Vertex transitive graphs with (very) small motion.

Symmetries of Discrete Objects
Auckland, NZ.
February 2024

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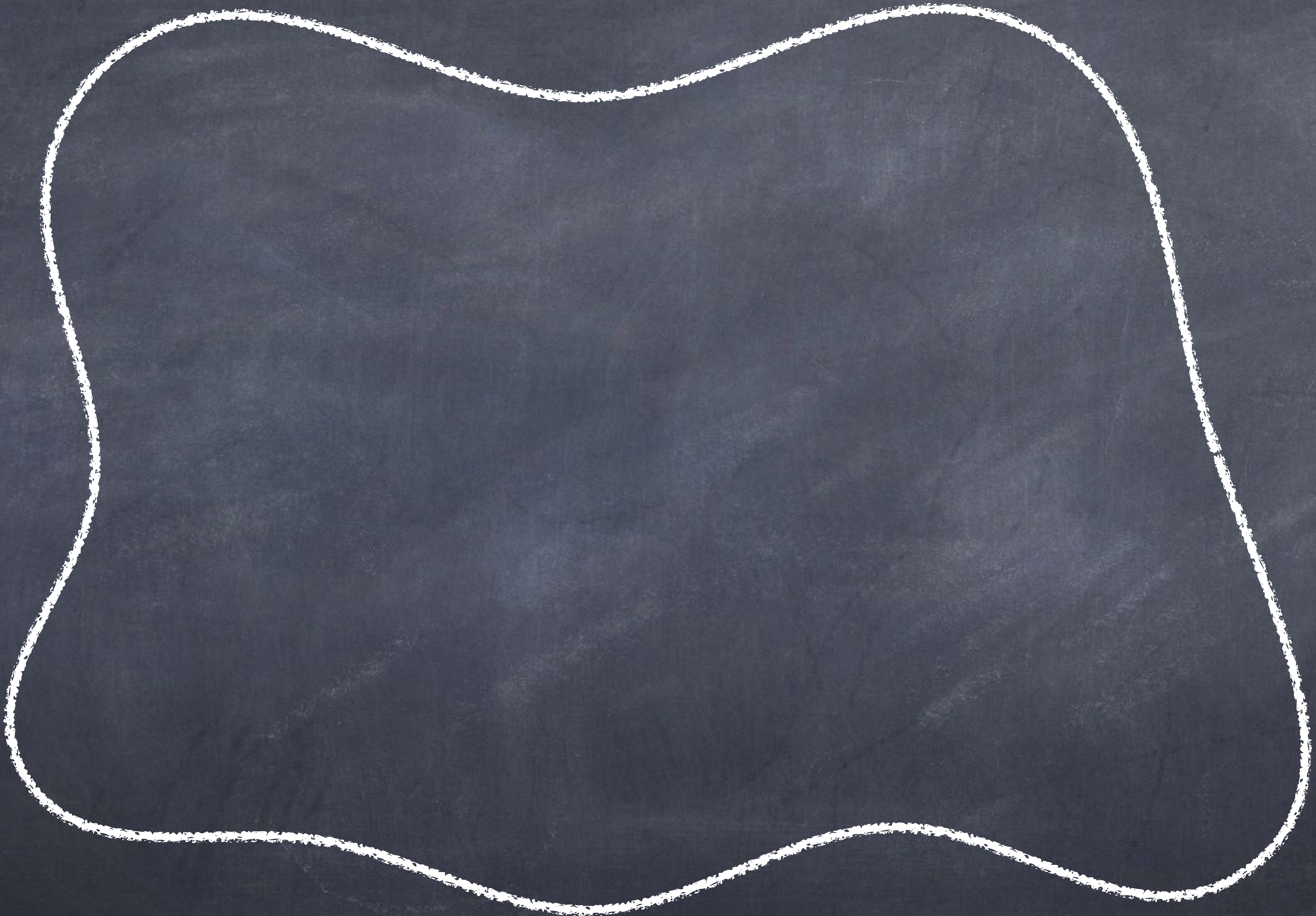
University of Ljubljana
Joint work with Primož Potočnik

$\Gamma =$



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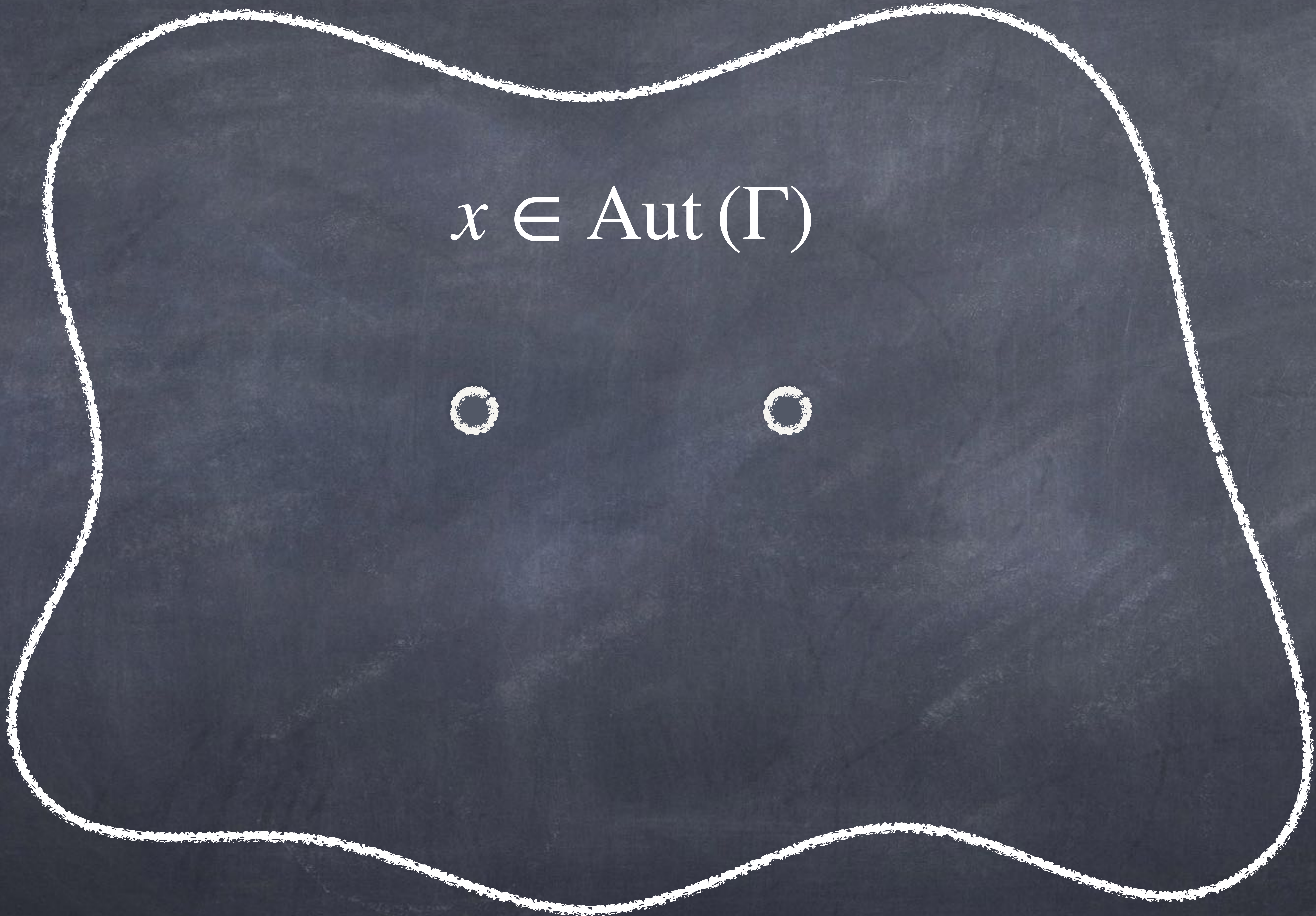

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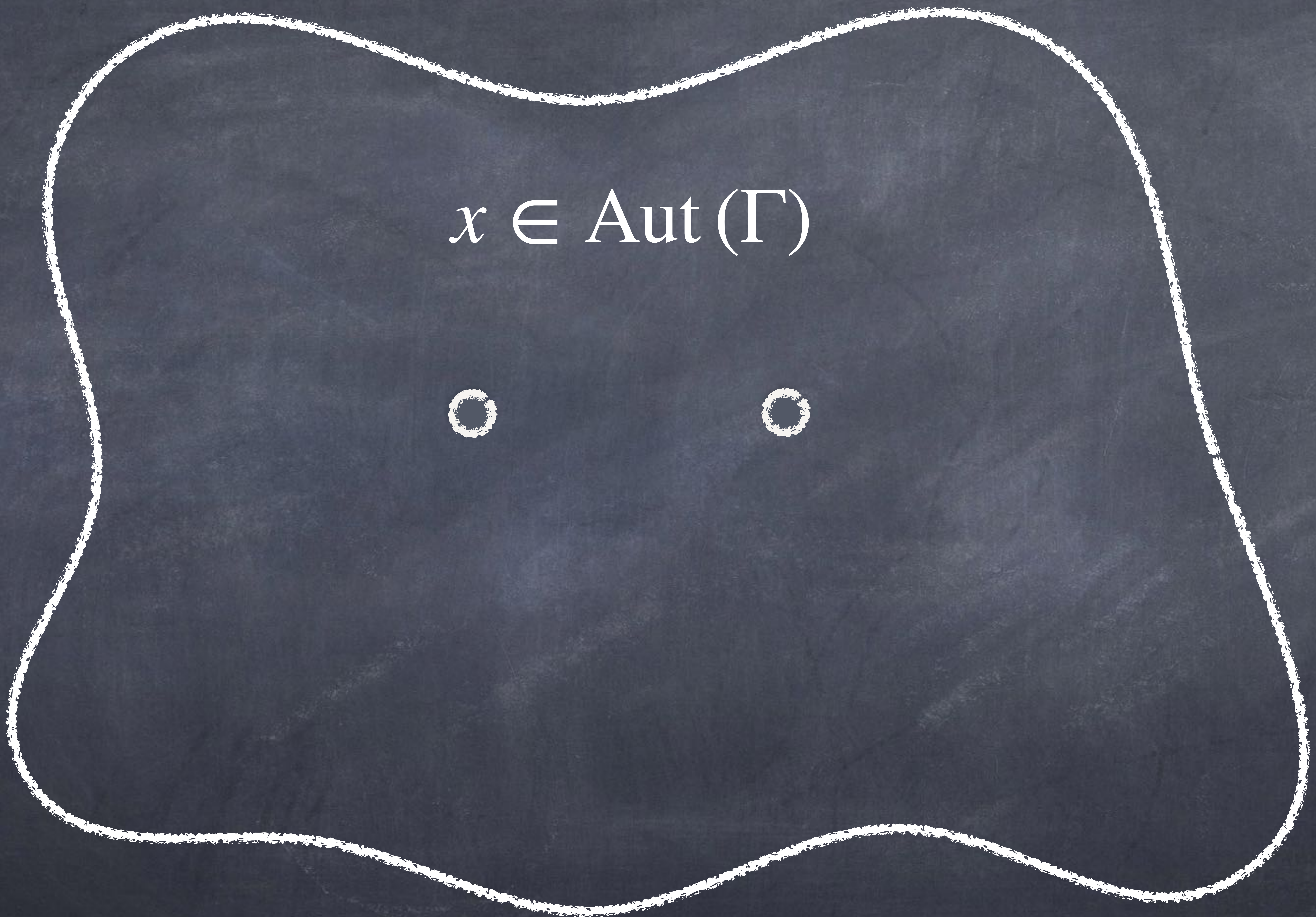

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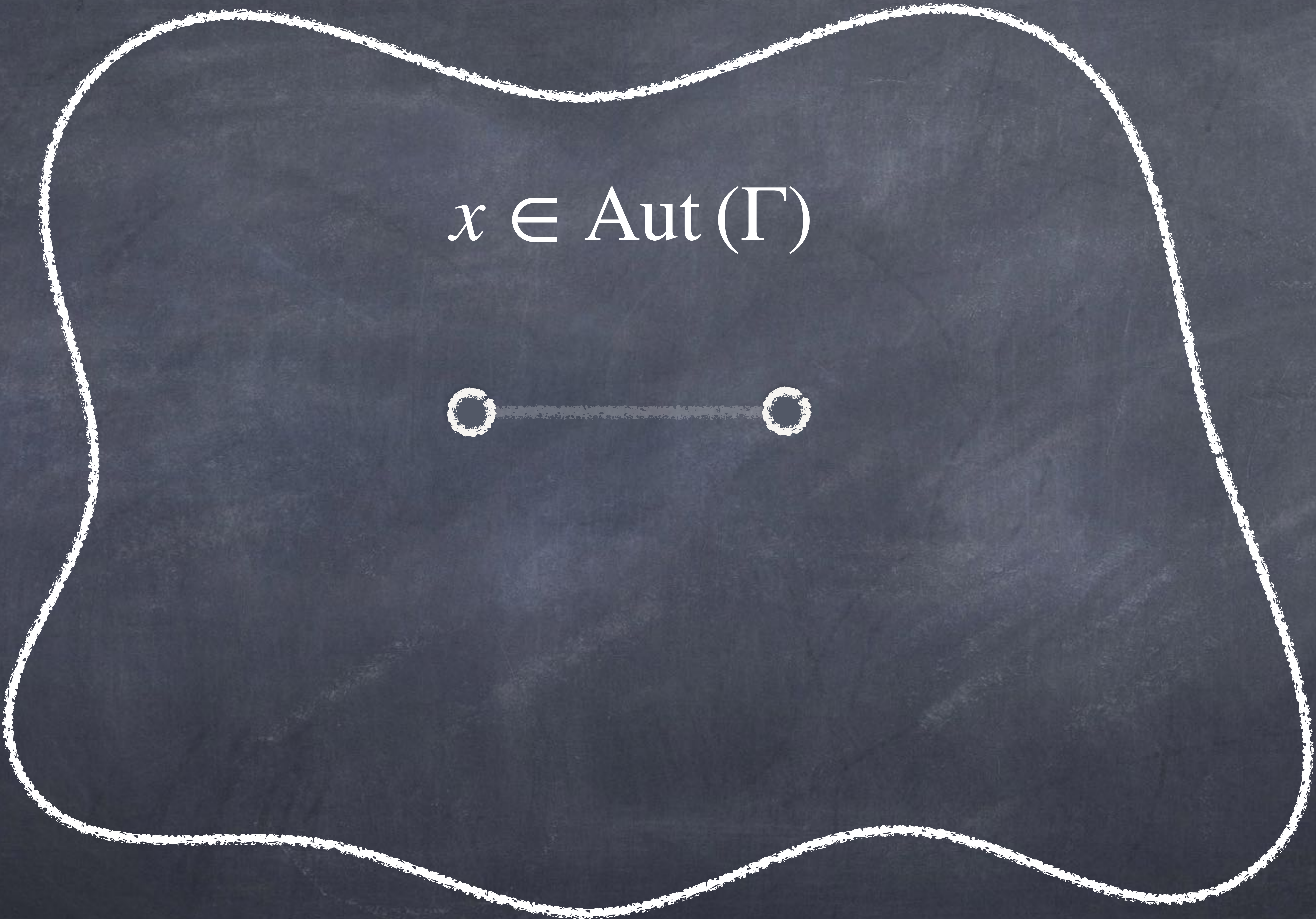

