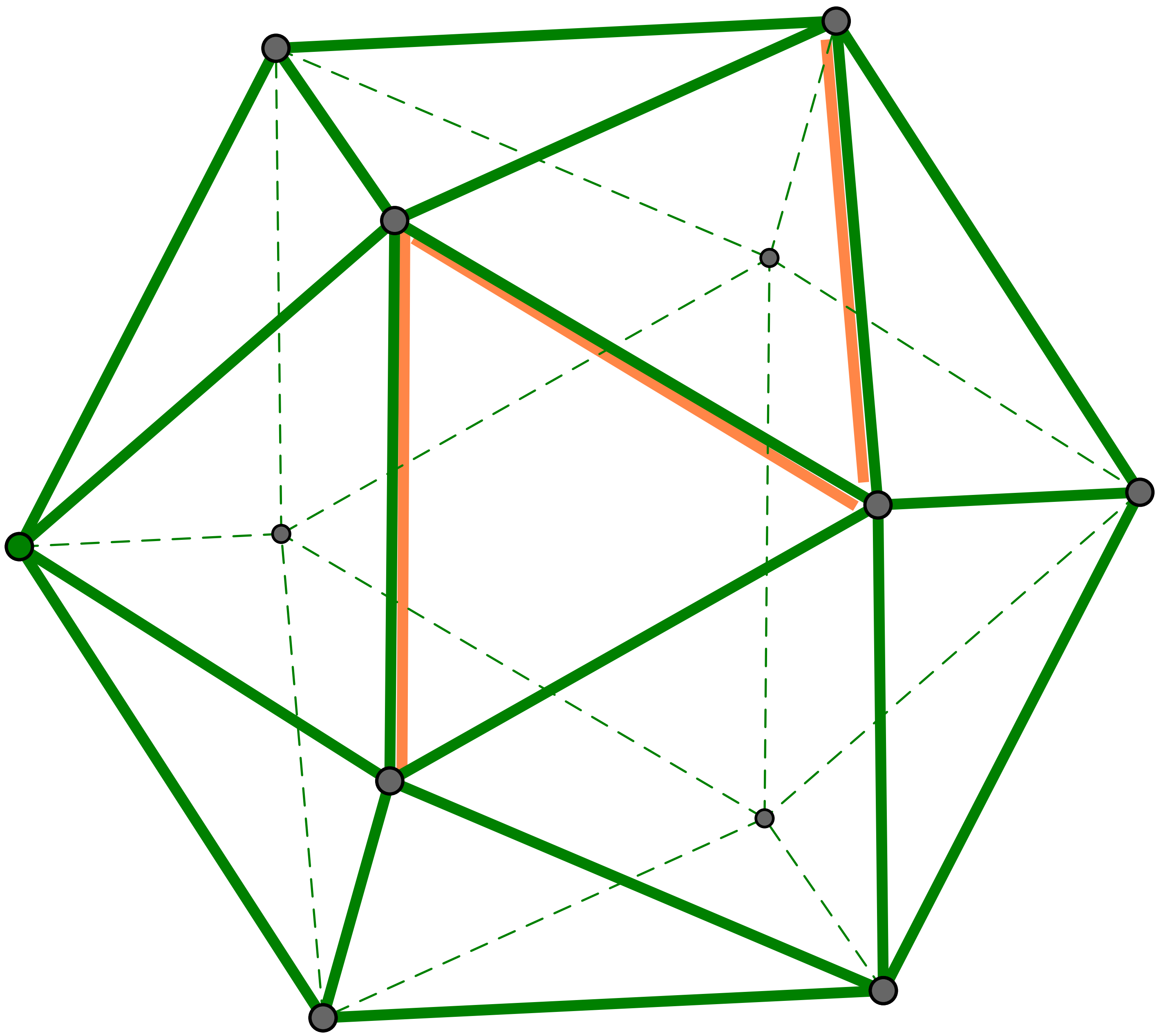

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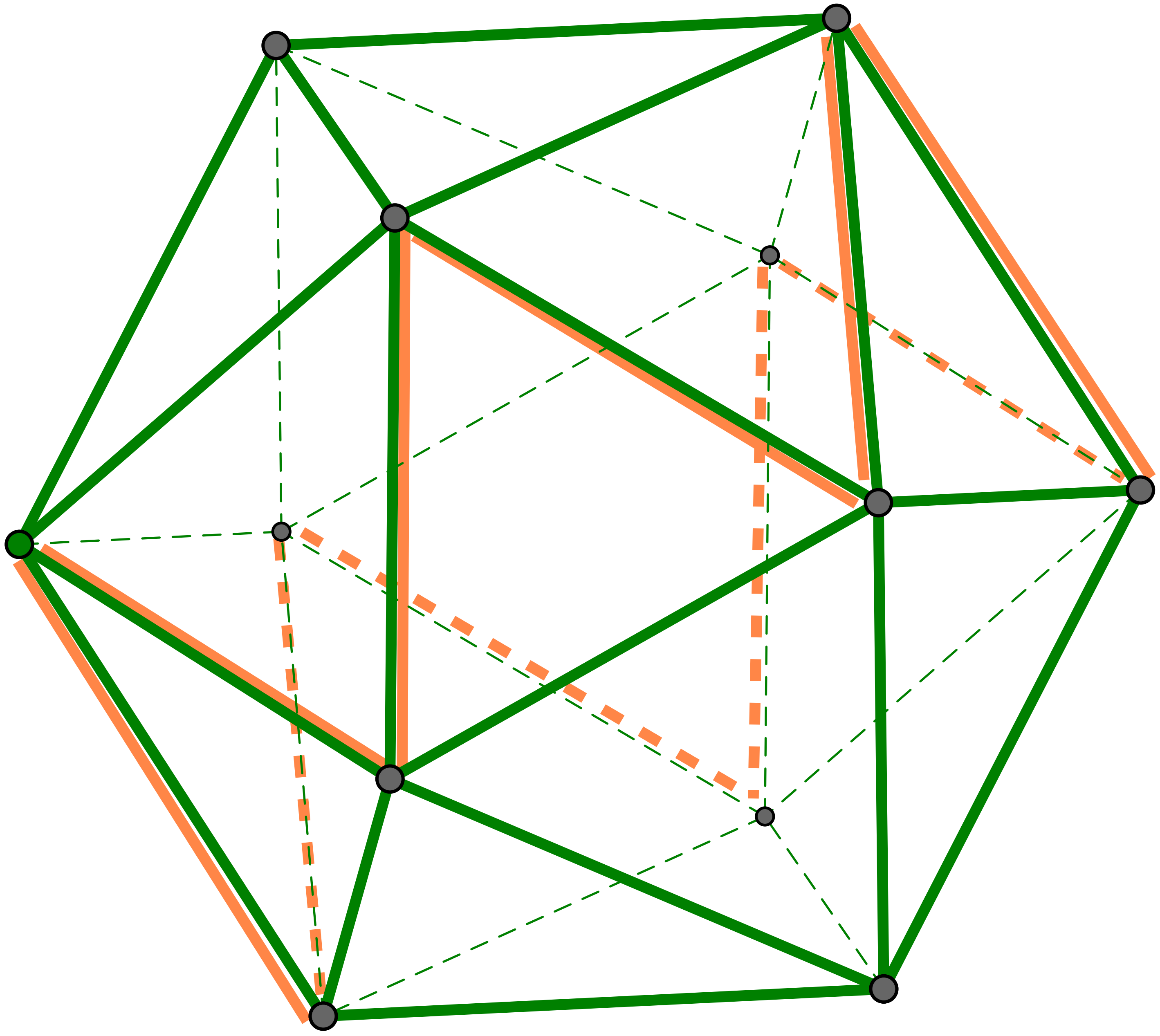

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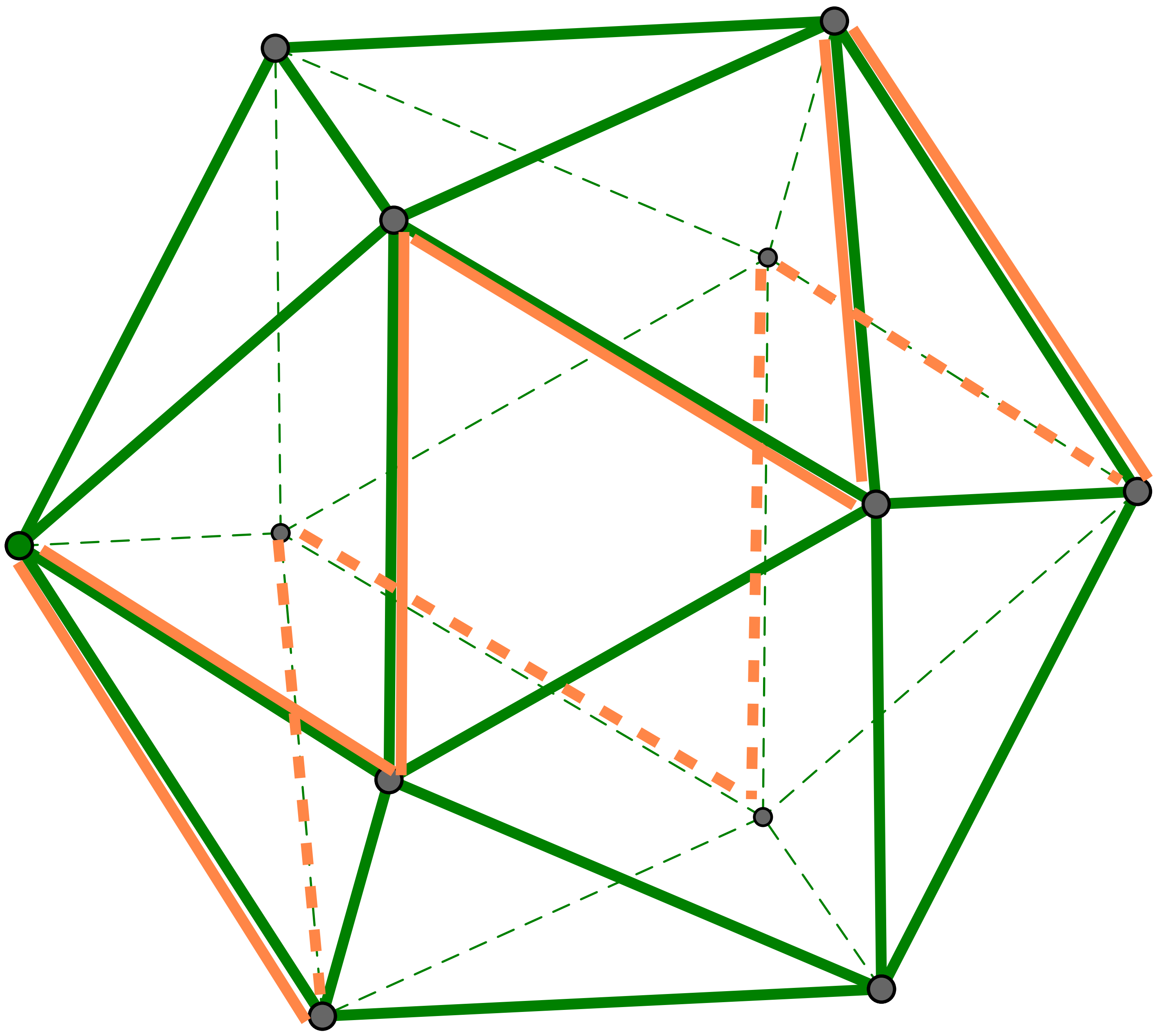