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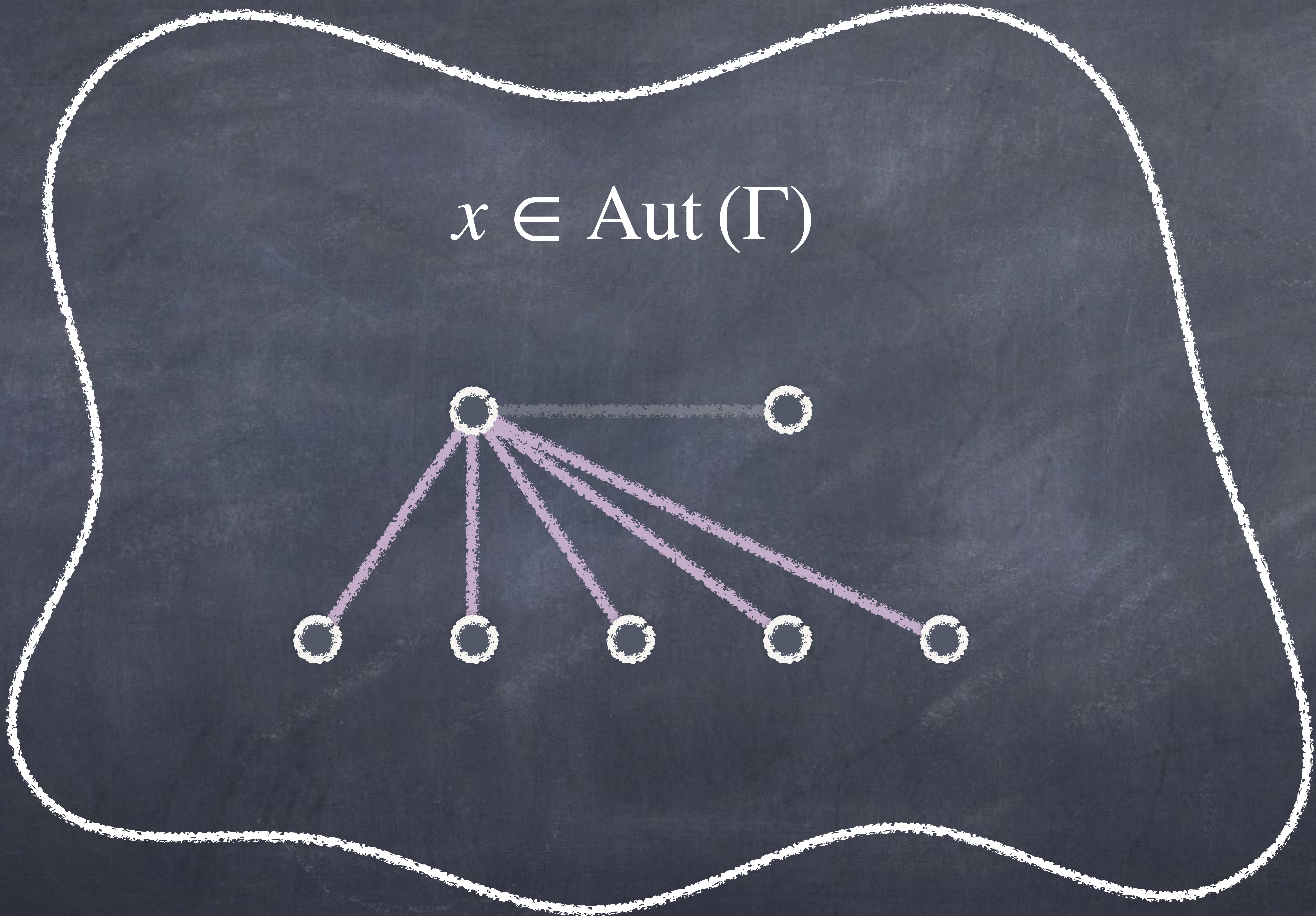

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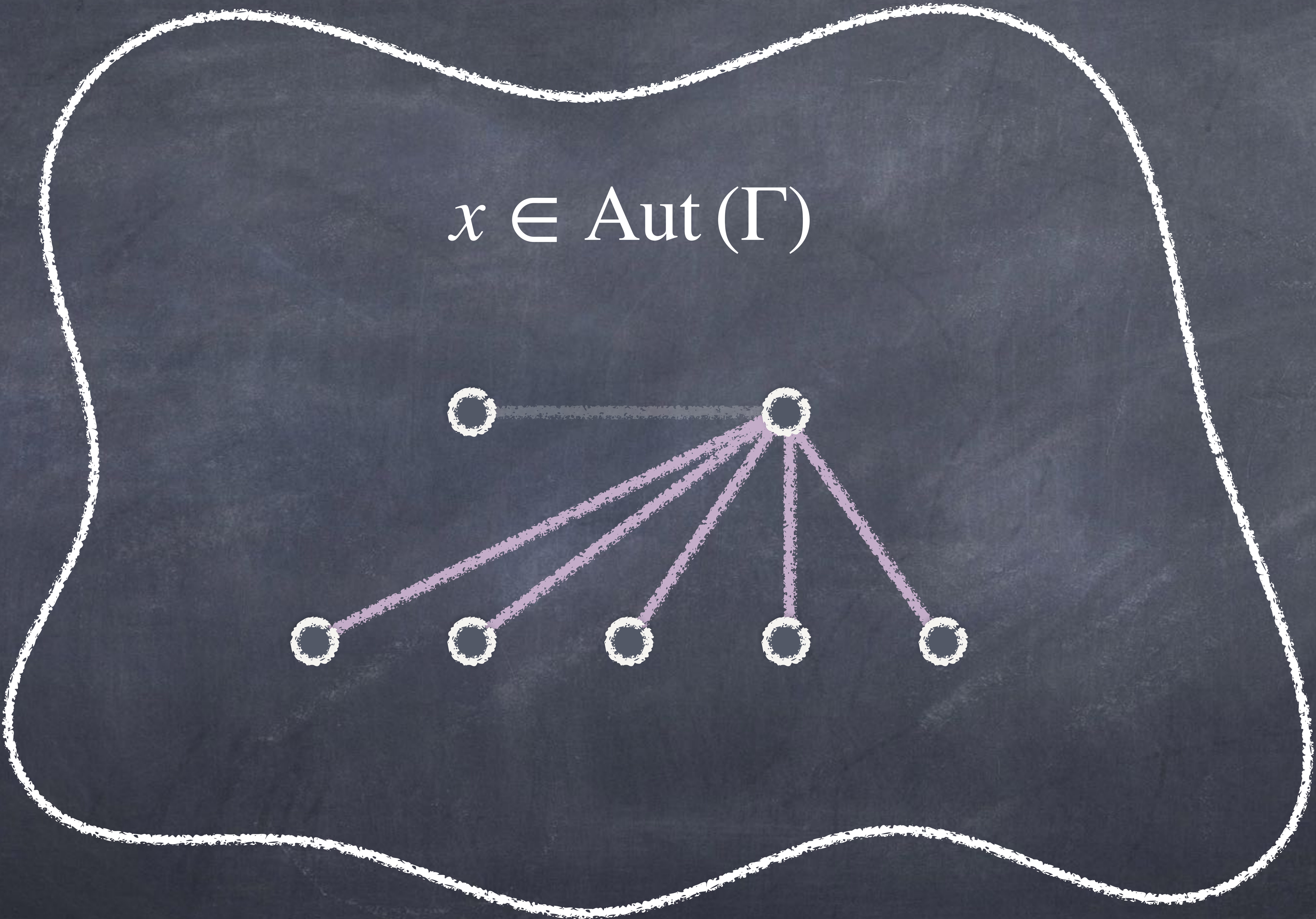

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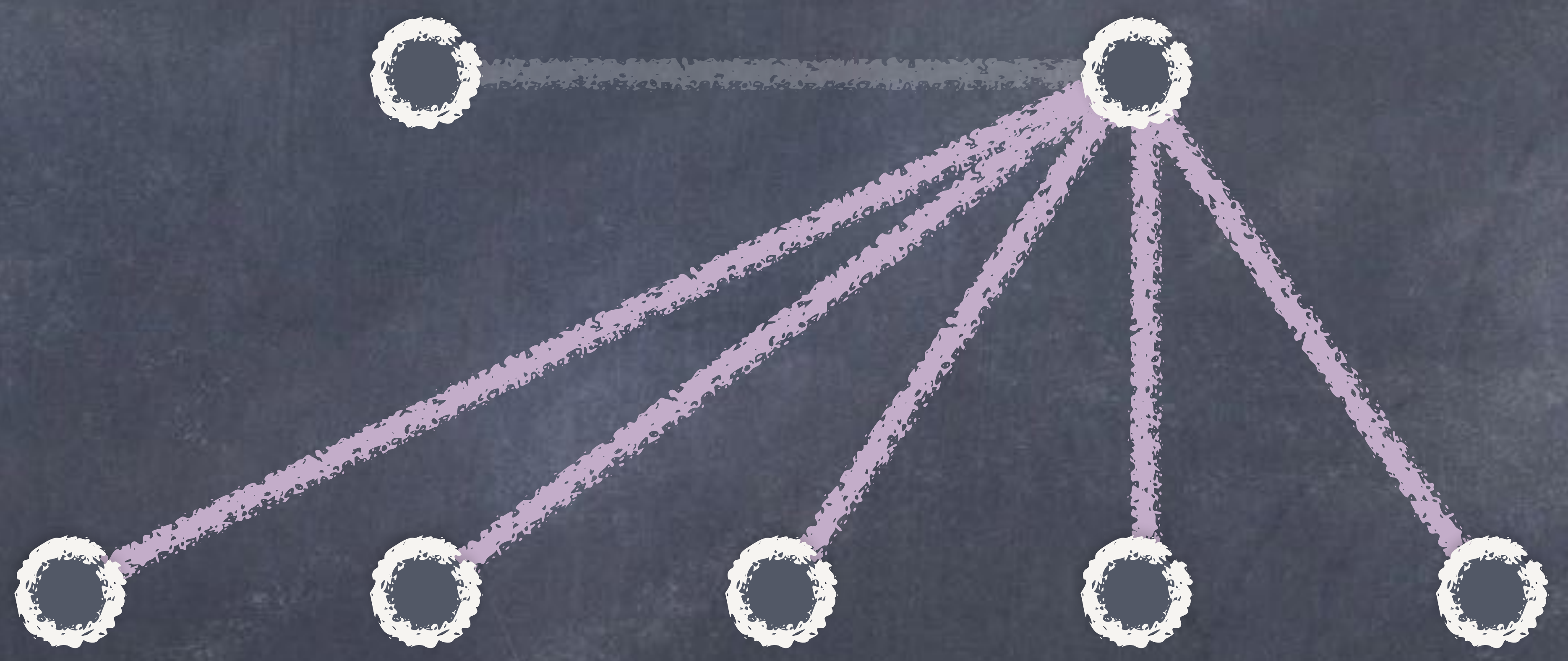
$x \in \mathrm{Aut}(\Gamma)$



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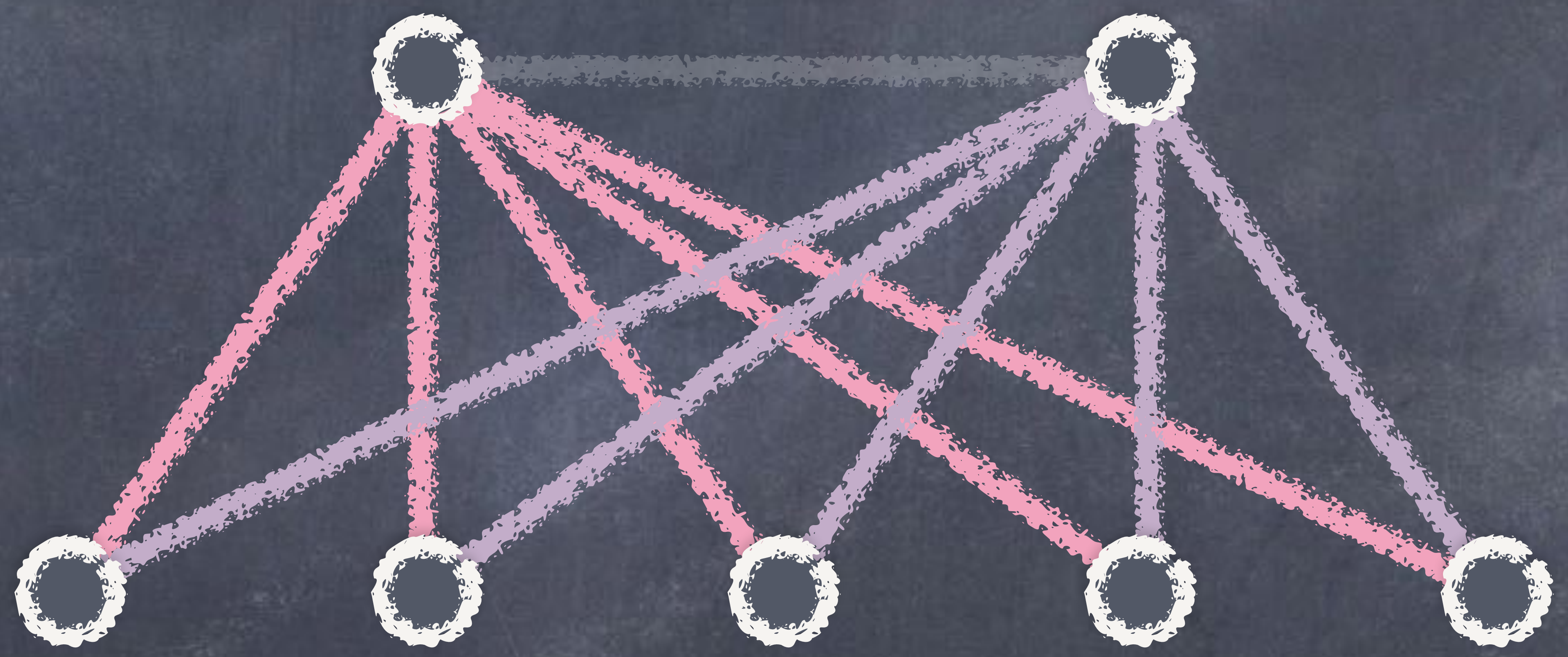

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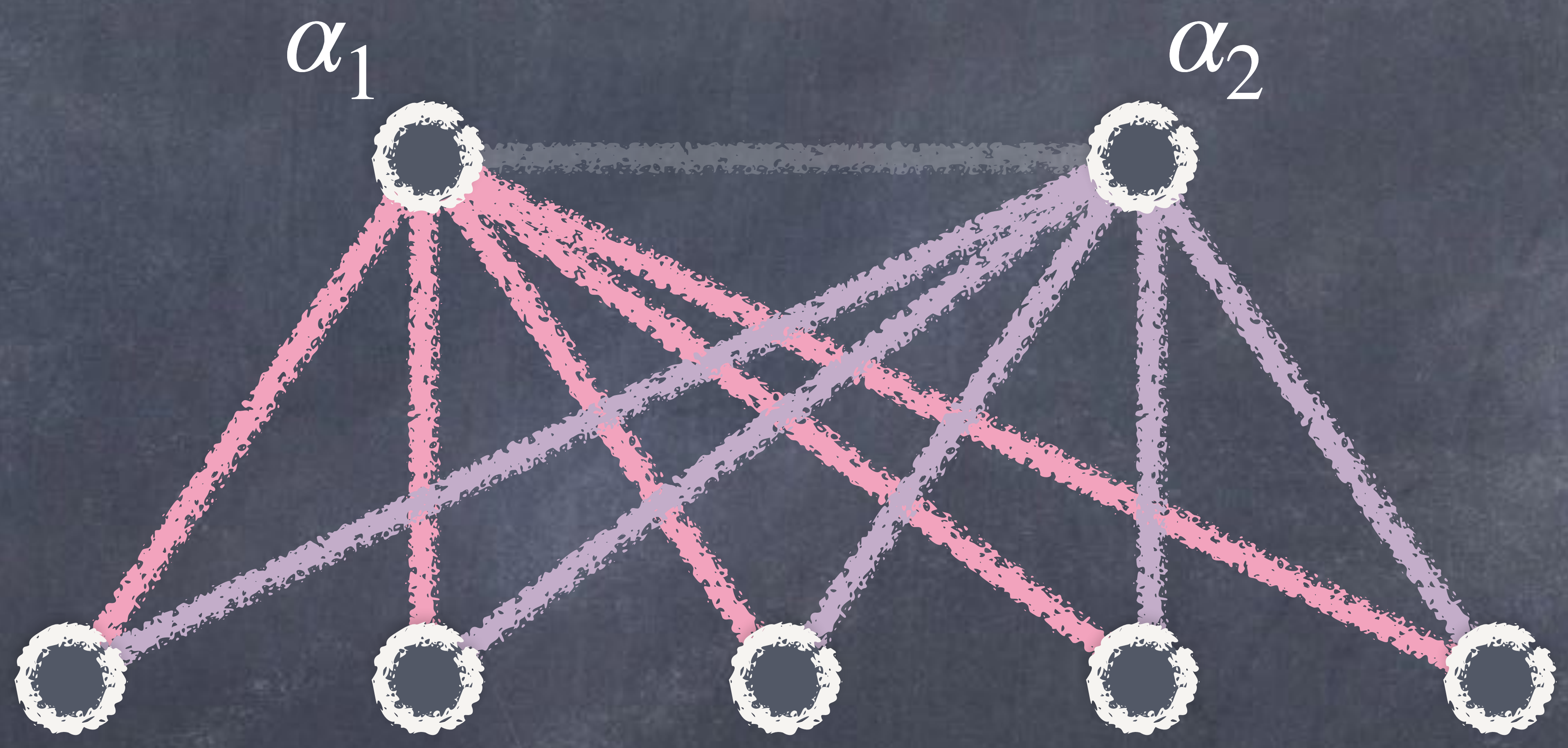

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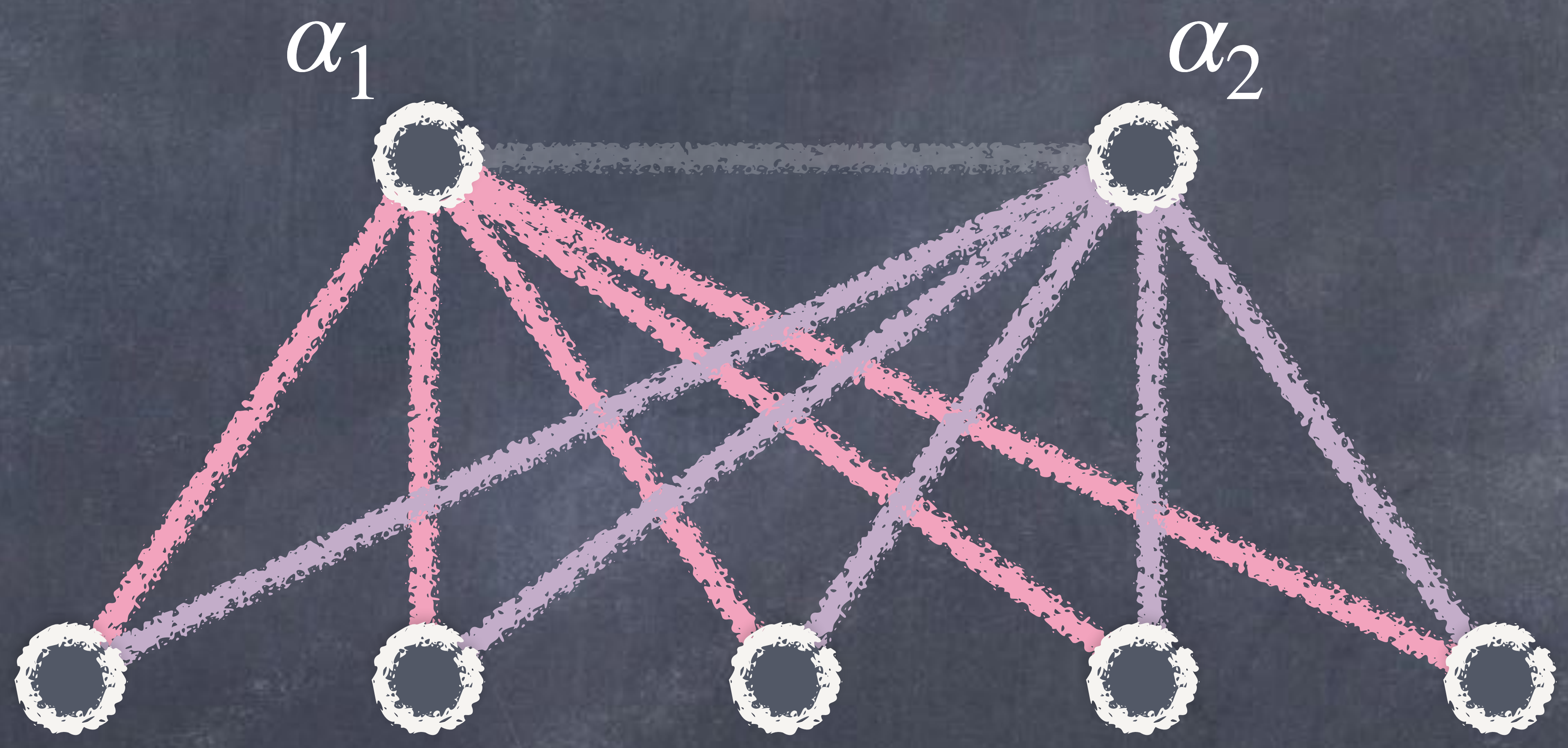

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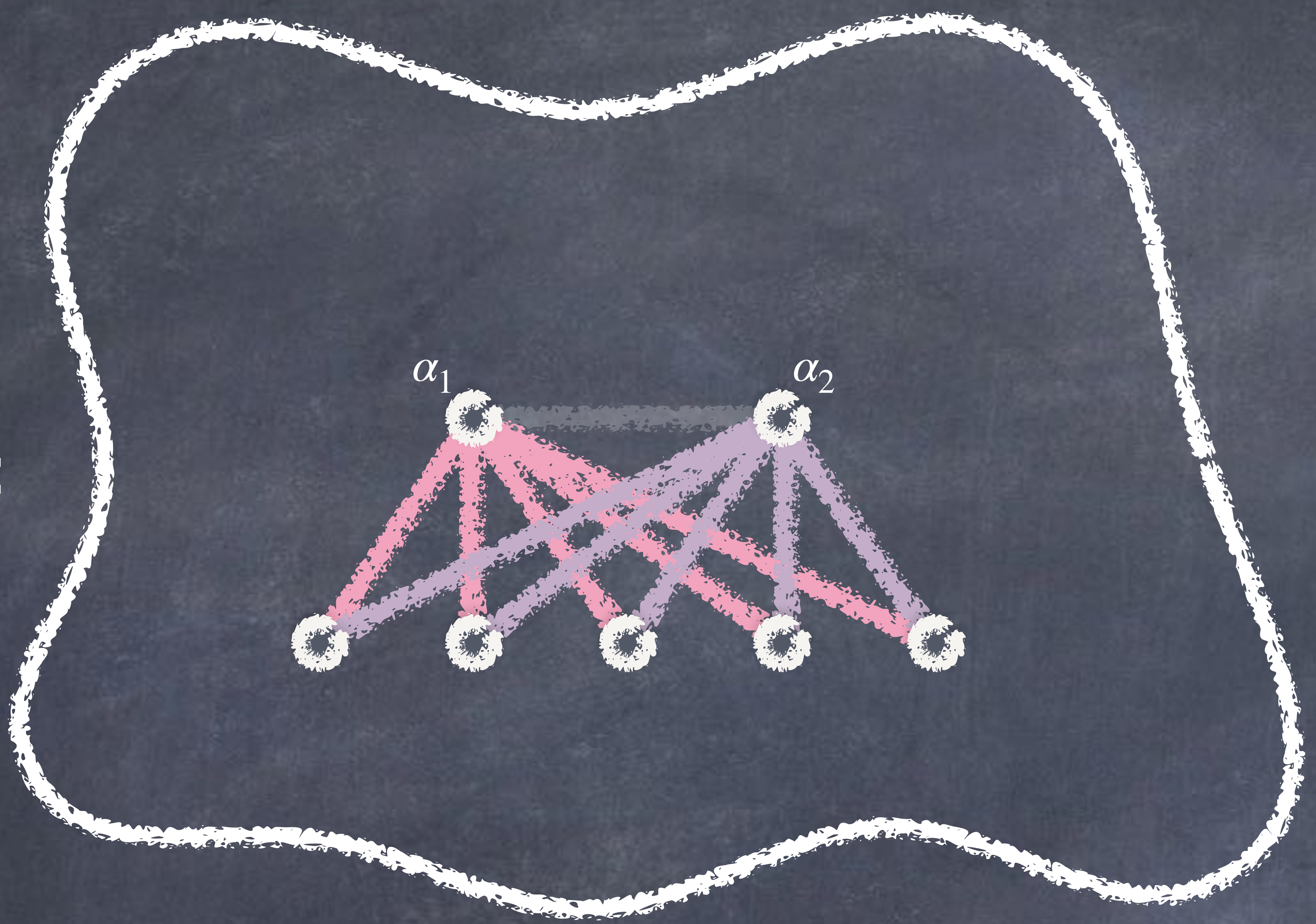