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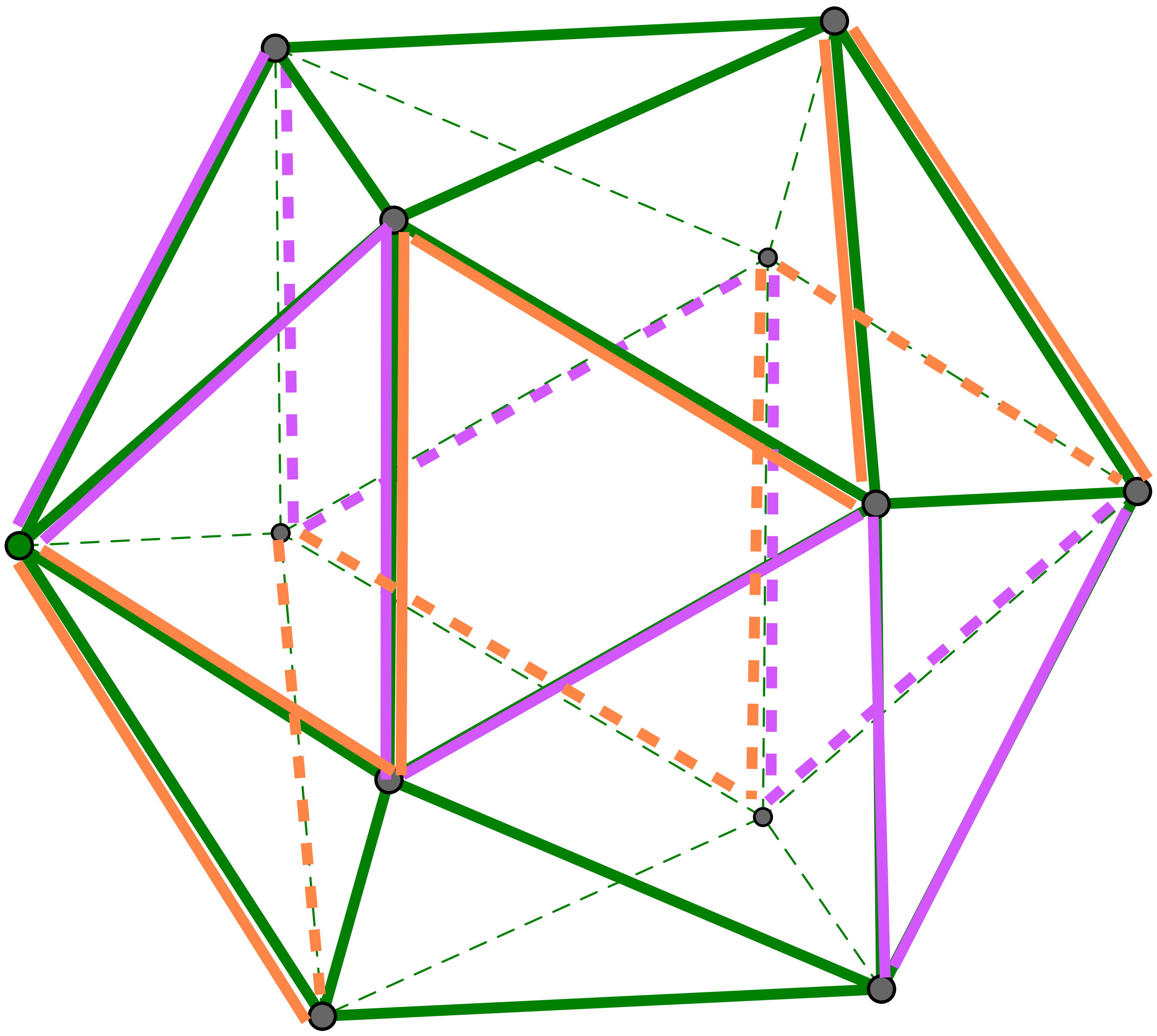

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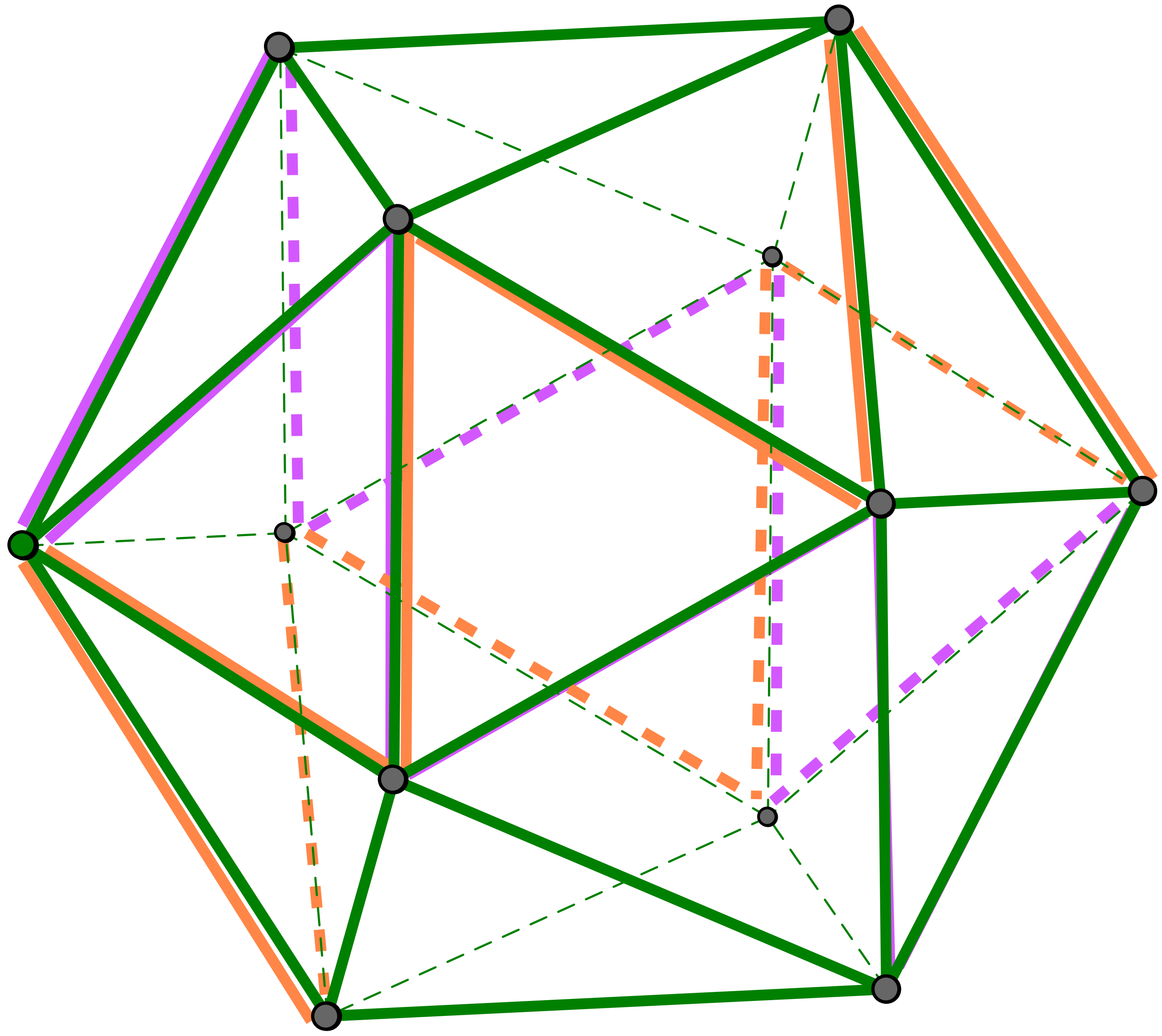

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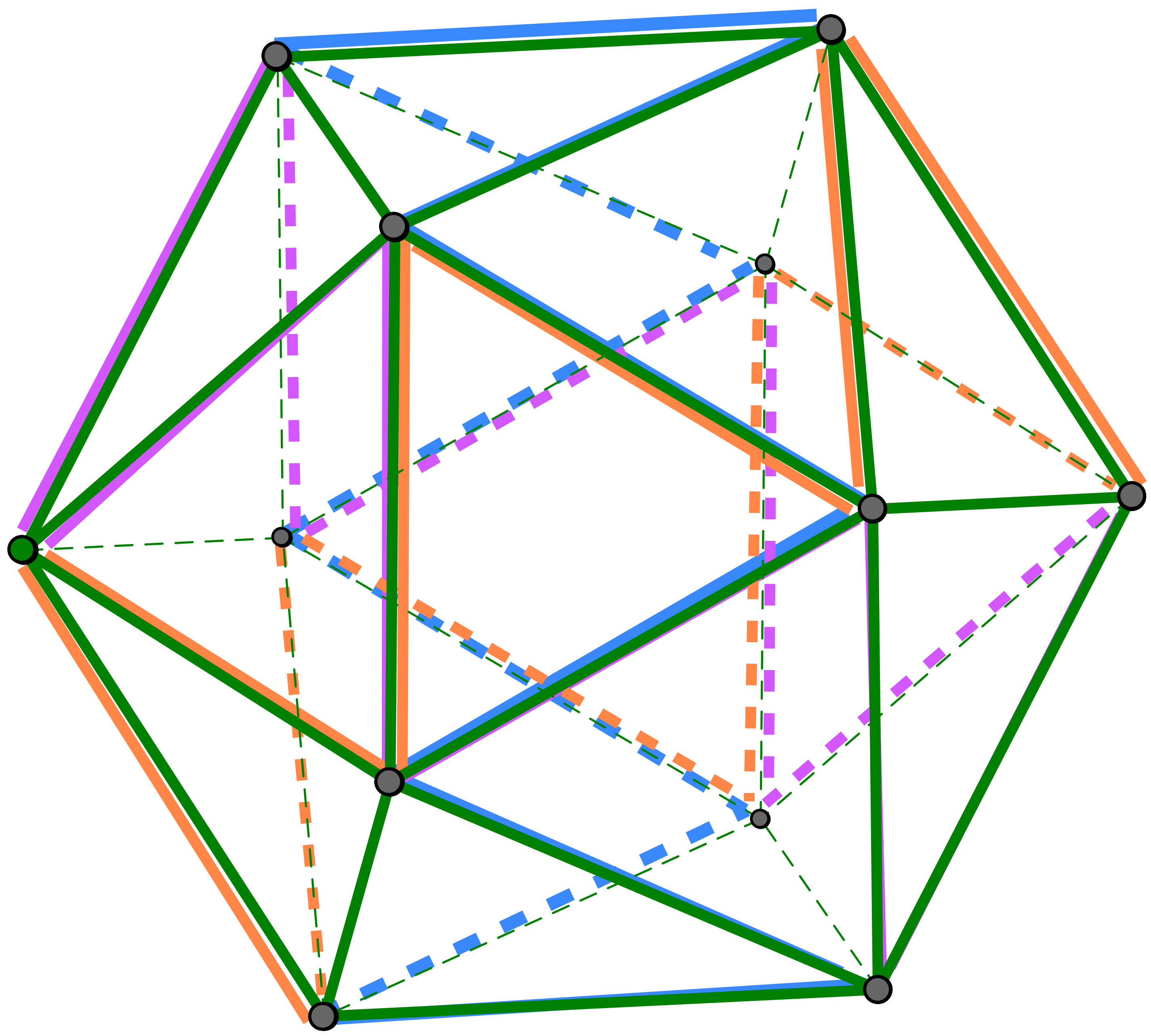

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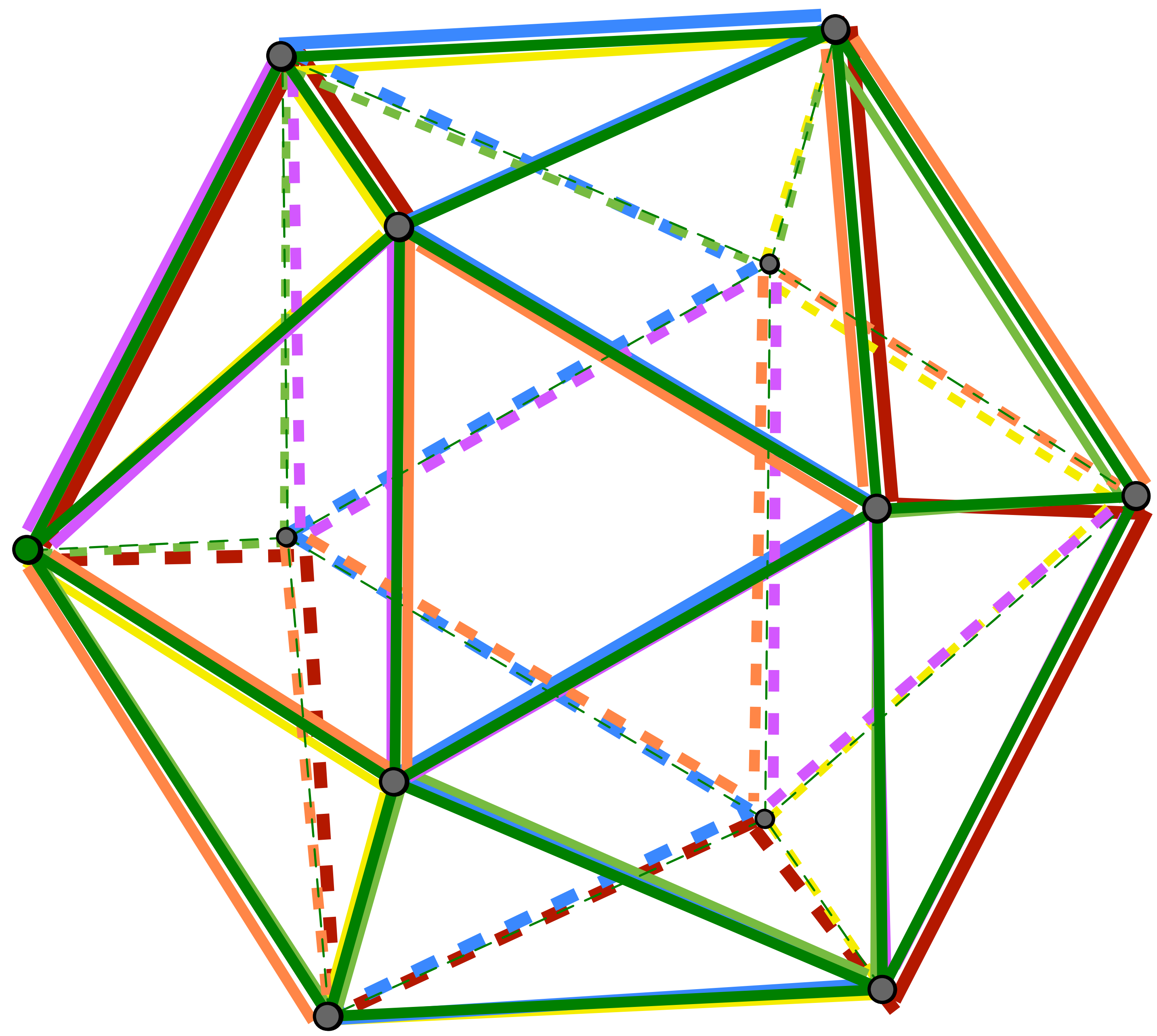