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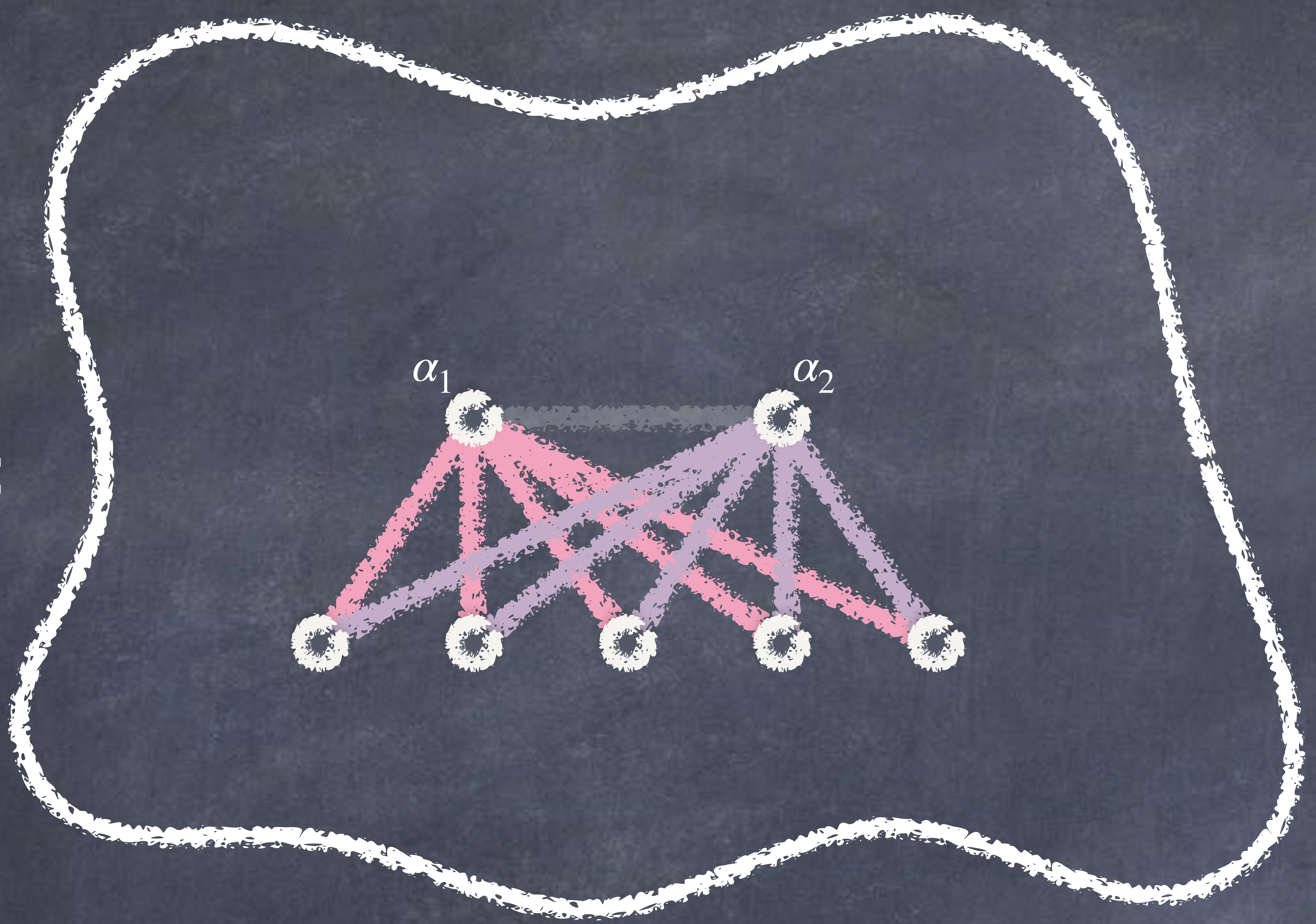
$\Gamma =$ 

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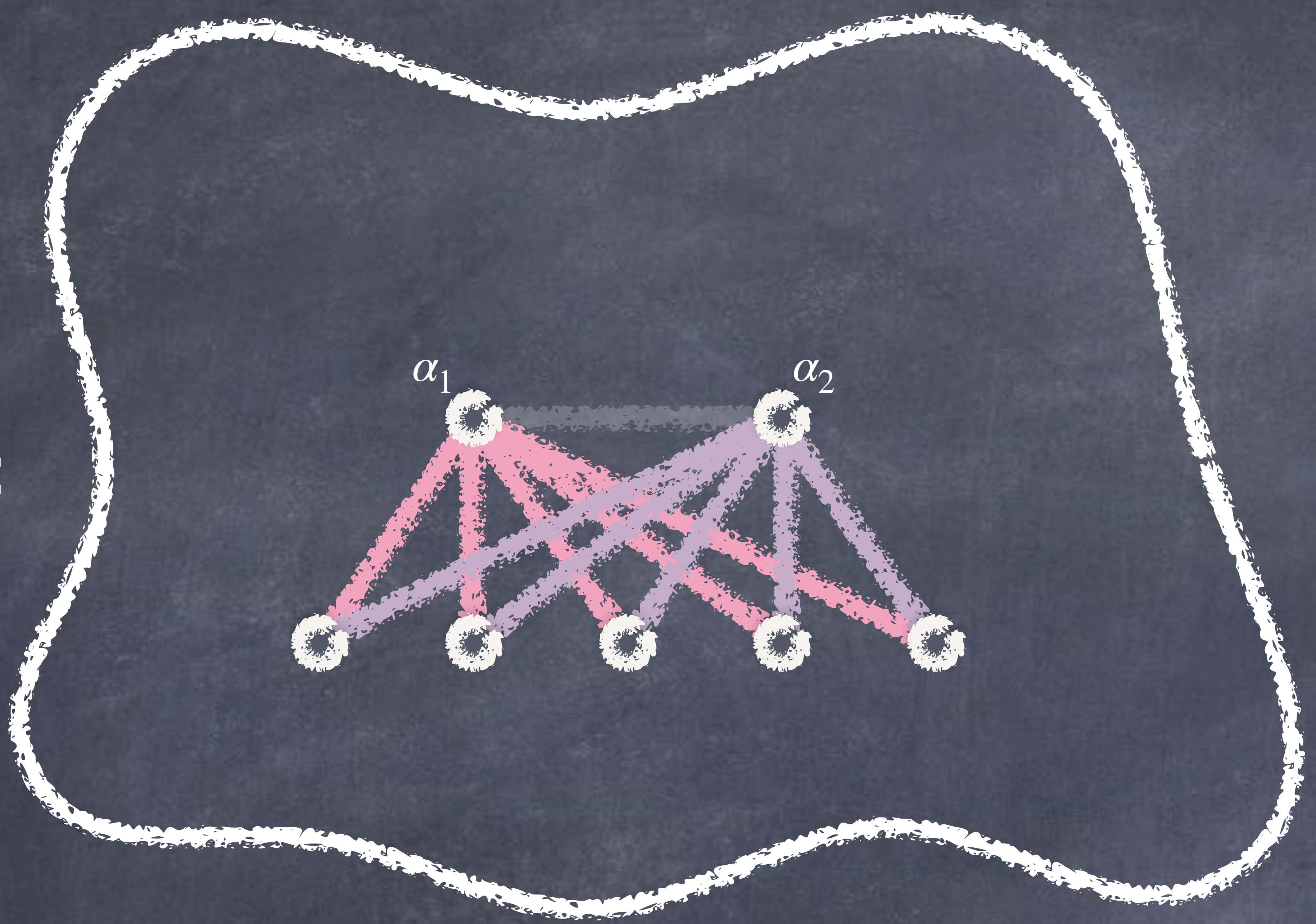

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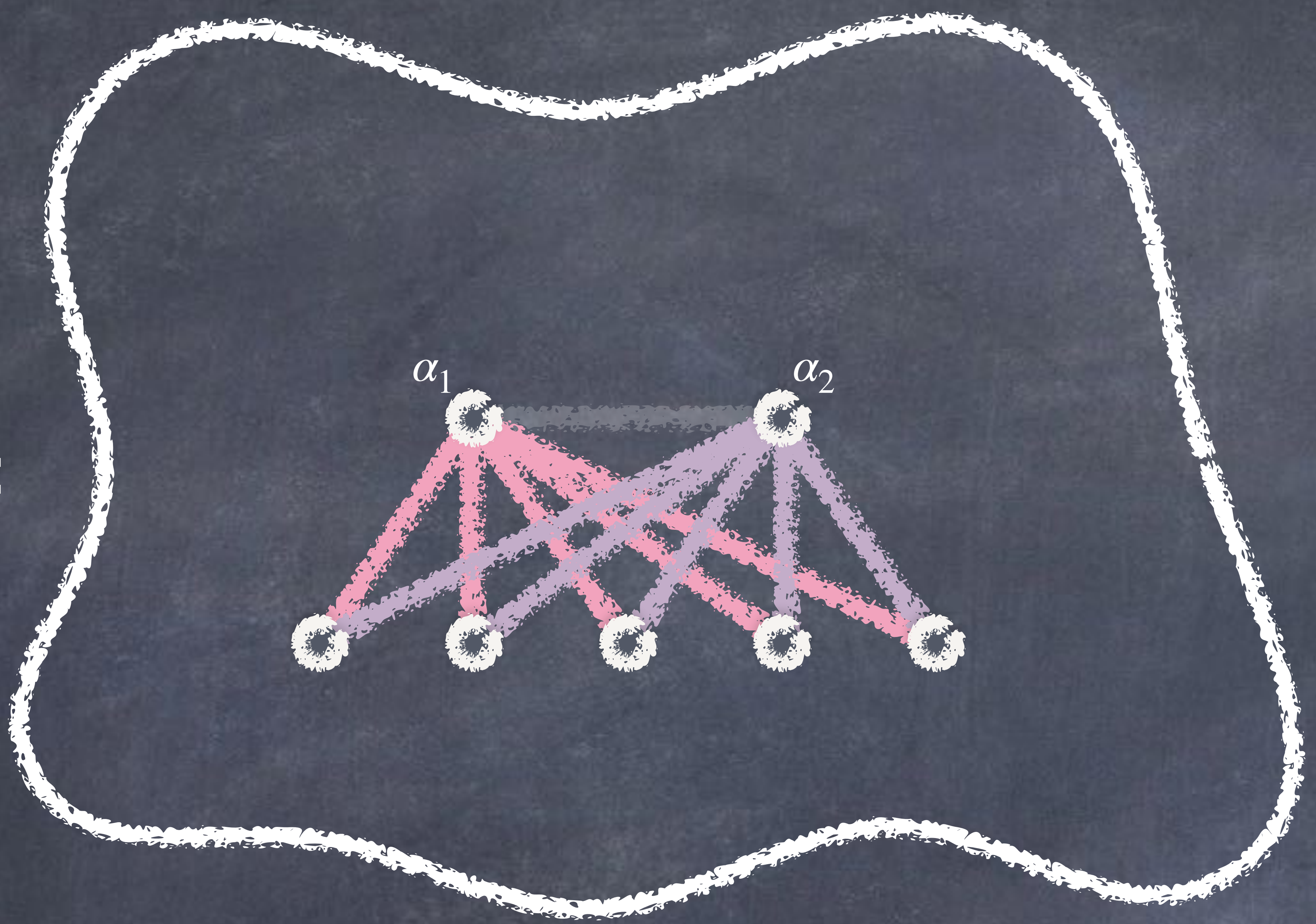

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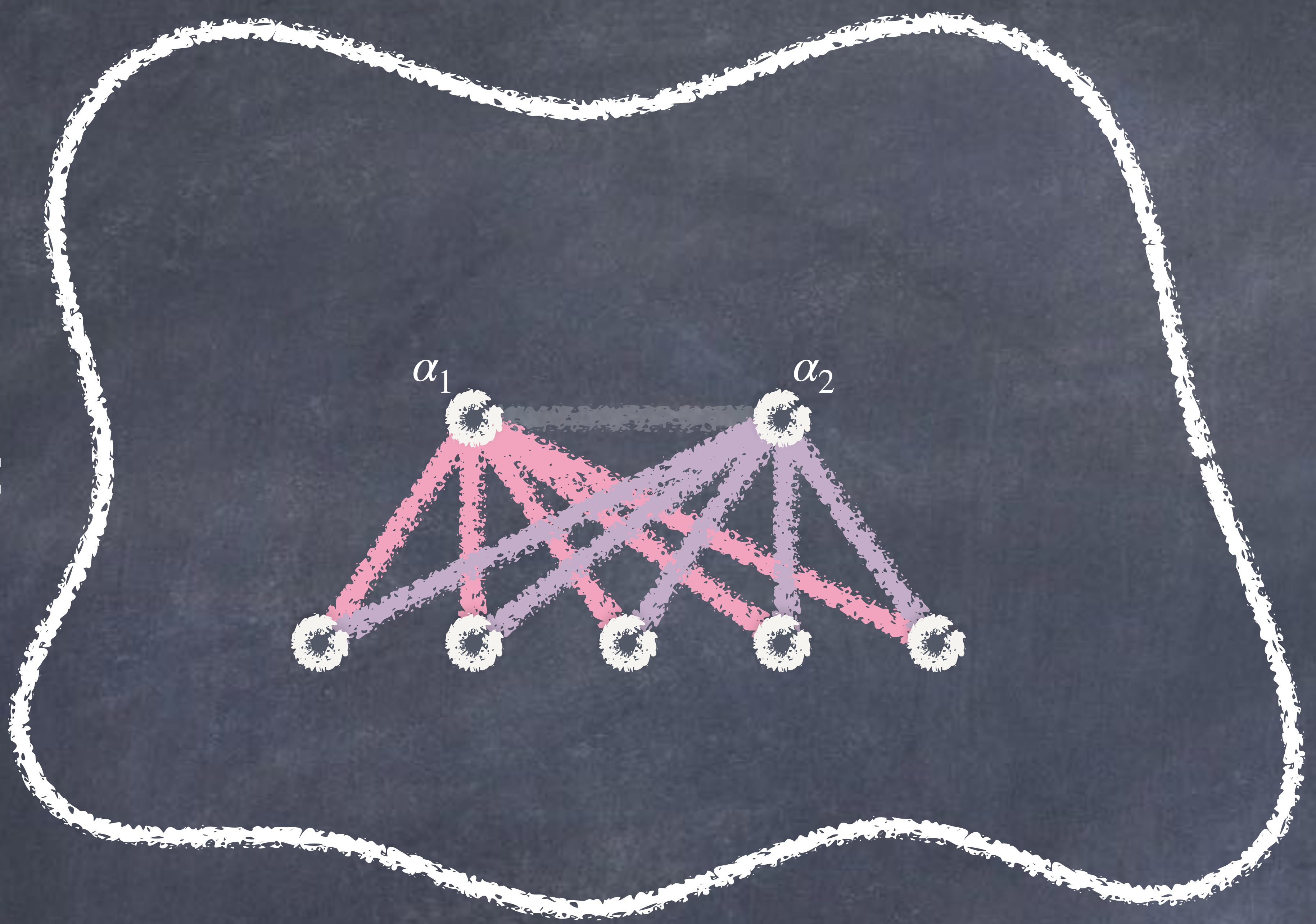

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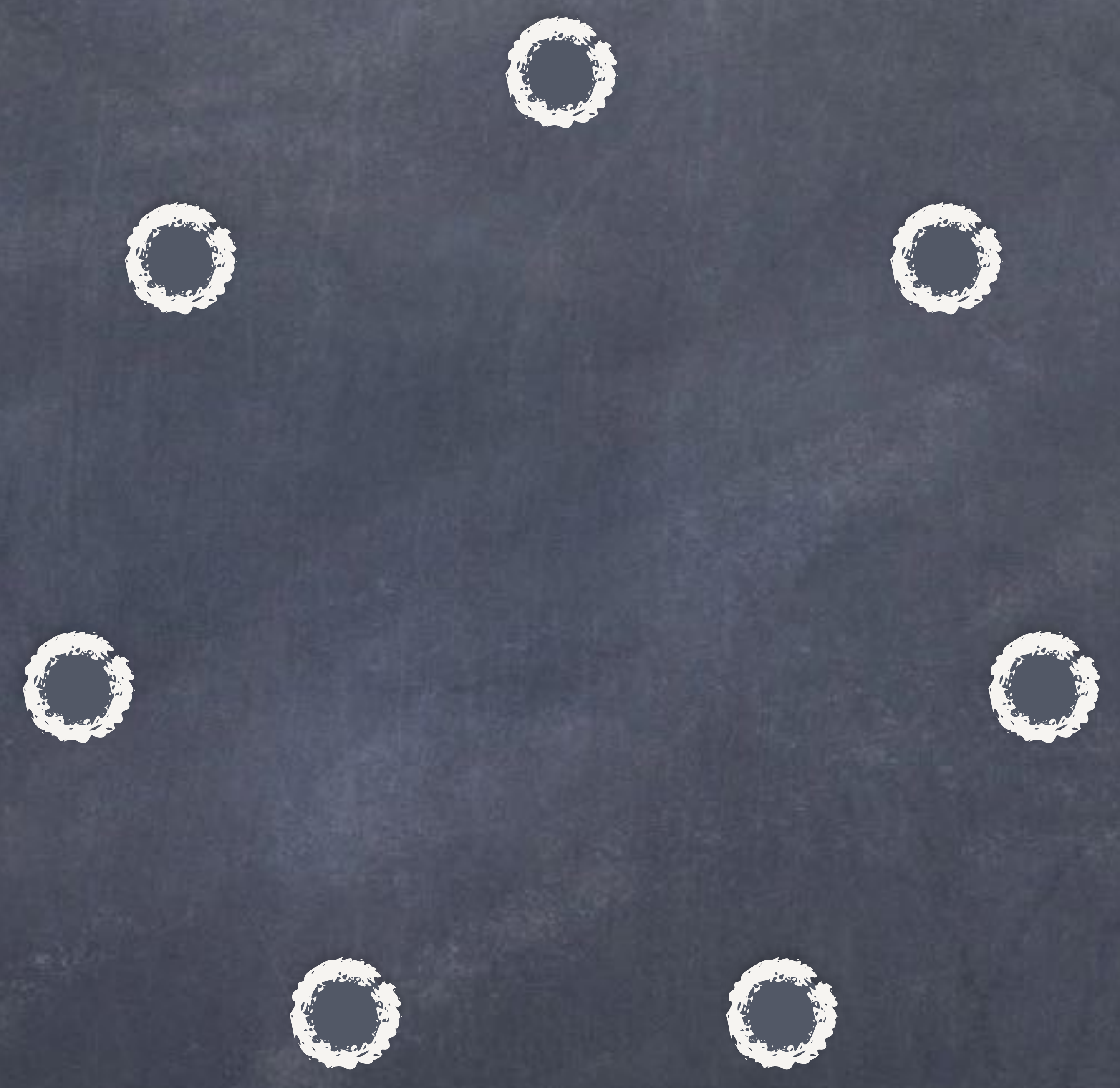
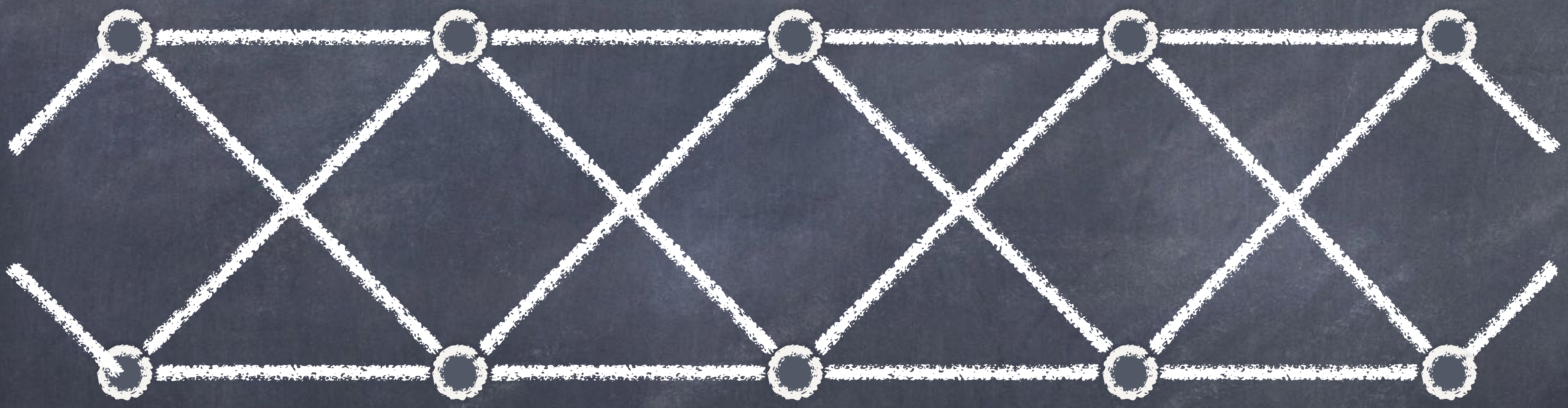
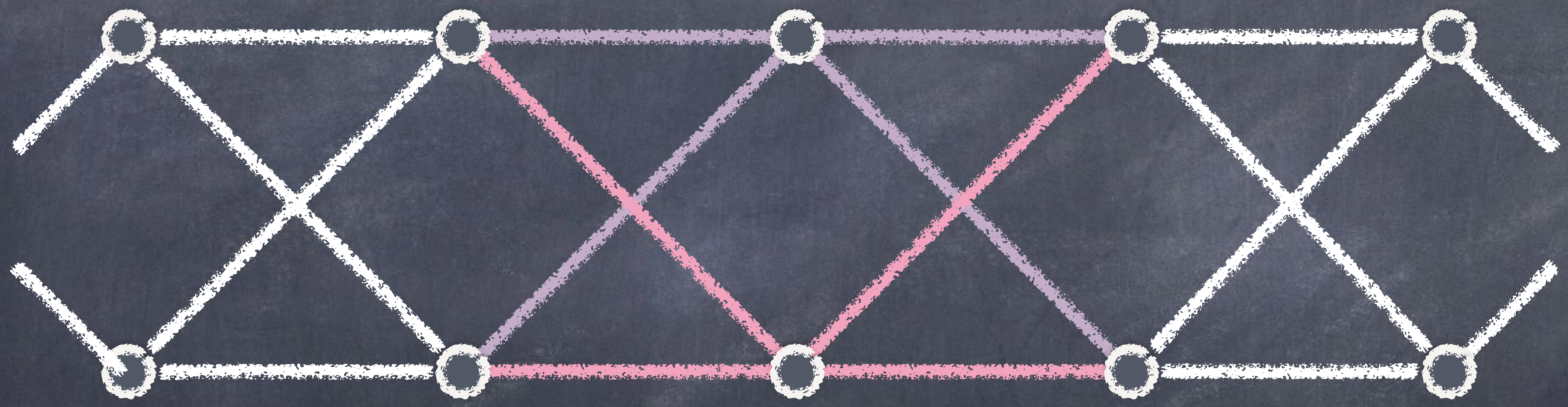
What if Γ is Vertex Transitive?





$\mathrm{PX}(r,1)$



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- The support $\text{supp}(x)$ of an automorphism x of a graph Γ is the subset of vertices of Γ moved by x

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- The support $\mathrm{supp}(x)$ of a permutation x of a set Ω is the subset of points in Ω moved by x

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- The motion $\mu(\Gamma)$ of a graph Γ is the minimum value of vertices moved by a non-trivial automorphism:

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- The mininimal degree $\mu(G)$ of a permutation group $G \leq \mathrm{Sym}(\Omega)$ is the minimum number of points moved by a non-trivial element of G:

$$\mu(G) = \min \{ |\mathrm{supp}(x)| : x \in G \}$$

The groups problem

Determine the transitive permutation groups $G \leq \mathrm{Sym}(\Omega)$ with $\mu(G) = k$

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$$\mu\left(\mathrm{Aut}(\Gamma)\right) = \mu(\Gamma)$$

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Determine the primitive permutation groups $G \leq \mathrm{Sym}(\Omega)$ with $\mu(G) = k$

Theorem (Jordan, 1871)

If $G \leq \mathrm{Sym}(\Omega)$ is primitive and $\mu(G) \leq 3$ then $\mathrm{Alt}(\Omega) \leq G$.

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If $G \leq \mathrm{Sym}(\Omega)$ is primitive and contains a p-cycle for some prime $p \leq |\Omega| - 3$ then $\mathrm{Alt}(\Omega) \leq G$.

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Theorem (Jones, 2014)

If $G \leq \mathrm{Sym}(\Omega)$ is primitive and contains a k-cycle for some $k \leq |\Omega|$ then $\mathrm{Alt}(\Omega) \leq G$ or G belongs to a list of well-known groups.

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- Herzog, Praeger (1976) - Minimal degree of PPG
- Liebeck, (1984) - On minimal degrees and base sizes of PPG
- Liebeck, Saxl (1991) - Min Deg of PPG with an application to monodromy groups of covers of Riemann Surfaces.
- Saxl, Shalev (1995) - The fixity of permutation groups.
- Guralnick, Magaard (1998) - On the minimal degree of a PPG.
- Lawther, Liebeck, Seitz (2014) - Fixed point rations in action of finite exceptional groups of Lie type.
- Liebeck, Shalev (2015) - On fixed points of elements in PPG.
- Burness, Guralnick. (2022) - Fixed point ratios for finite primitive groups and applications.

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- Lehner, Potočnik, Spiga (2021) - On fixity of arc-transitive graphs.
- Potočnik, Spiga (2021) - On the number of fixed points of automorphisms of vertex-transitive graphs.
- Barbieri, Grazian, Spiga (2023) - On the number of fixed edges of automorphisms of vertex-transitive graphs of small valency.

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- Γ is one of six exceptions; or

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- Γ is isomorphic to a Praeger-Xu graph $\mathrm{PX}(r, s)$ with $1 \leq s \leq 2r/3$ and $r \geq 3$

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Classify the vertex transitive graphs Γ with $\mu(\Gamma) = k$

The graphs problem

Theorem: Potočnik, M.

Let Γ be a vertex transitive graph with $\mu(\Gamma) = k$, then

- If $k = 2$ ...
- If k is an odd prime number...
- If $k = 4$ ...