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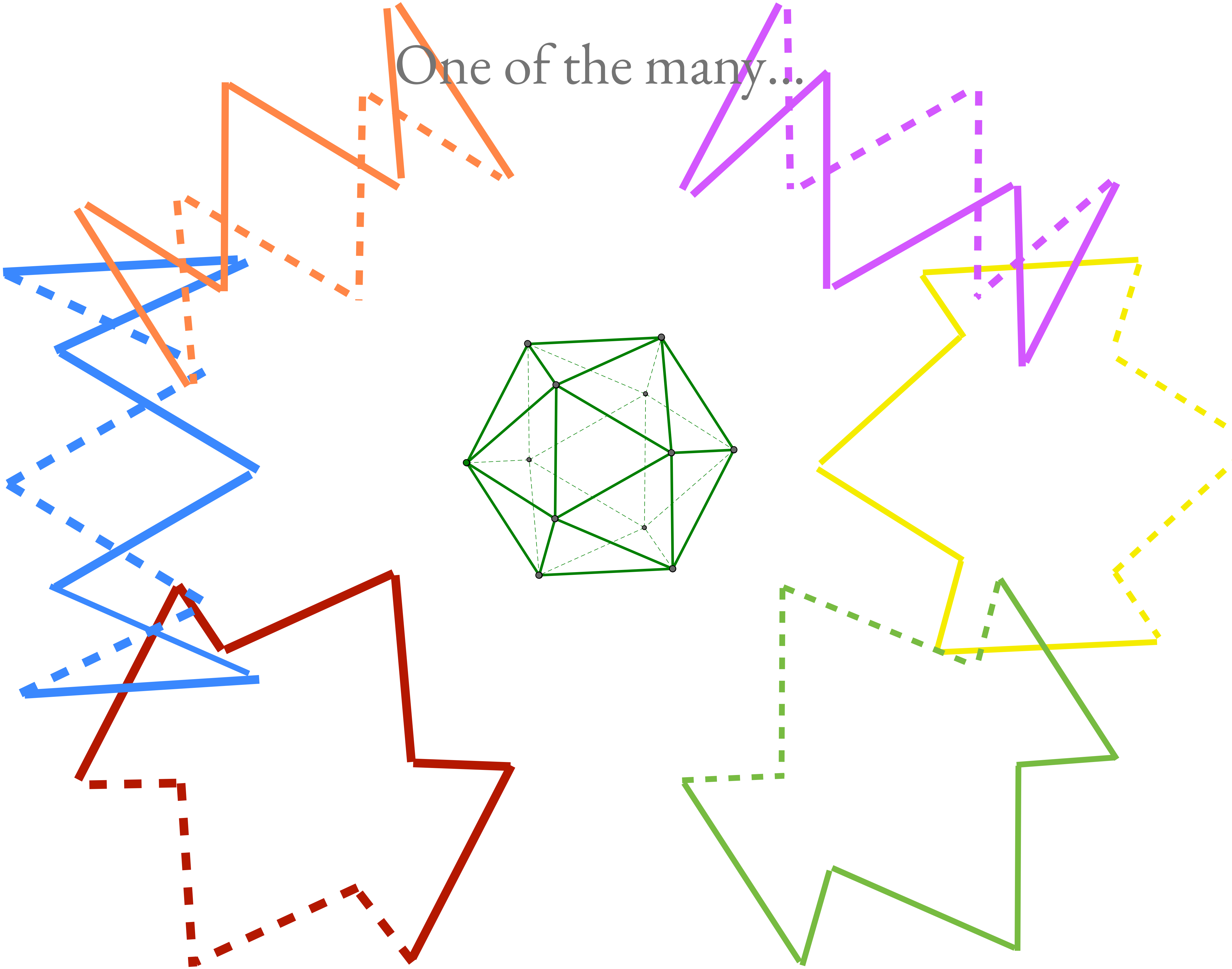

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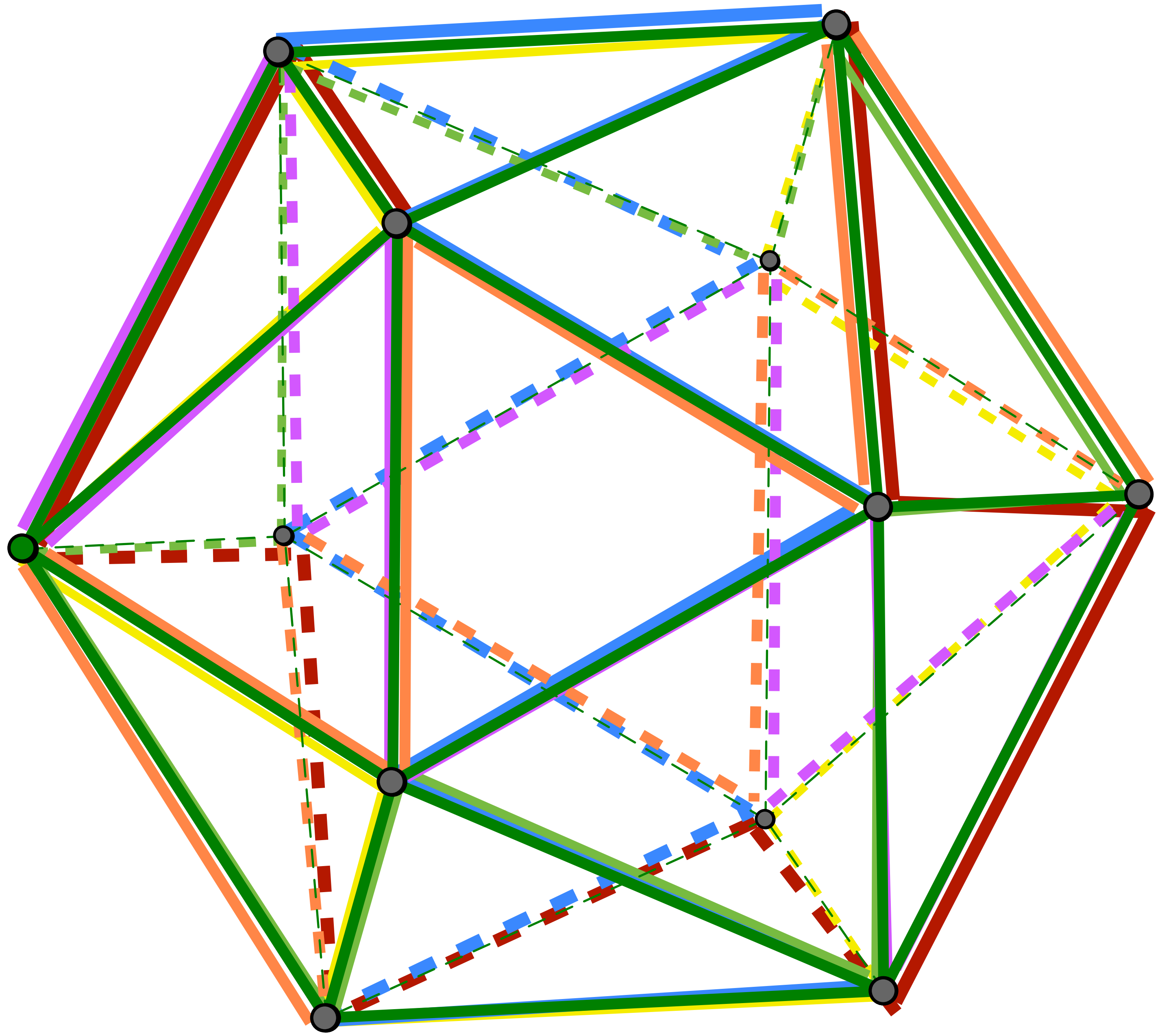
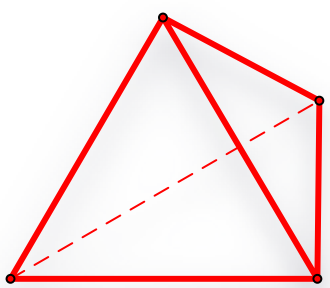
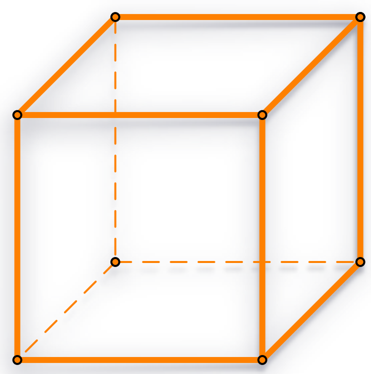
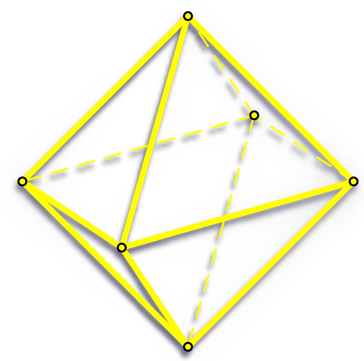
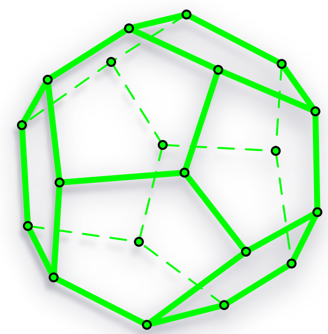
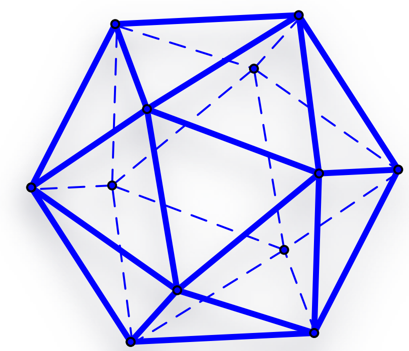
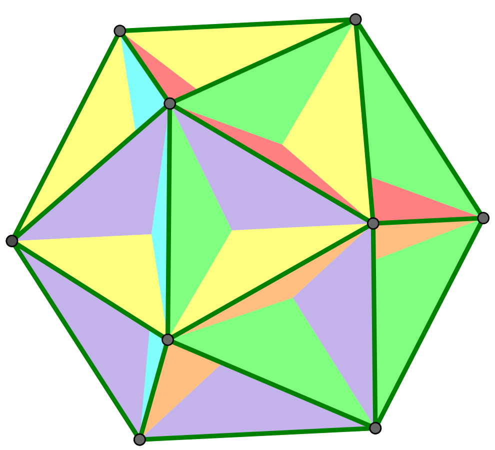
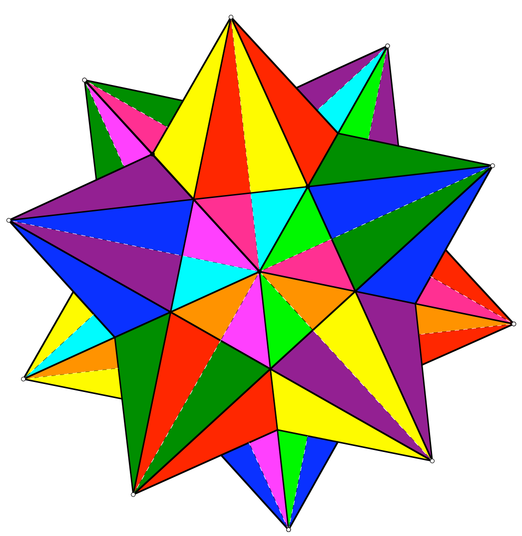
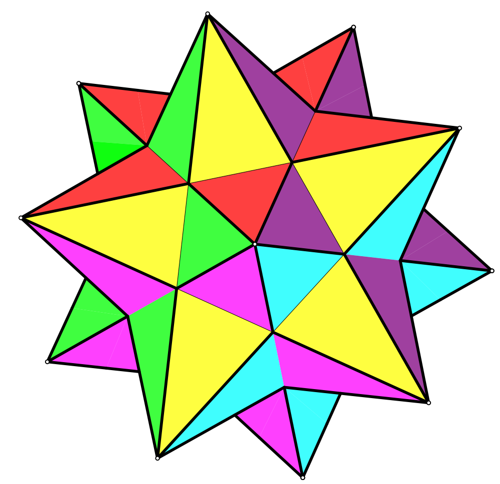
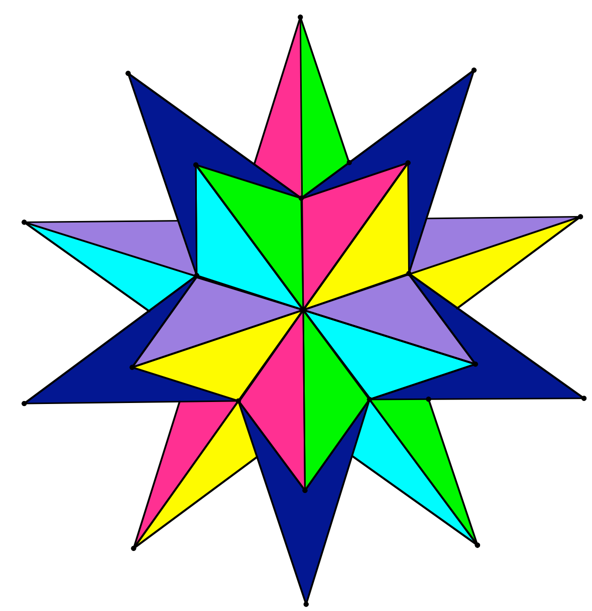