The lexicographic product:

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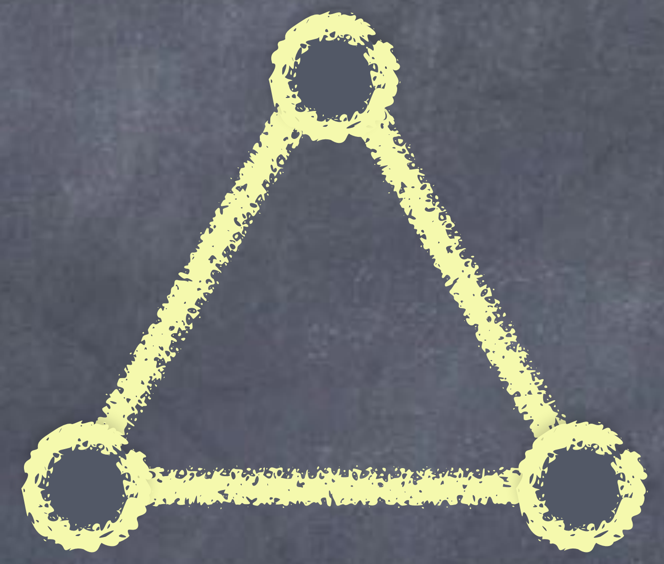

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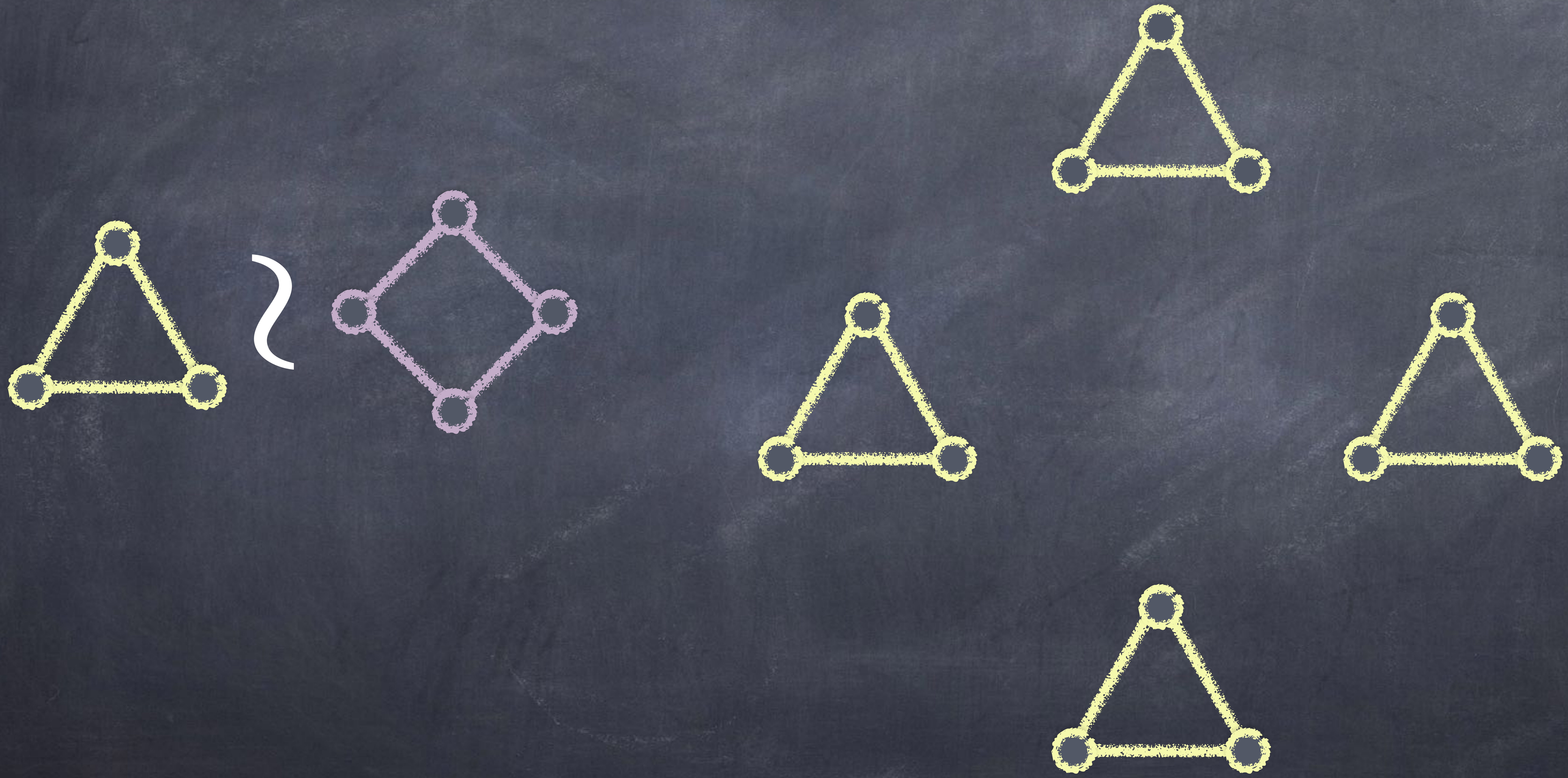

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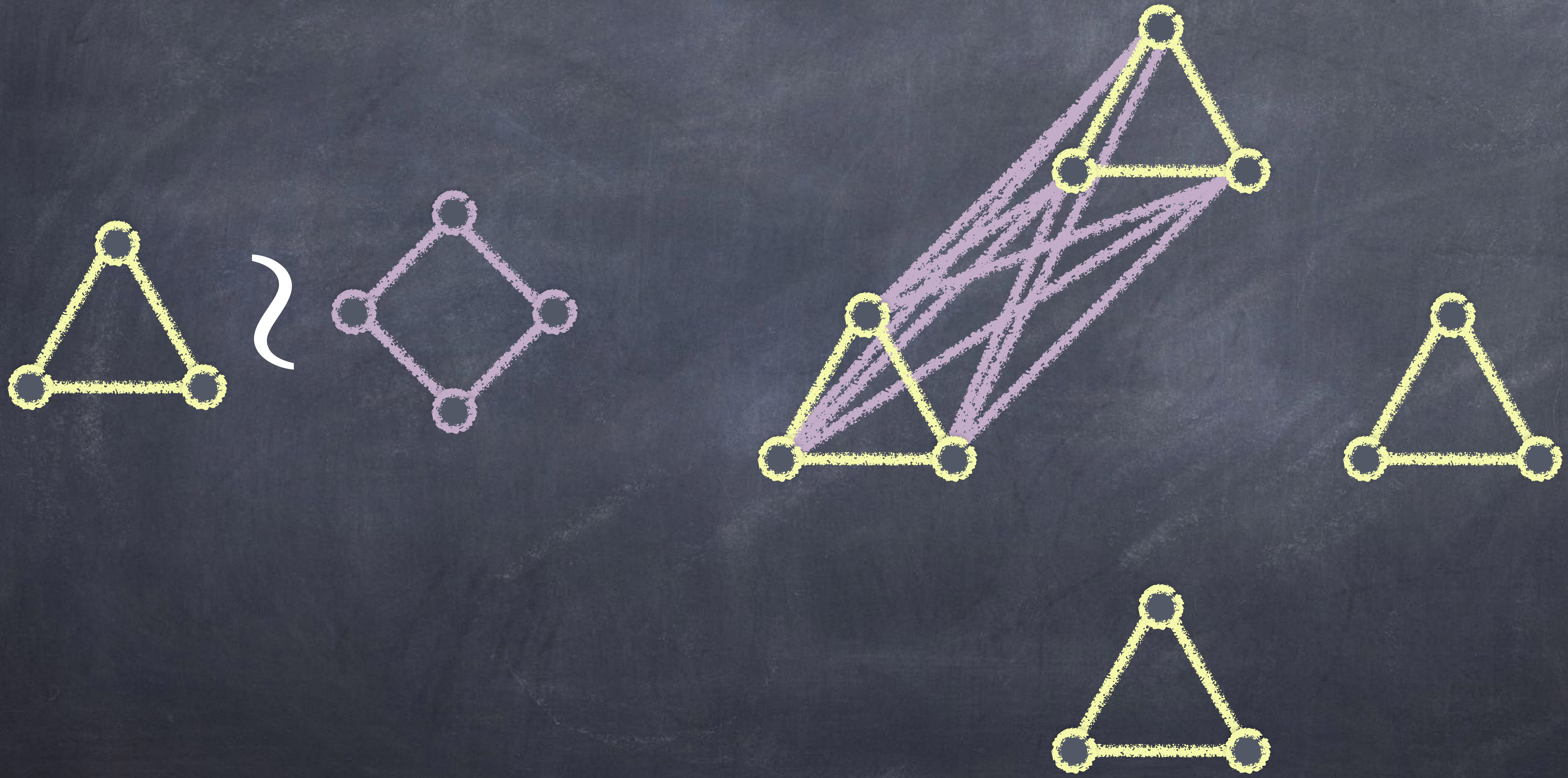

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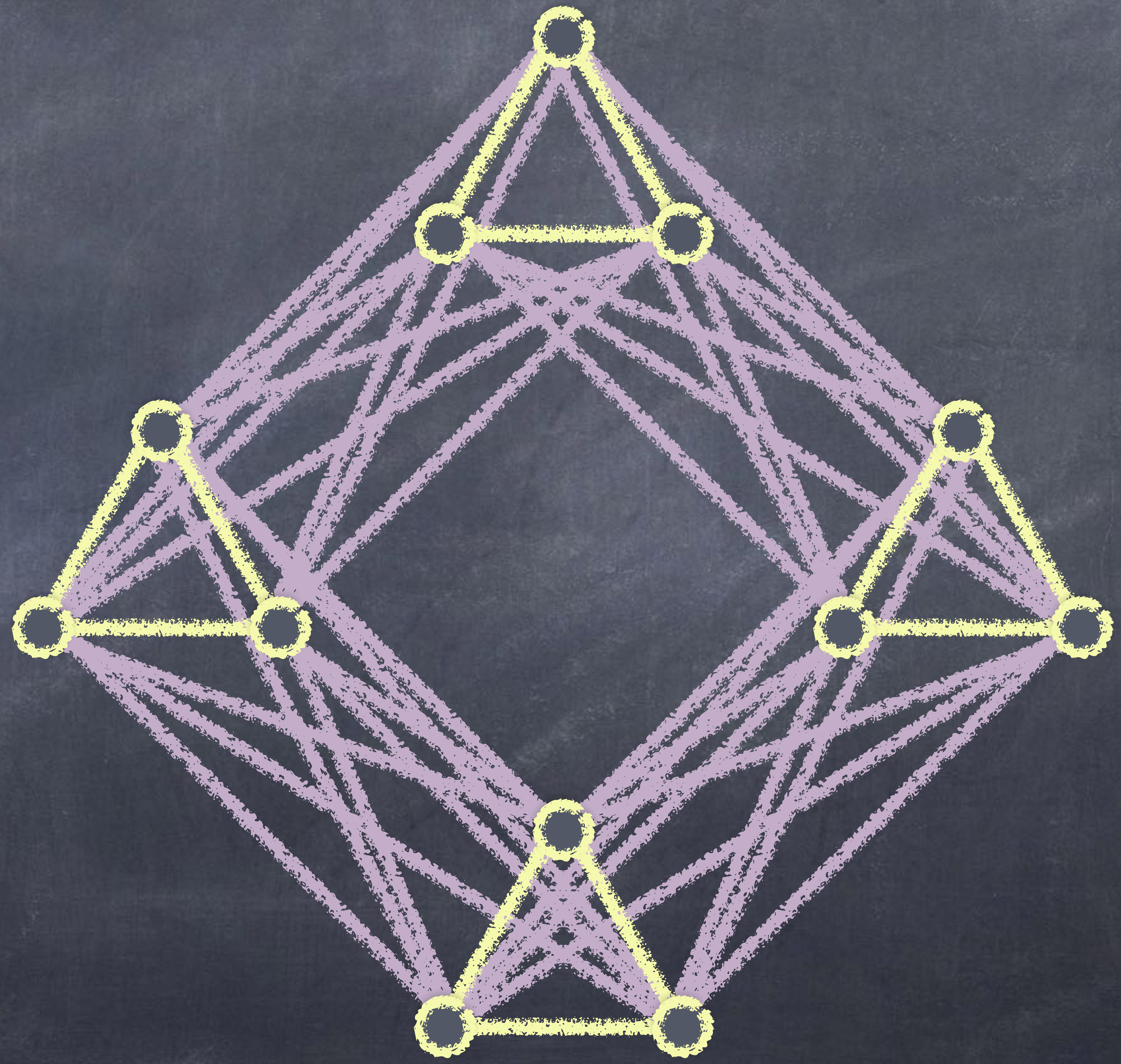

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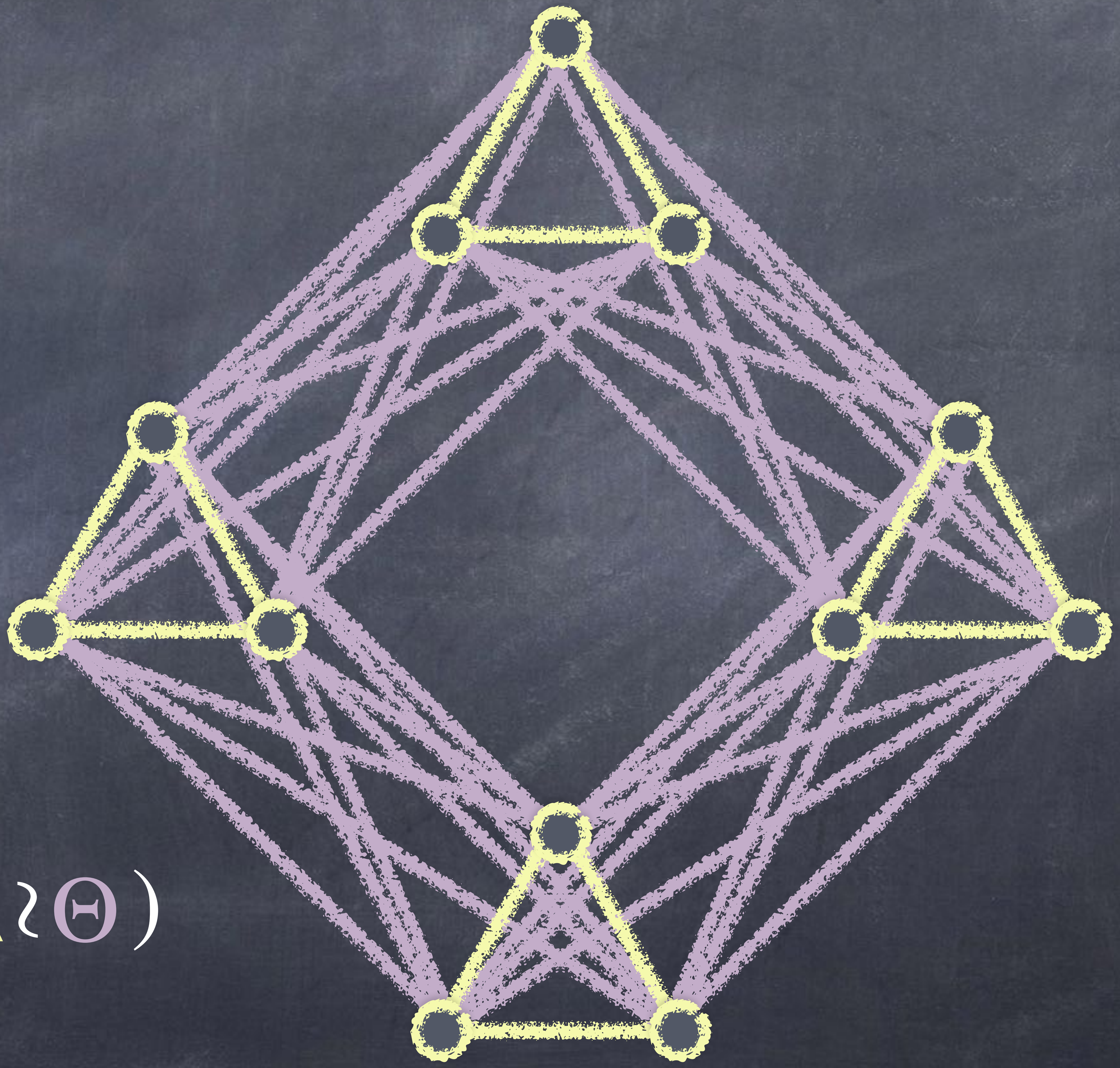

$\mathrm{Aut}(\Delta)\ \mathrm{wr}\ \mathrm{Aut}(\Theta) \leq \mathrm{Aut}(\Delta \wr \Theta)$

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for some $m \geq 2$ and Θ a vertex-transitive graph.