Theorem (Grünbaum-Dress)

There are exactly 18 finite regular polyhedra in the 3-space $\mathbb{E}^3$

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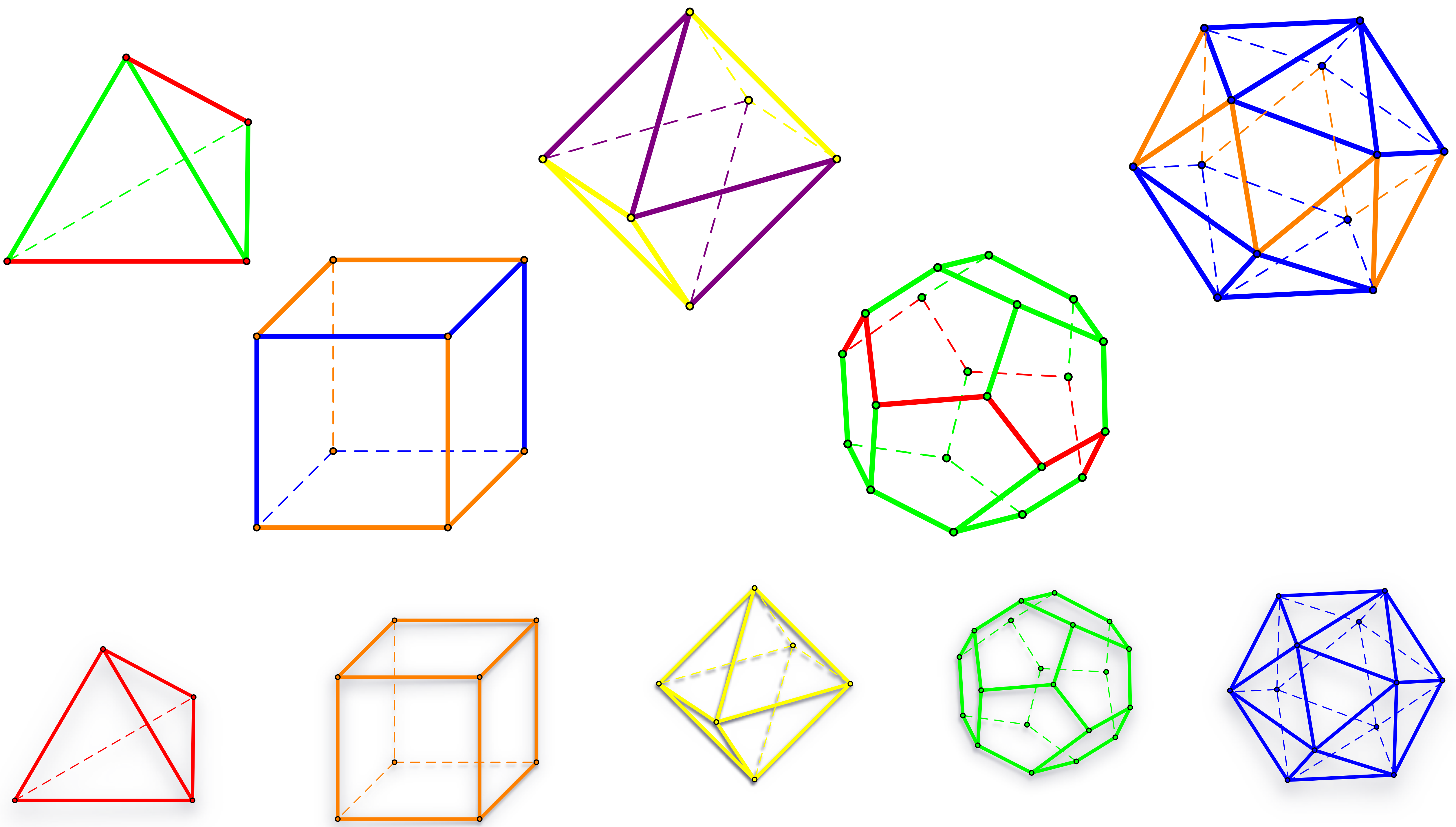

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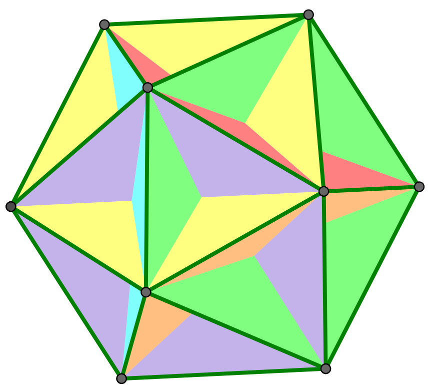

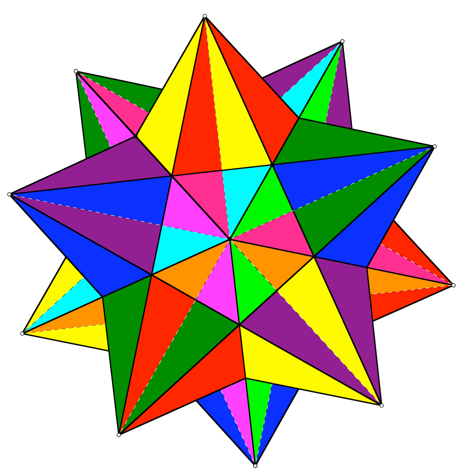

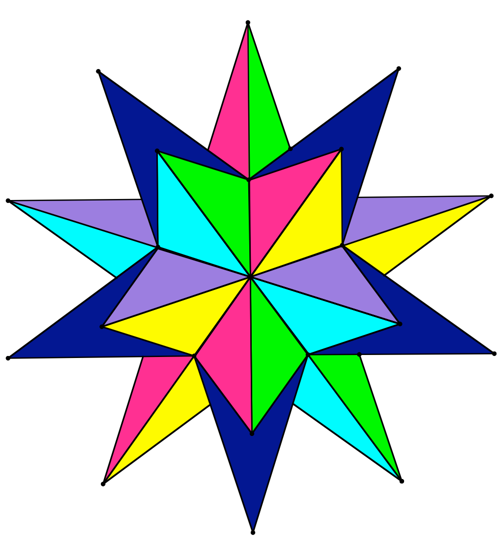

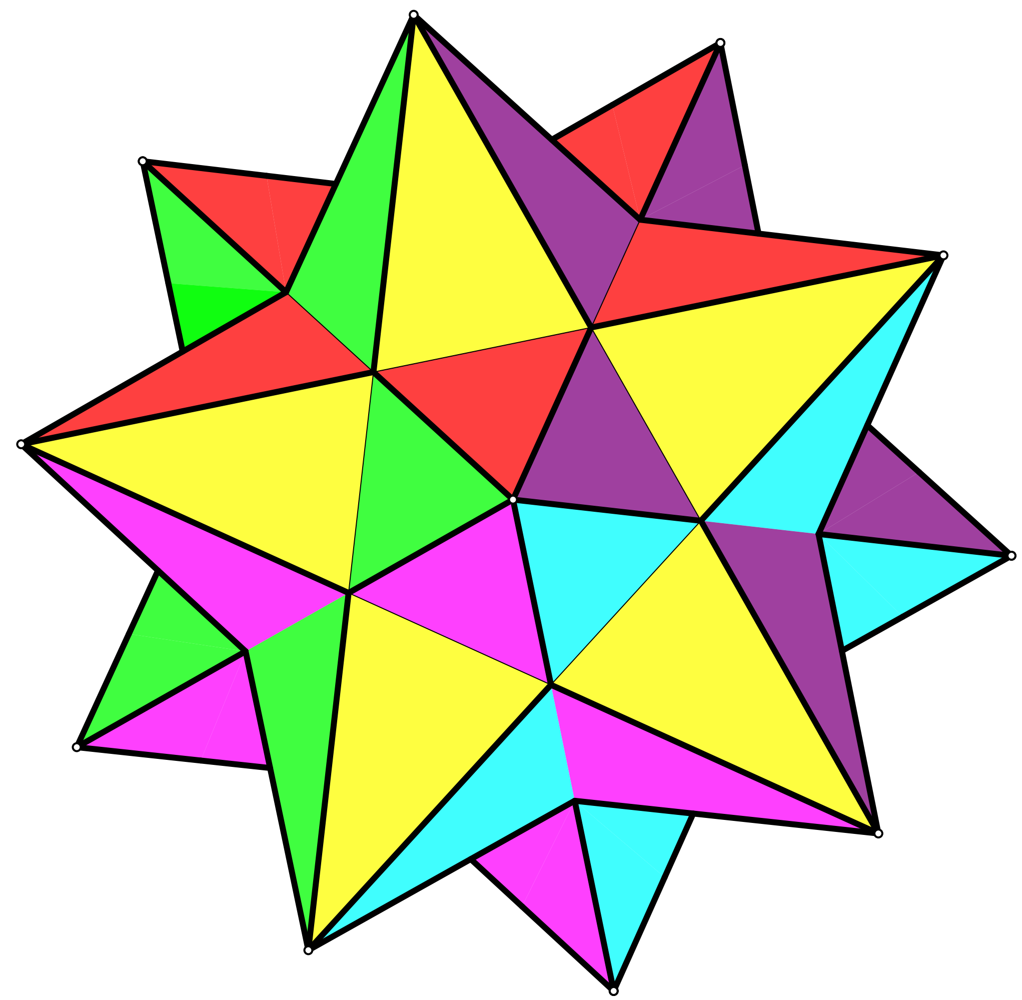

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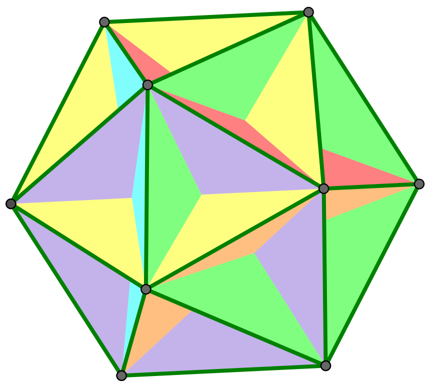

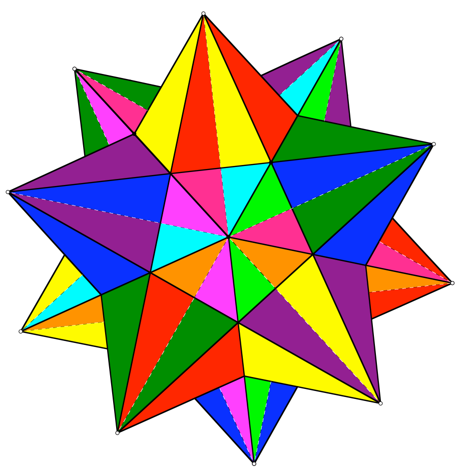

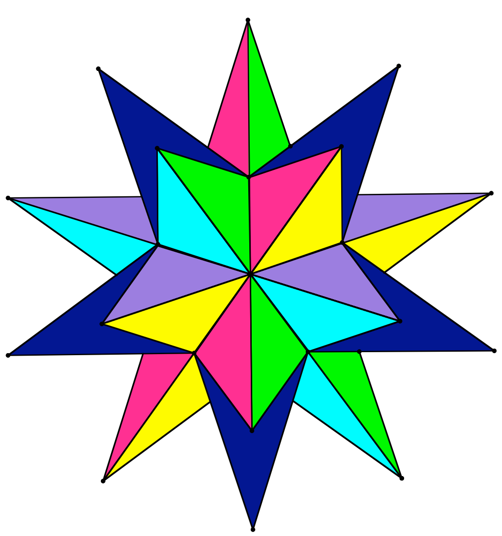

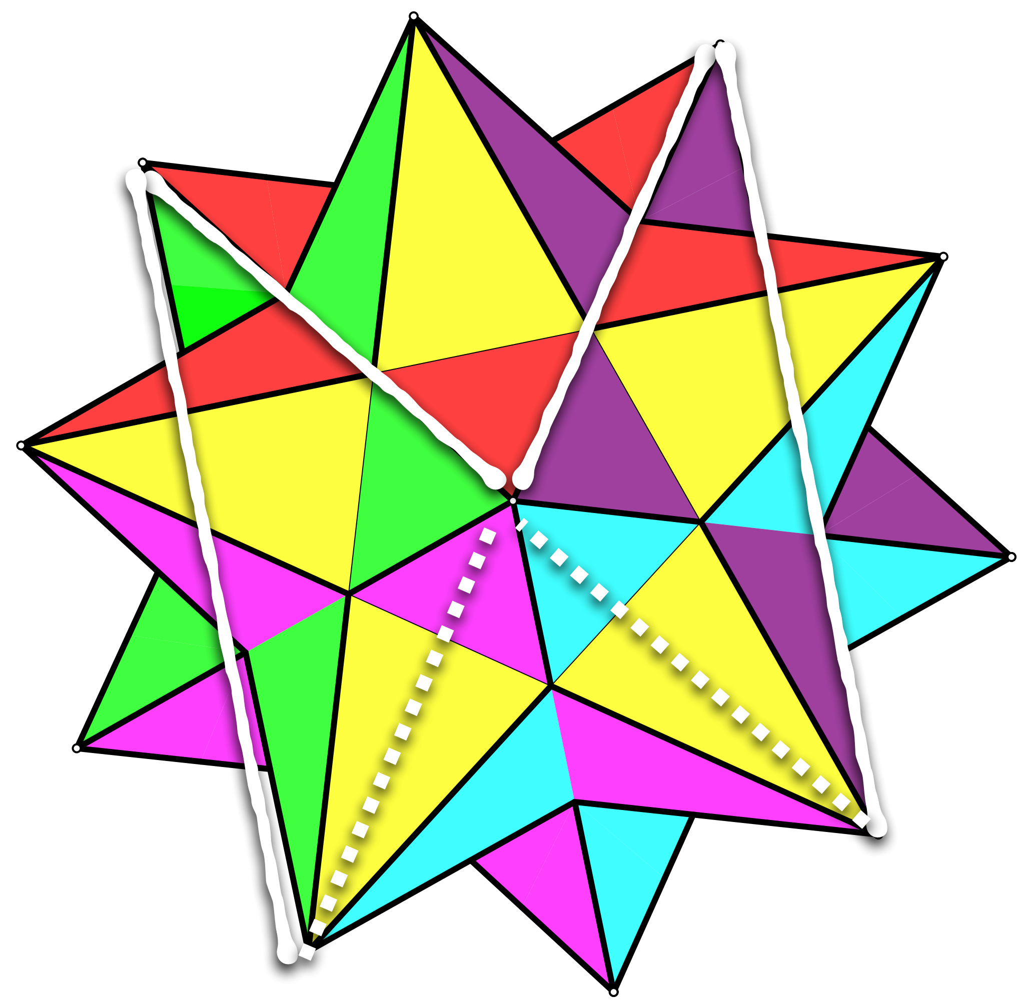

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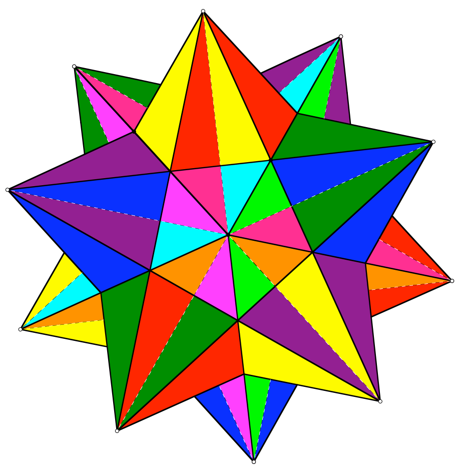

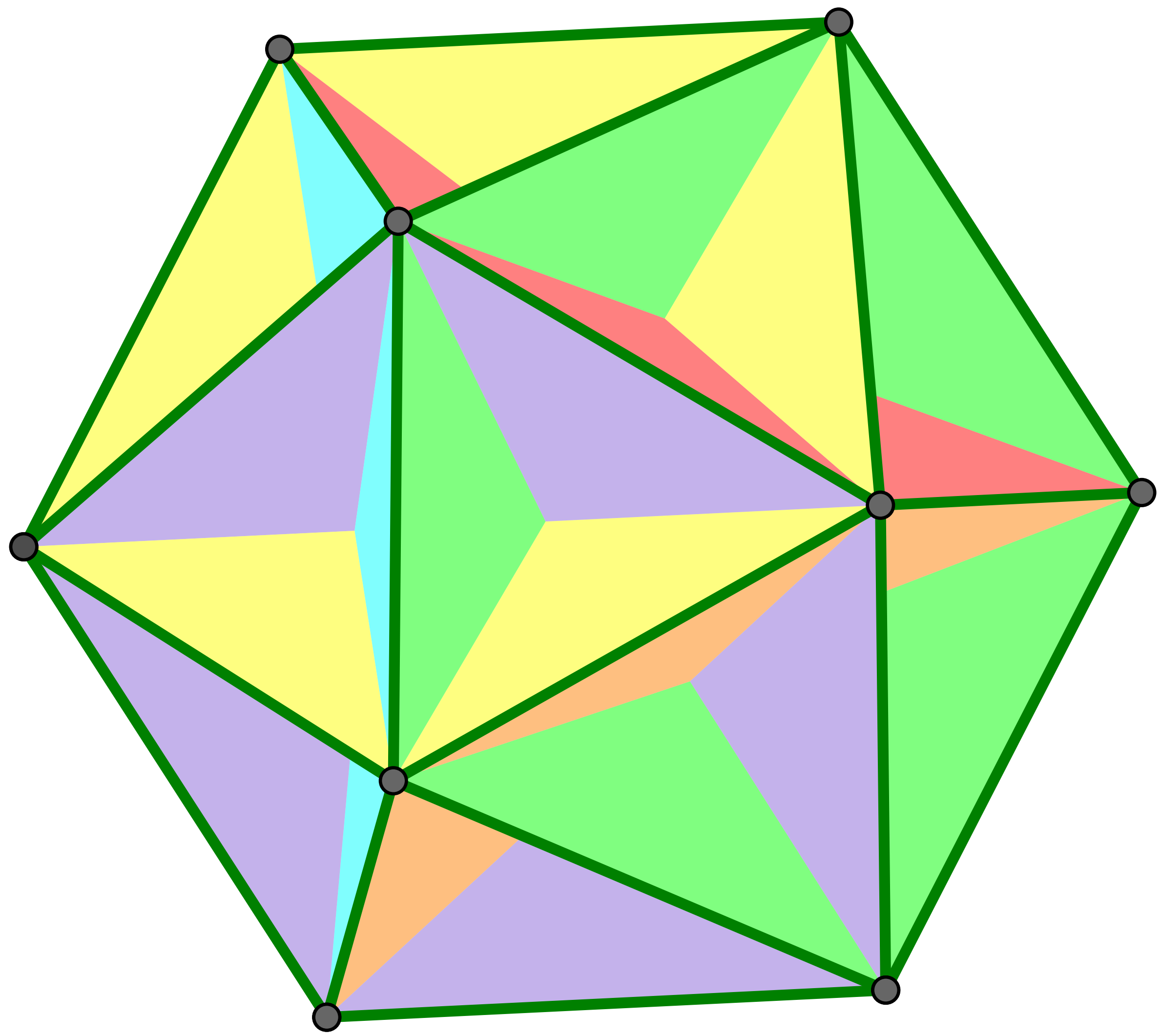

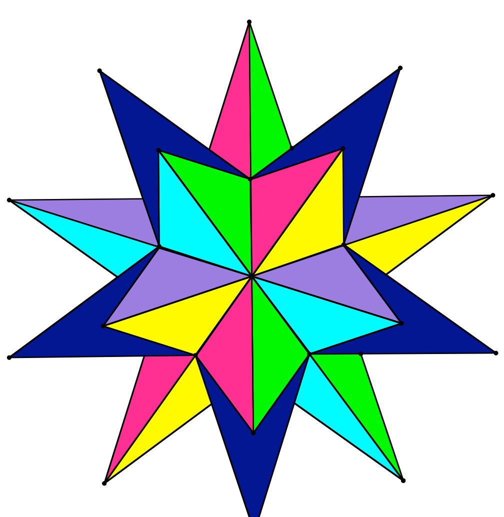

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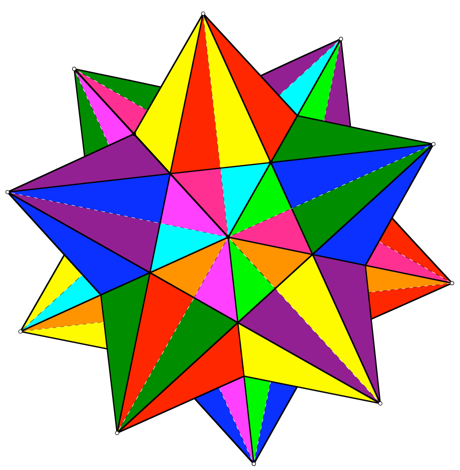

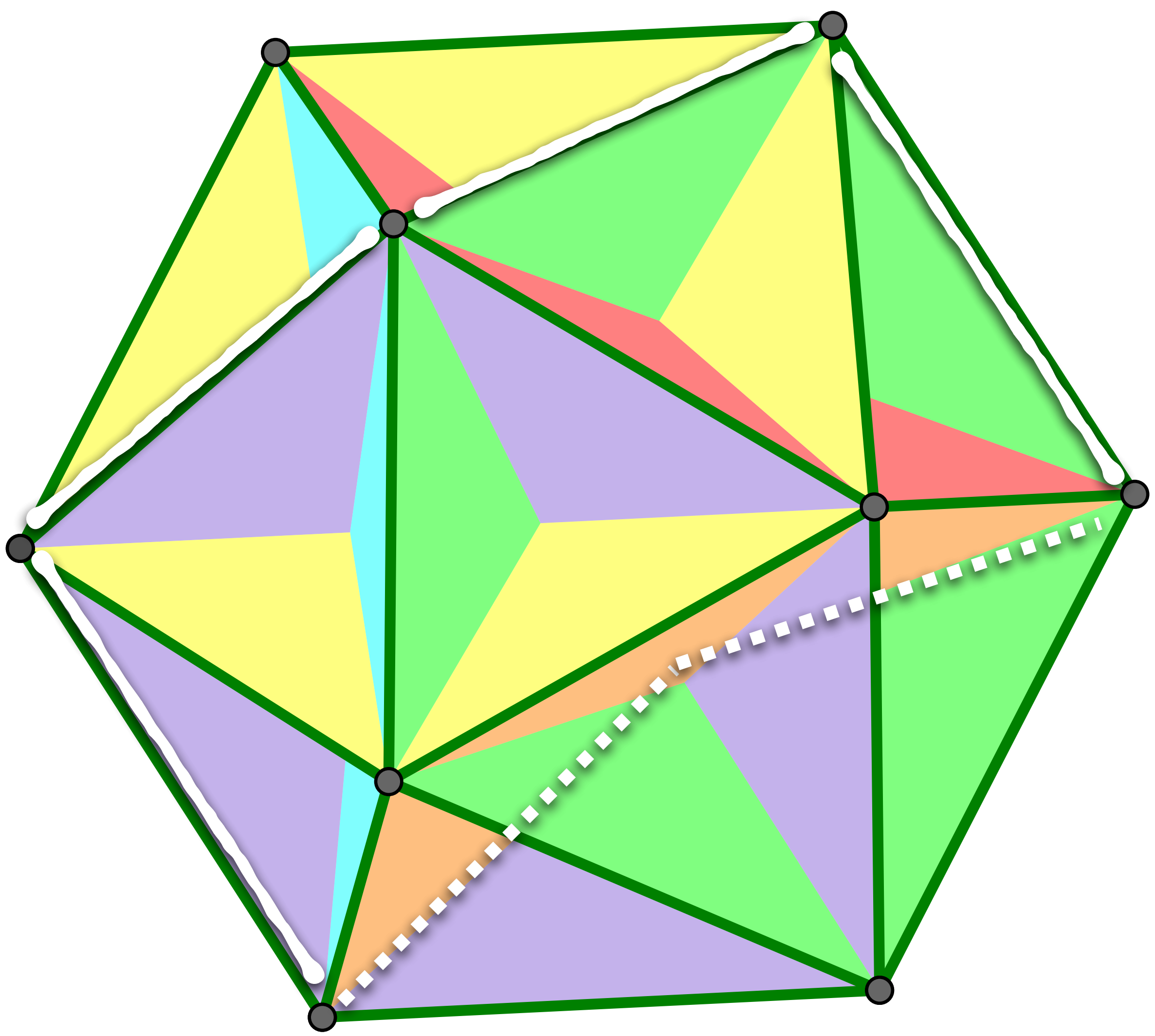

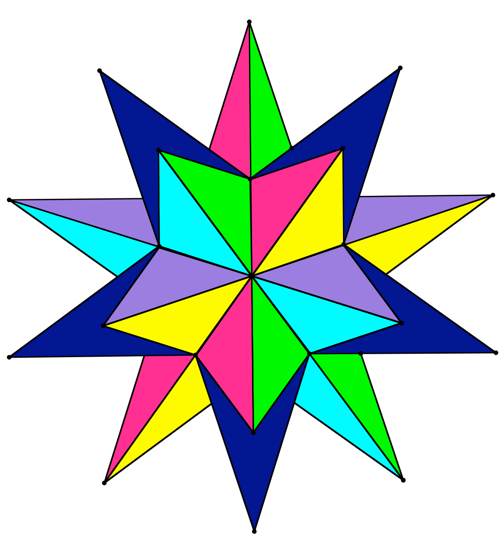

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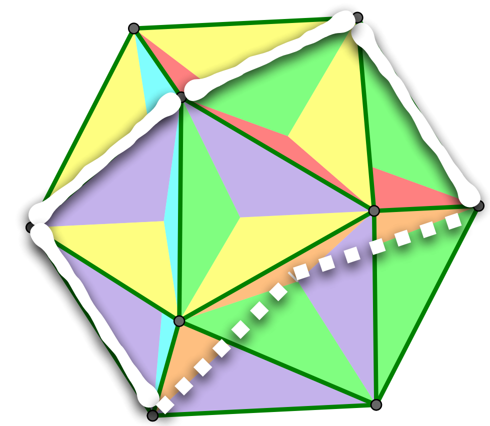