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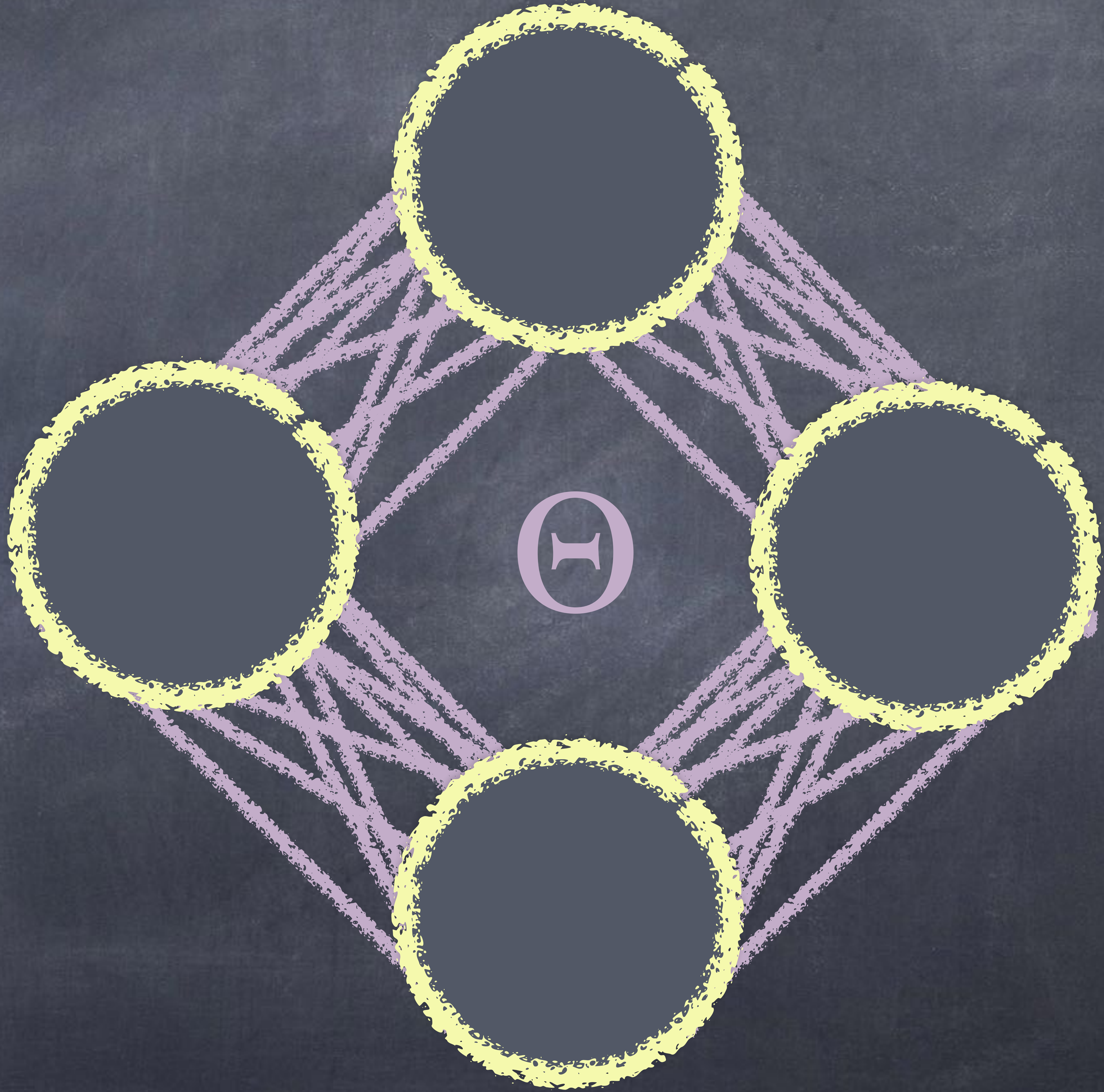

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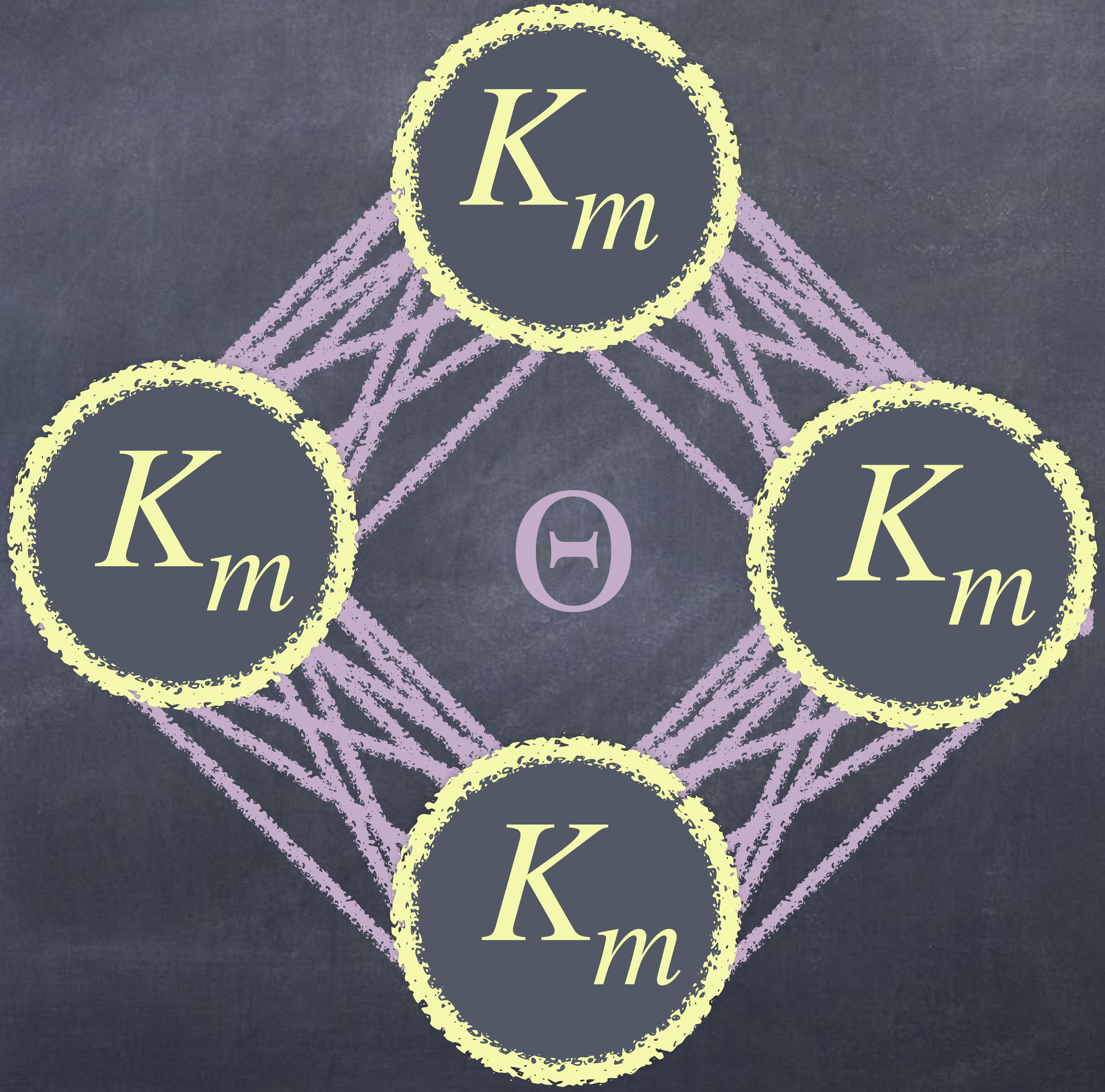

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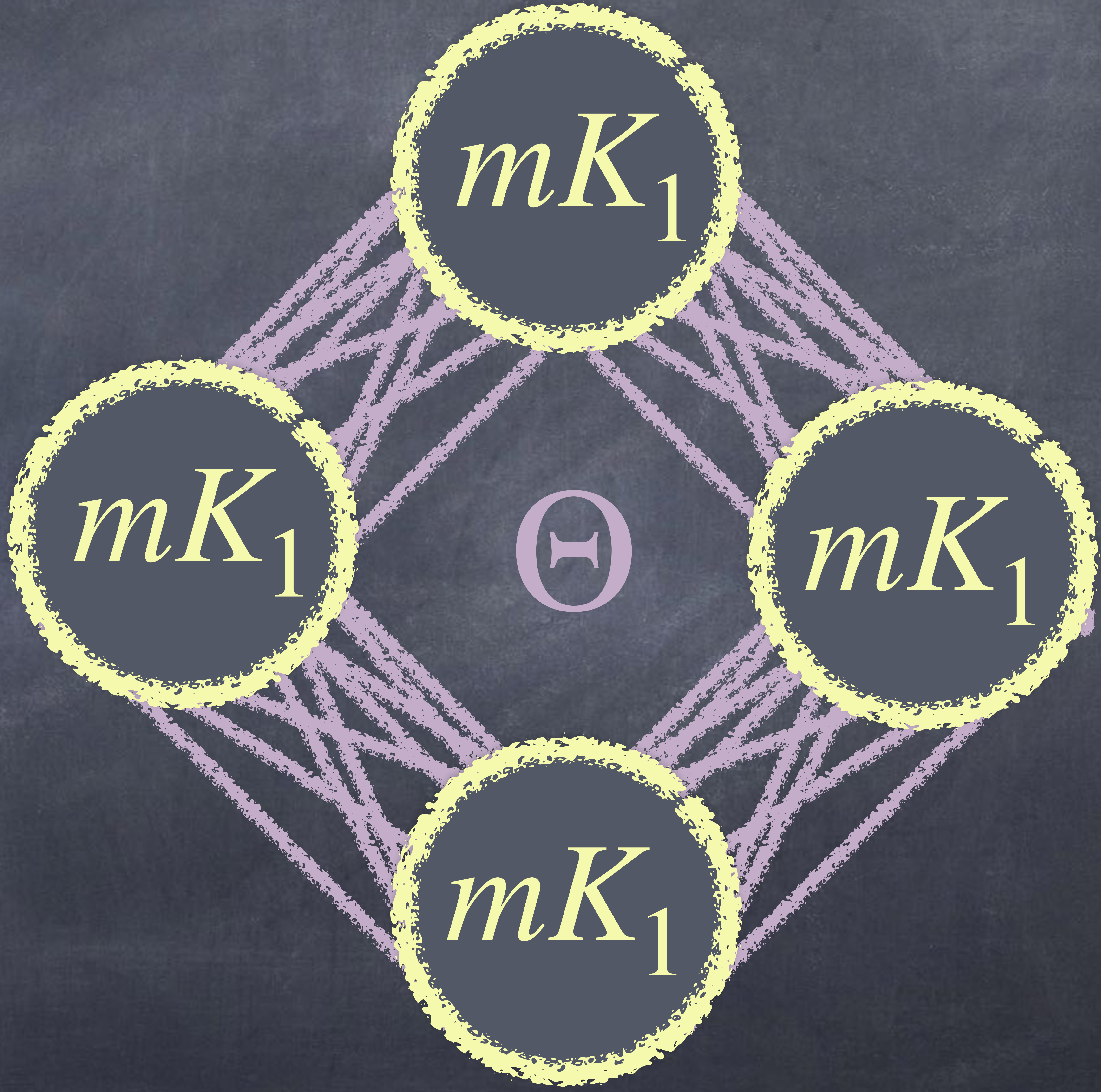

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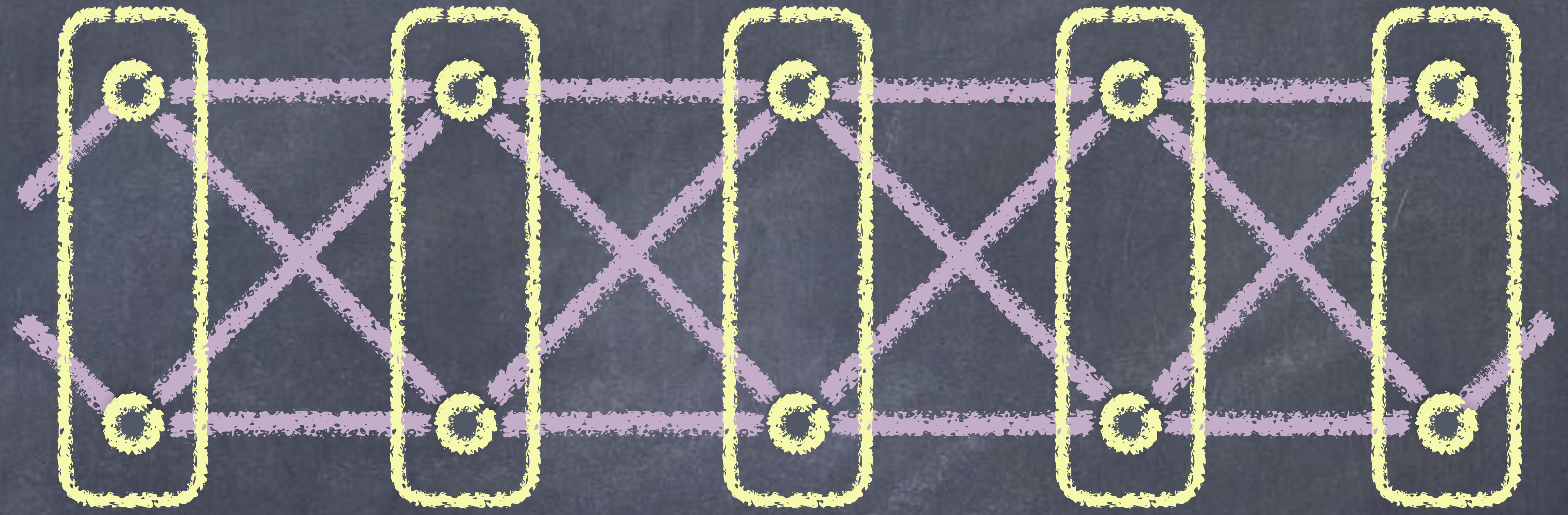

$\mathrm{PX}(r,1) \cong (2K_1) \wr C_r$

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- $\Gamma \cong K_m \wr \Theta$;
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- $\Gamma \cong \Sigma_p \wr \Theta$;

for some $m \geq 2$, Θ a vertex-transitive graph and Σ_p a circulant graph with p vertices.