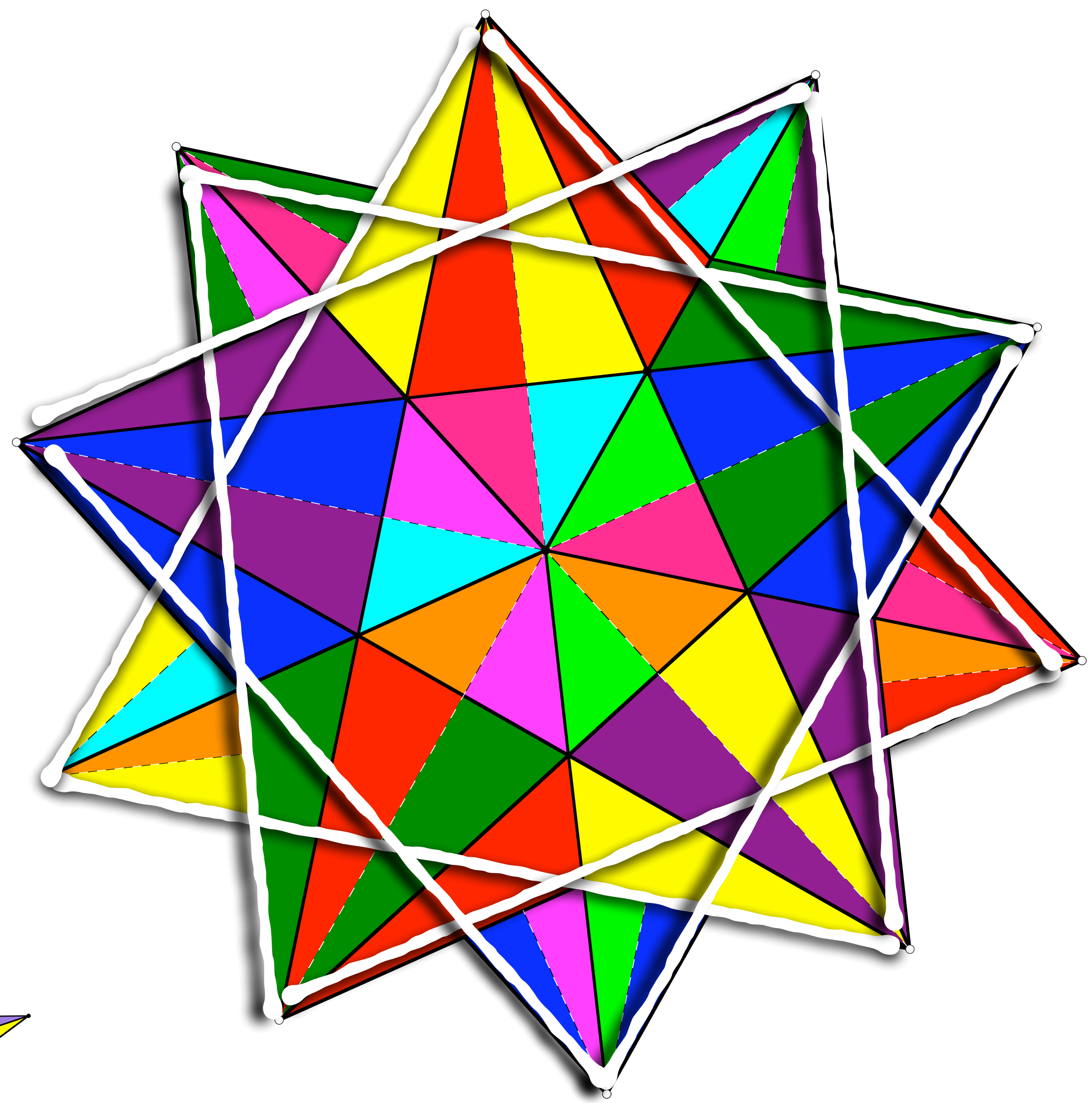

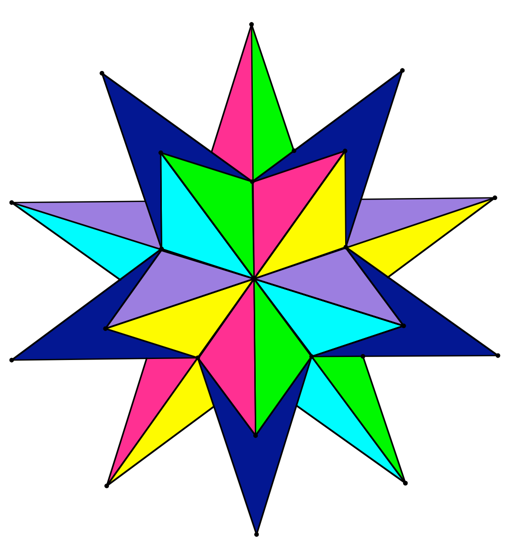

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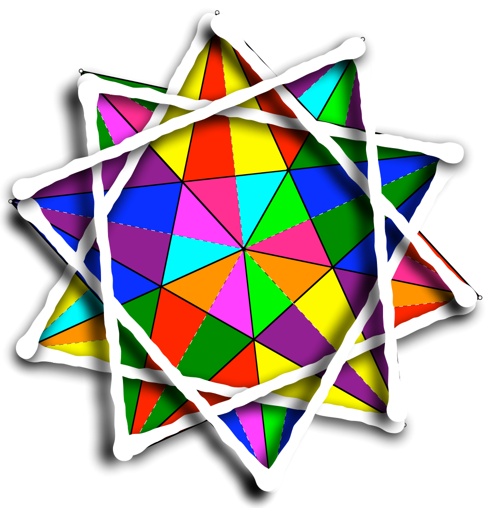

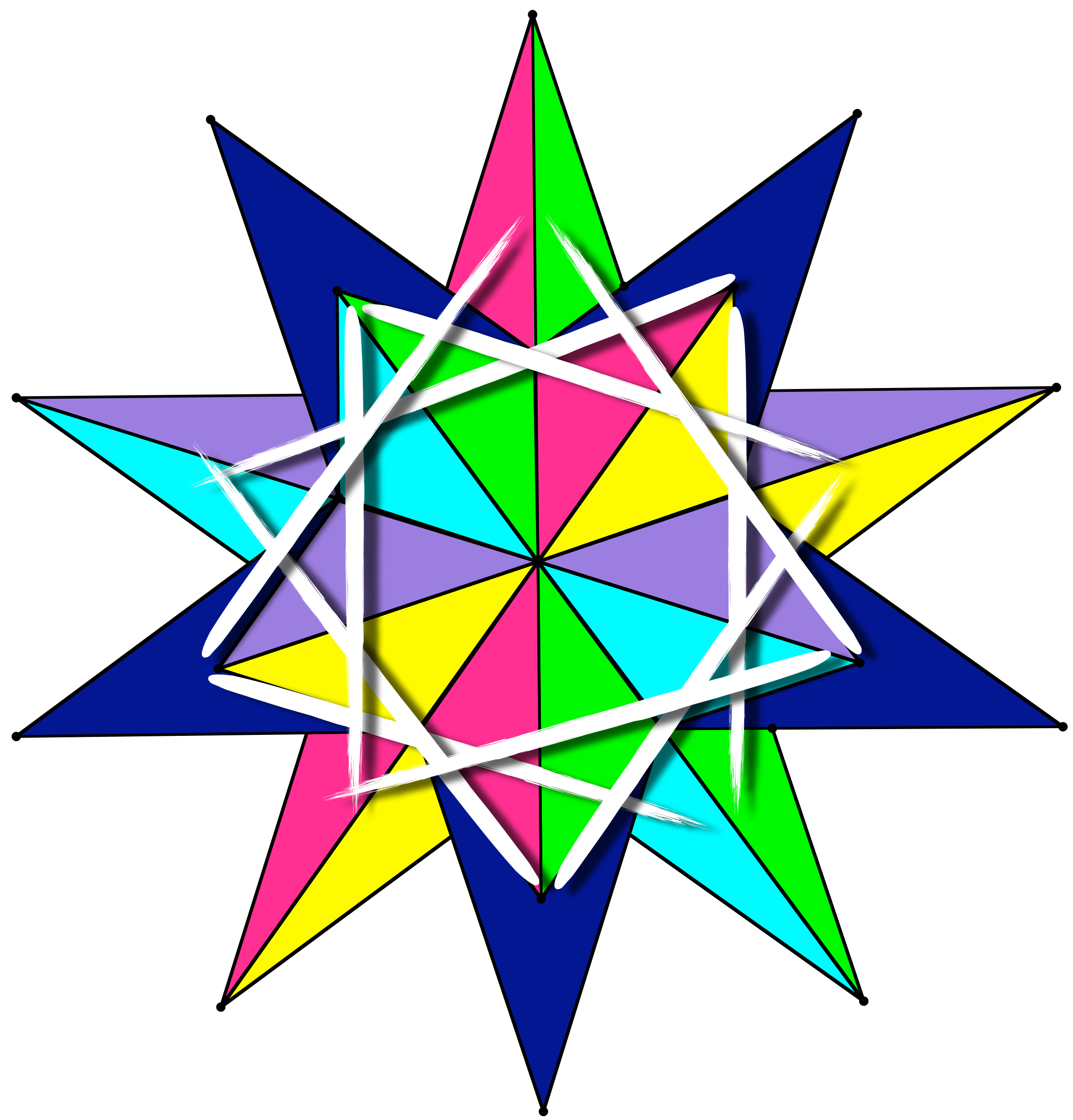

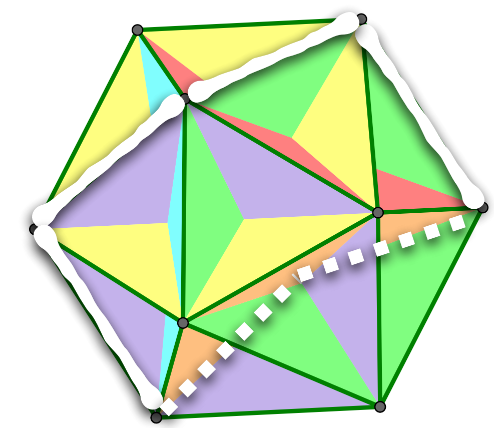

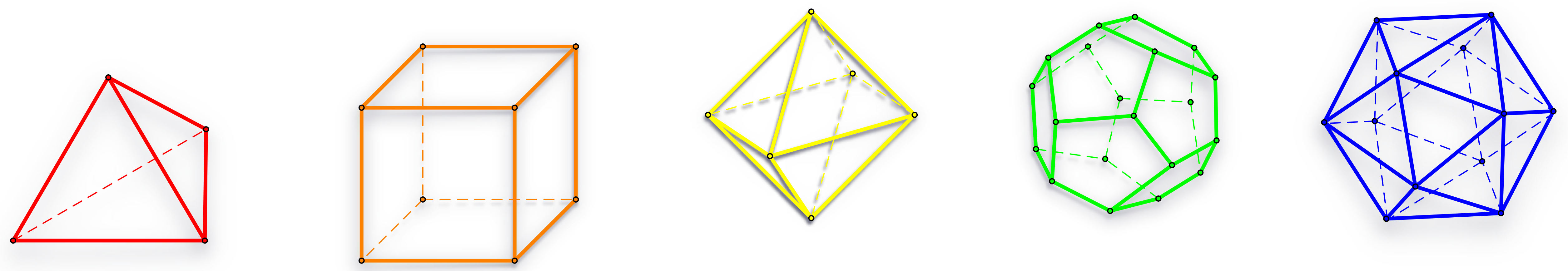

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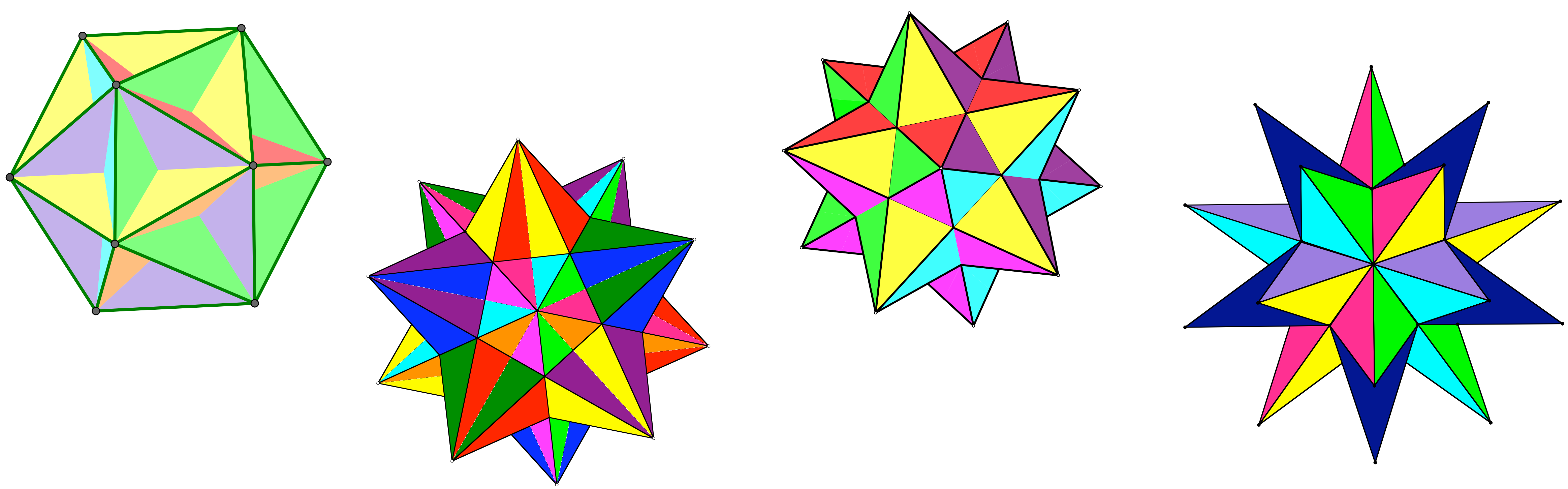

Theorem (Grünbaum-Dress)

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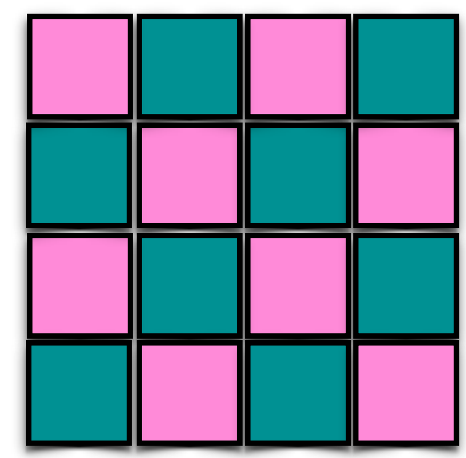

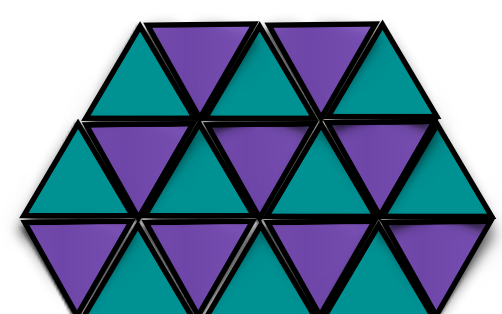

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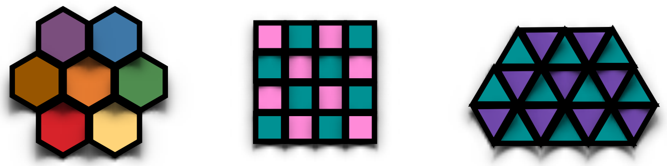

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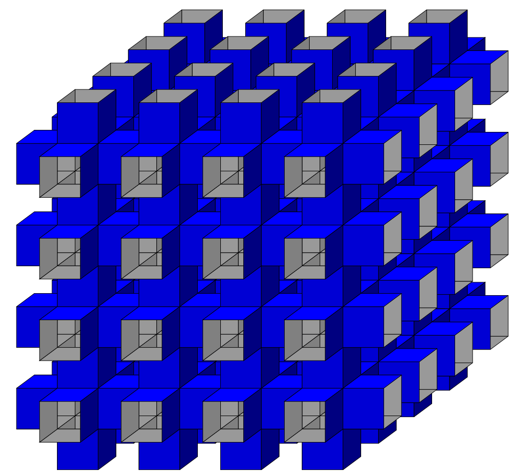
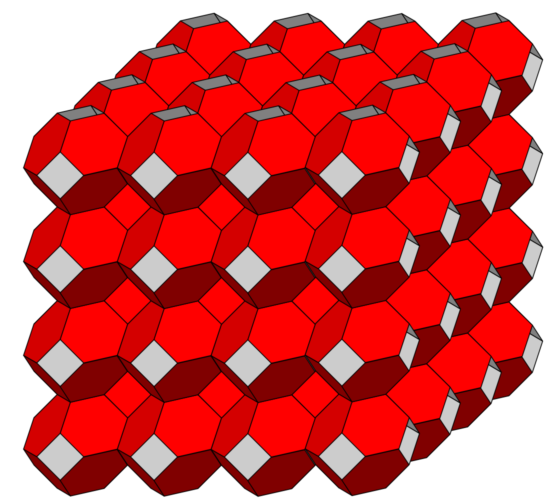
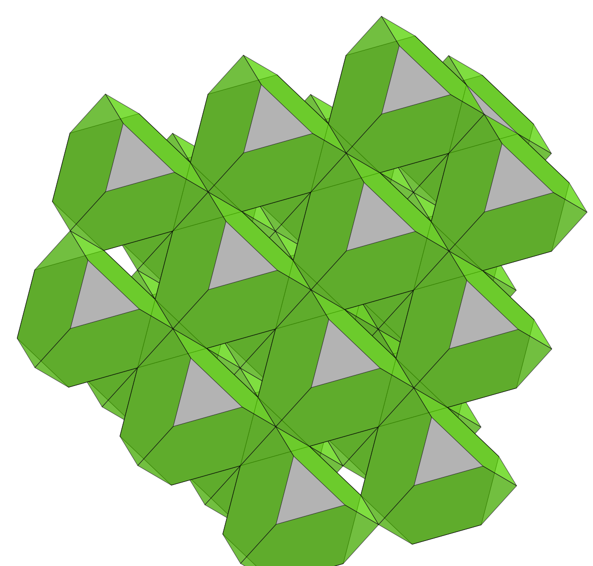
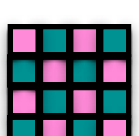
- Planar regular polyhedra

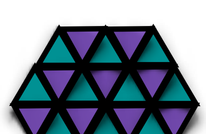
- Petrie - Coxeter polyhedra







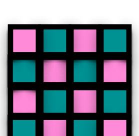
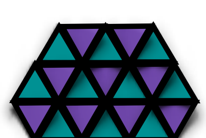
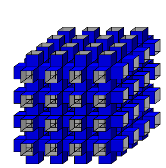
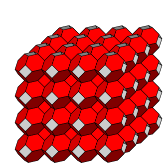
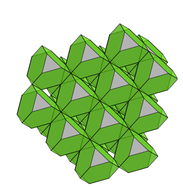




Theorem (Grünbaum-Dress)

There are exactly 30 infinite regular polyhedra in the 3-space $\mathbb{E}^3$

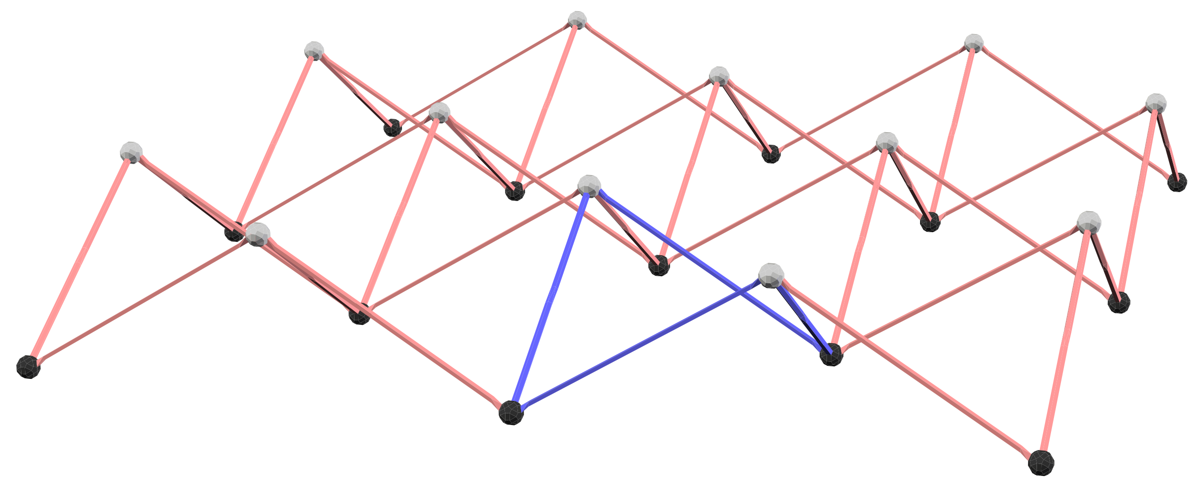
- Planar regular polyhedra
- Petrie - Coxeter polyhedra

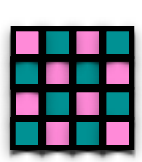
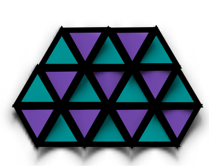
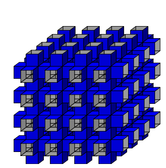
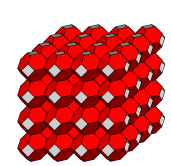
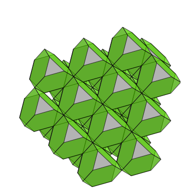
     

Theorem (Grünbaum-Dress)

There are exactly 30 infinite regular polyhedra in the 3-space $\mathbb{E}^3$

- Planar regular polyhedra
- Petrie - Coxeter polyhedra
- Almost planar polyhedra



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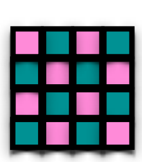
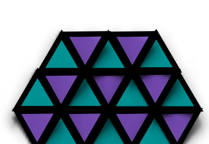
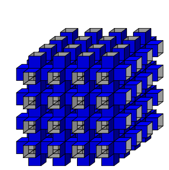
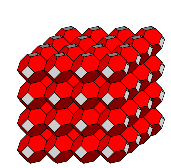
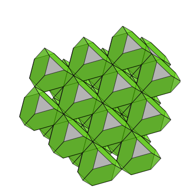
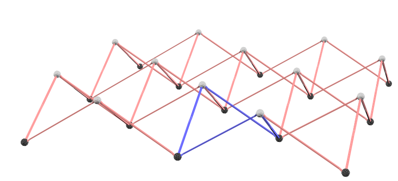
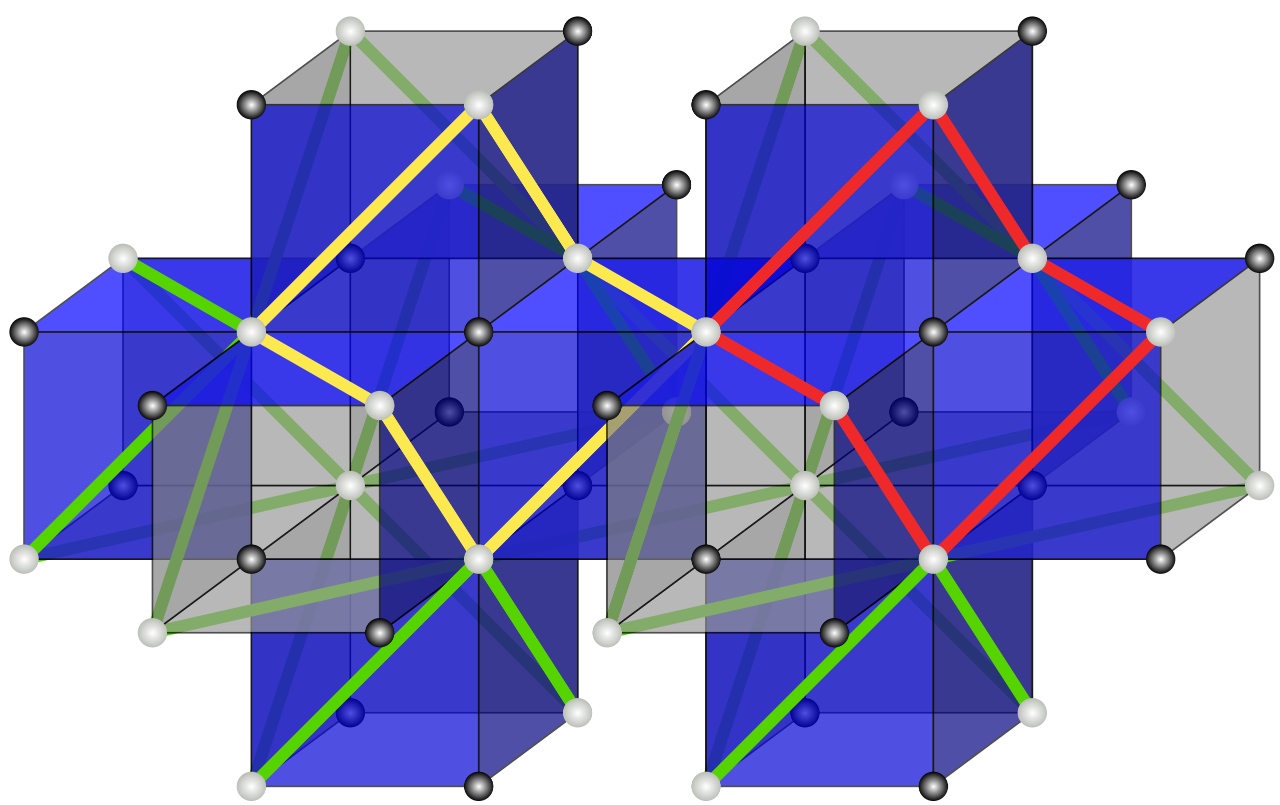
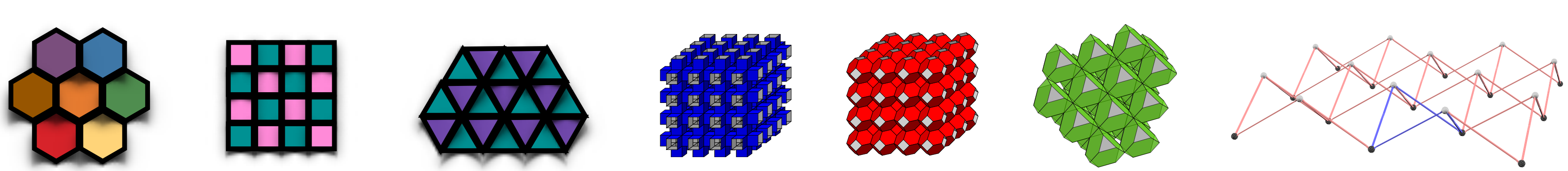







Theorem (Grünbaum-Dress)

There are exactly 30 infinite regular polyhedra in the 3-space $\mathbb{E}^3$

- Planar regular polyhedra
- Petrie - Coxeter polyhedra
- Almost planar polyhedra
- Polyhedra with skew faces





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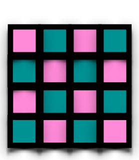
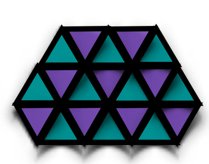
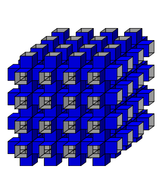
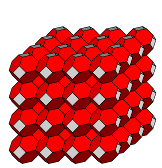
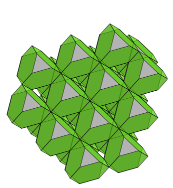
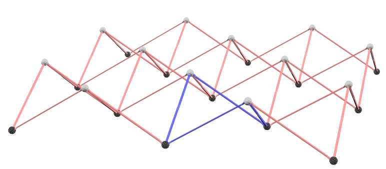
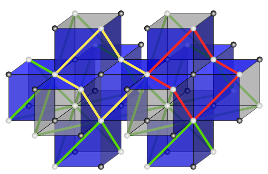
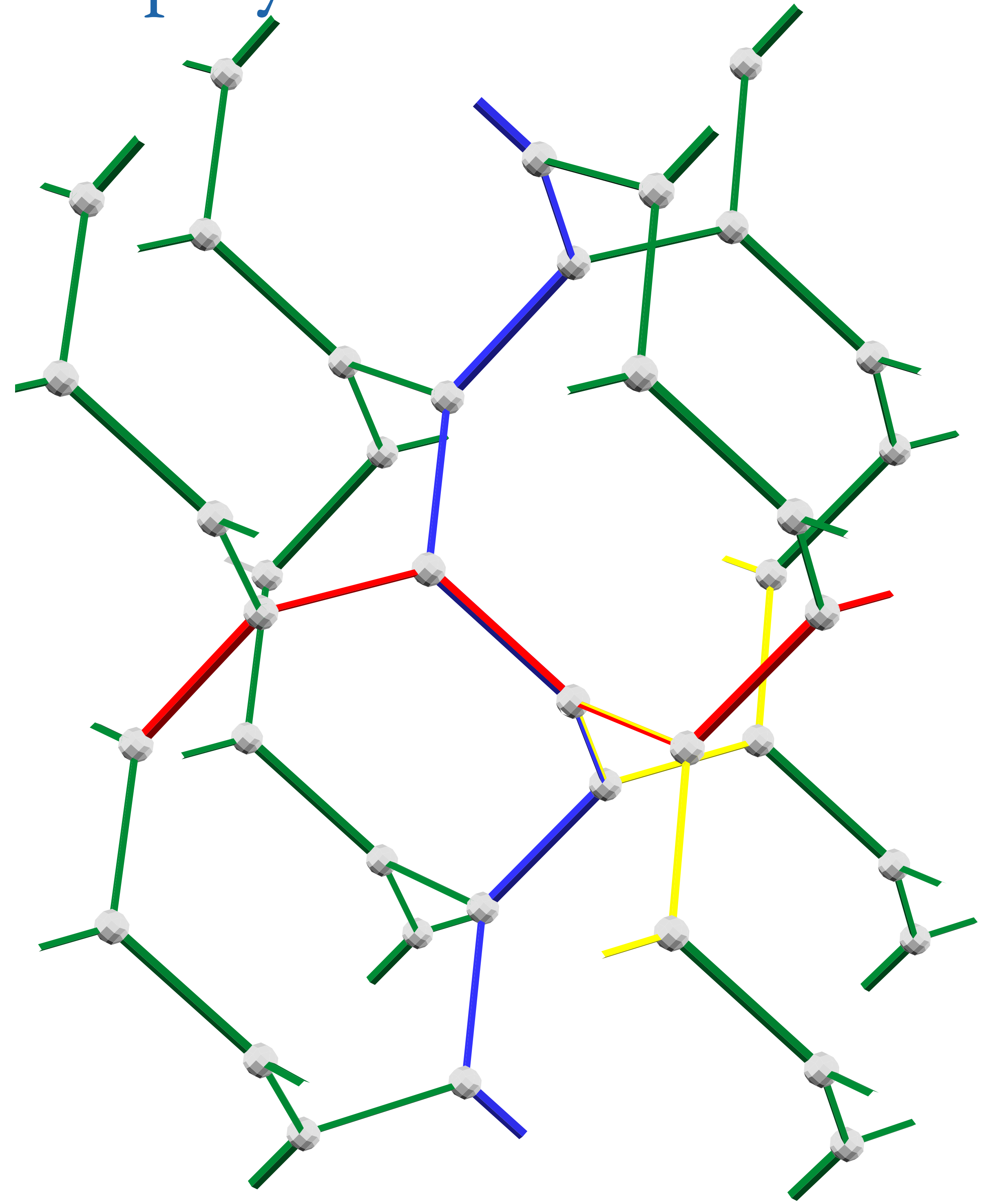
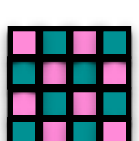
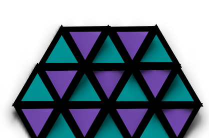
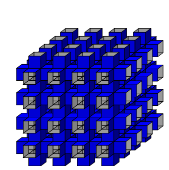
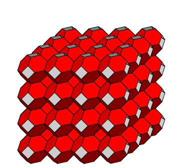
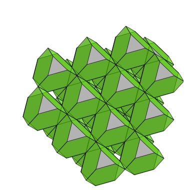
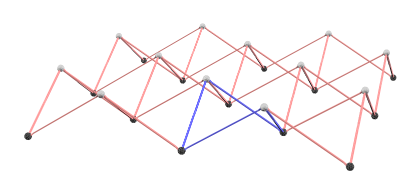
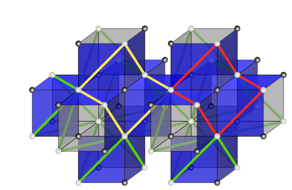
       

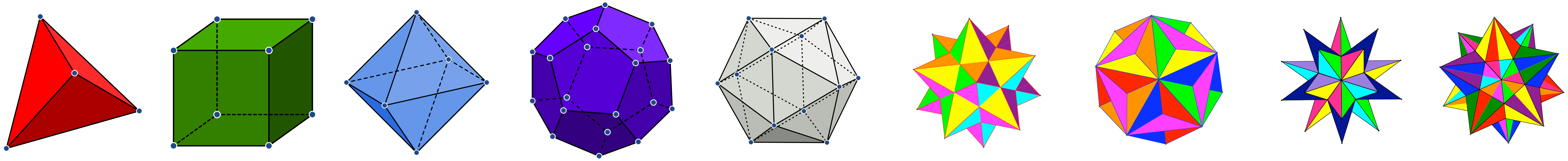
Theorem (Grünbaum-Dress)

There are exactly 30 infinite regular polyhedra in the 3-space $\mathbb{E}^3$



- Planar regular polyhedra
- Petrie - Coxeter polyhedra
- Almost planar polyhedra
- Polyhedra with skew faces
- Polyhedra with helicoidal faces

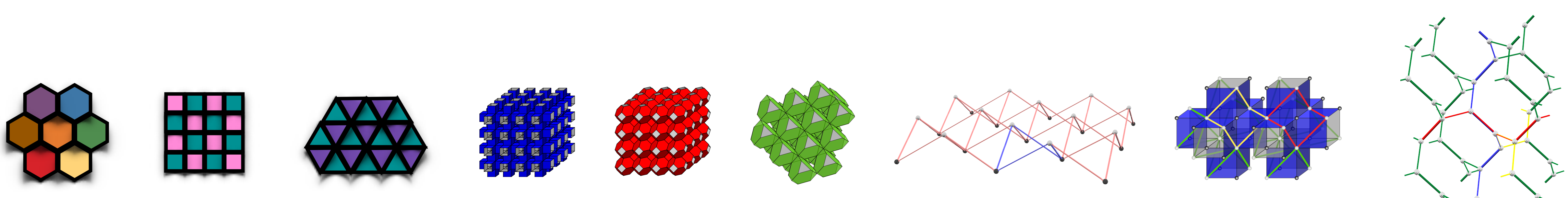


Theorem (Grünbaum-Dress)

There are exactly 18 finite regular polyhedra in the 3-space $\mathbb{E}^3$

Theorem (Grünbaum-Dress)

There are exactly 30 infinite regular polyhedra in the 3-space $\mathbb{E}^3$



Are Your Polyhedra the Same as My Polyhedra?

Branko Grünbaum

1 Introduction

"Polyhedron" means different things to different people. There is very little in common between the meaning of the word in topology and in geometry. But even if we confine attention to geometry of the 3-dimensional Euclidean space – as we shall do from now on – "polyhedron" can mean either a solid (as in "Platonic solids", convex polyhedron, and other contexts), or a surface (such as the polyhedral models constructed from cardboard using "nets", which were introduced by Albrecht Dürer [17] in 1525, or, in a more modern version, by Aleksandrov [1]), or the 1-dimensional complex consisting of points ("vertices") and line-segments ("edges") organized in a suitable way into polygons ("faces") subject to certain restrictions ("skeletal polyhedra", diagrams of which have been presented first by Luca Pacioli [44] in 1498 and attributed to Leonardo da Vinci). The last alternative is the least usual one – but it is close to what seems to be the most useful approach to the theory

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Before deciding on the particular choice of definition, the following facts – which I often mention at the start of courses or lectures on polyhedra – should be considered. The regular polyhedra were enumerated by the mathematicians of ancient Greece; an account of these five "Platonic solids" is the final topic of Euclid's "Elements" [18]. Although this list was considered to be complete, two millennia later Kepler [38] found two additional regular polyhedra, and in the early 1800's Poinsot [45] found these two as well as two more; Cauchy [7] soon proved that there are no others. But in the 1920's Petrie and Coxeter found (see [8]) three new regular polyhedra, and proved the completeness of that enumeration. However, in 1977 I found [21] a whole lot of new regular polyhedra, and soon thereafter Dress proved [15], [16] that one needs to add just one more polyhedron to make my list complete. Then, about ten years ago I found [22] a whole slew of new regular polyhedra, and so far nobody claimed to have found them all.

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How come that results established by such accomplished mathematicians as Euclid, Cauchy, Coxeter, Dress were seemingly disproved after a while? The answer is simple – all the results mentioned are completely valid; what changed is the meaning in which the word “polyhedron” is used. As long as different people interpret the concept in different ways there is always the possibility that results true under one interpretation are false with other understandings. As a matter of fact, even slight variations in the definitions of concepts often entail significant changes in results.

In some ways the present situation concerning polyhedra is somewhat analogous to the one that developed in ancient Greece after the discovery of incommensurable quantities. Although many of the results in geometry were not affected by the existence of such quantities, it was philosophically and logically important to find a reasonable and effective approach for dealing with them. In recent years, several papers dealing with more or less general polyhedra appeared. However, the precise boundaries of the concept of polyhedra are mostly not explicitly stated, and even if explanations are given – they appear rather arbitrary and tailored to the needs of the moment [12] or else aimed at objects with great symmetry [40]. The main purpose of this paper is to present an internally consistent and quite general approach, and to illustrate its effectiveness by a number of examples.

In the detailed discussions presented in the following sections we shall

the completeness of that enumeration. However, in 1977 I found [21] a whole lot of new regular polyhedra, and soon thereafter Dress proved [15], [16] that one needs to add just one more polyhedron to make my list complete. Then, about ten years ago I found [22] a whole slew of new regular polyhedra, and so far nobody claimed to have found them all.

How come that results established by such accomplished mathematicians as Euclid, Cauchy, Coxeter, Dress were seemingly disproved after a while? The answer is simple – all the results mentioned are completely valid; what changed is the meaning in which the word "polyhedron" is used. As long as different people interpret the concept in different ways there is always the possibility that results true under one interpretation are false with other understandings. As a matter of fact, even slight variations in the definitions of concepts often entail significant changes in results.

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Thank you!

Gracias!

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