The graphs problem

There are no VT graphs Γ with $\mu(\Gamma) = p$ for an odd prime p.

The graphs problem

Theorem (Potočnik, M.)

Let Γ be a vertex-transitive graph with $\mu(\Gamma) = 4$, then

- $\Gamma \cong C_5 \wr \Theta$;
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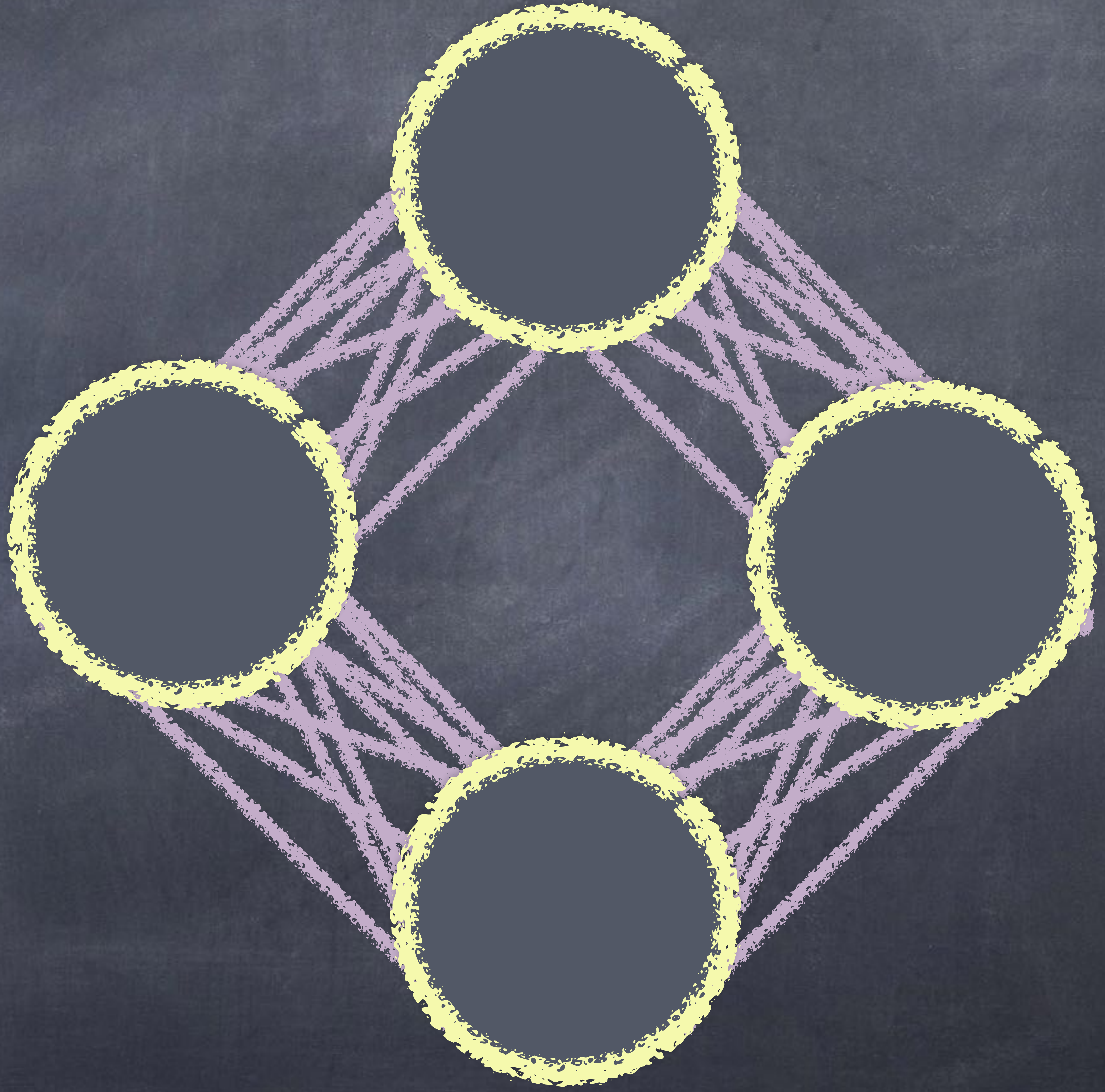

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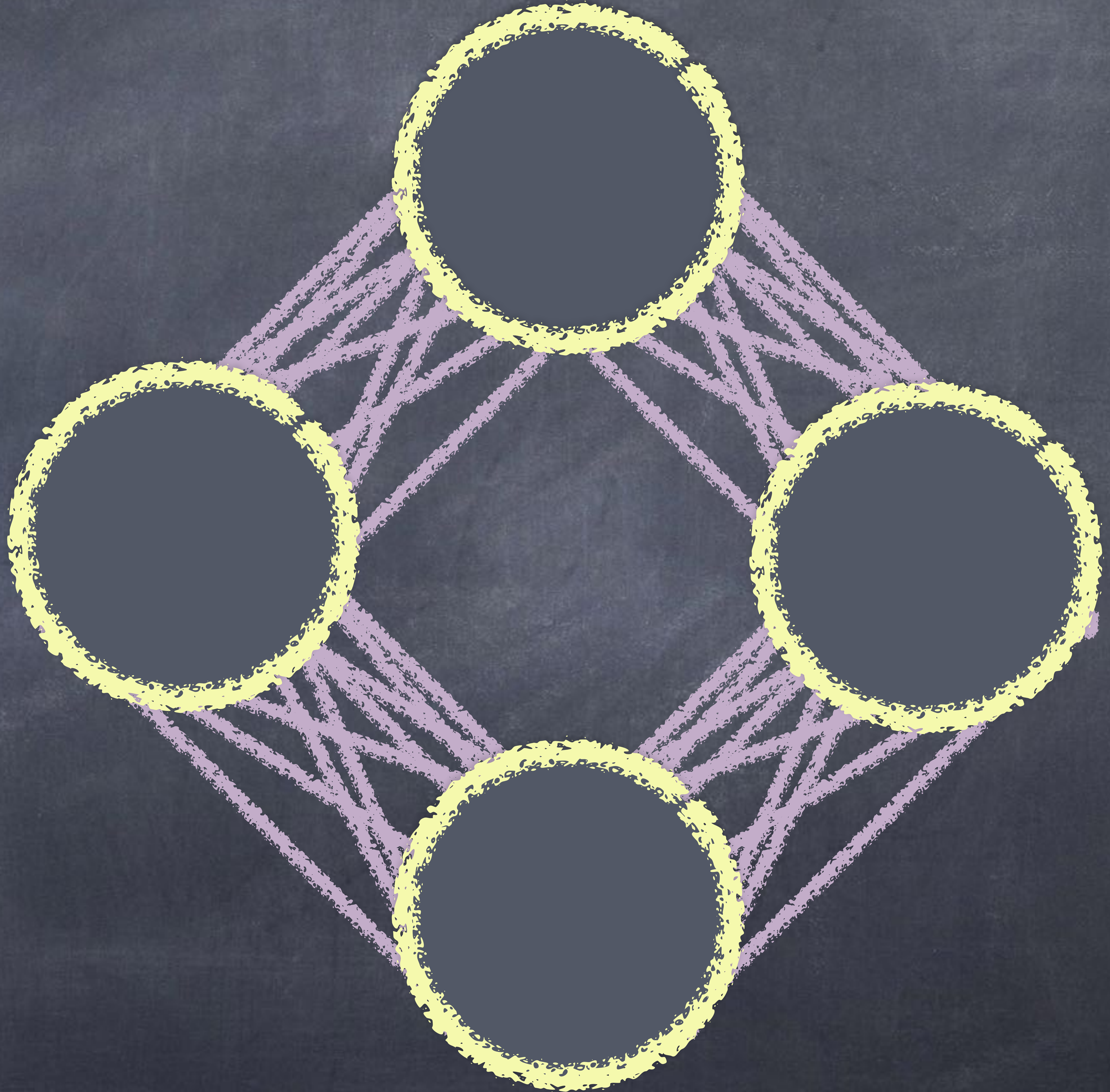

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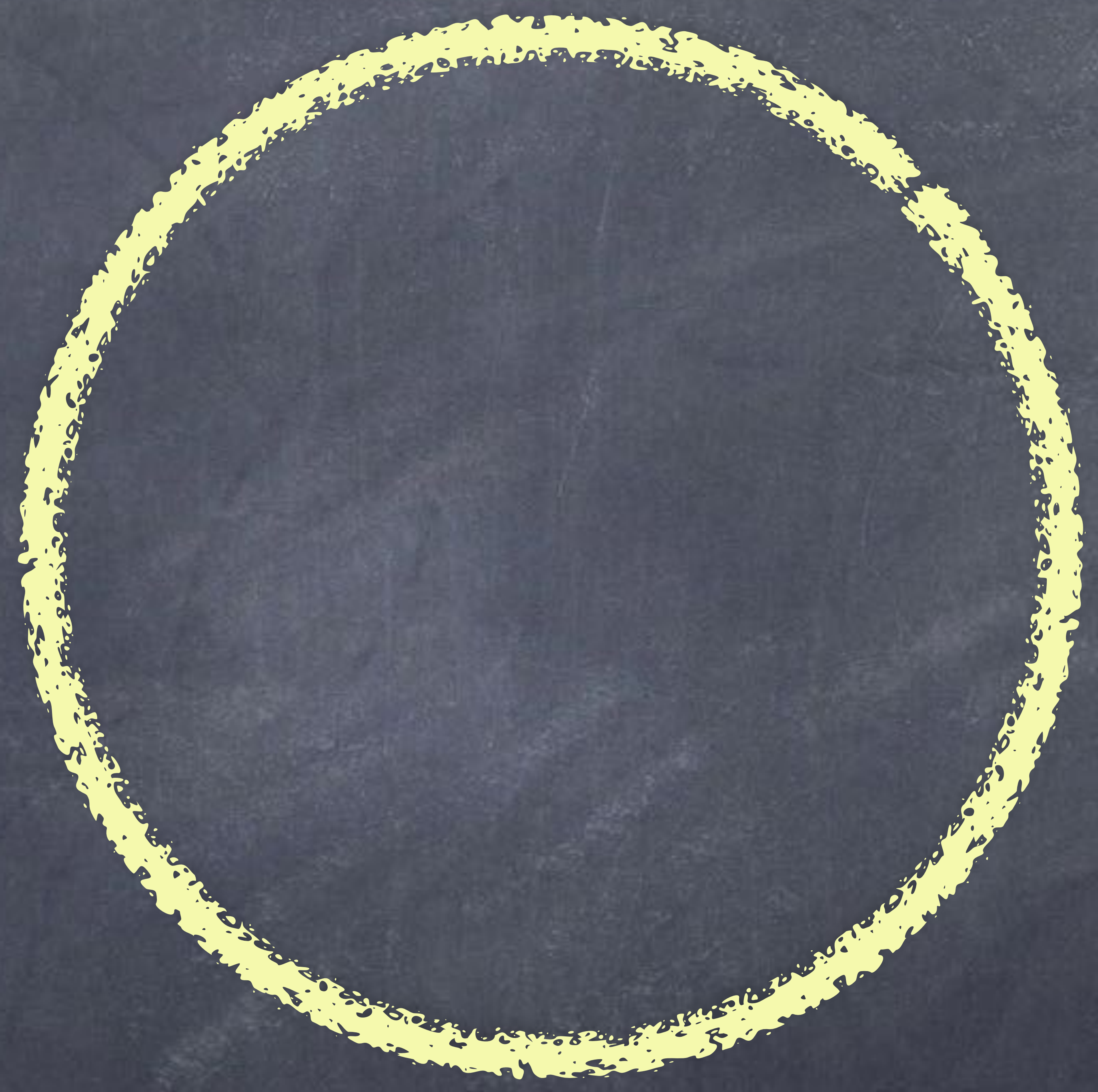

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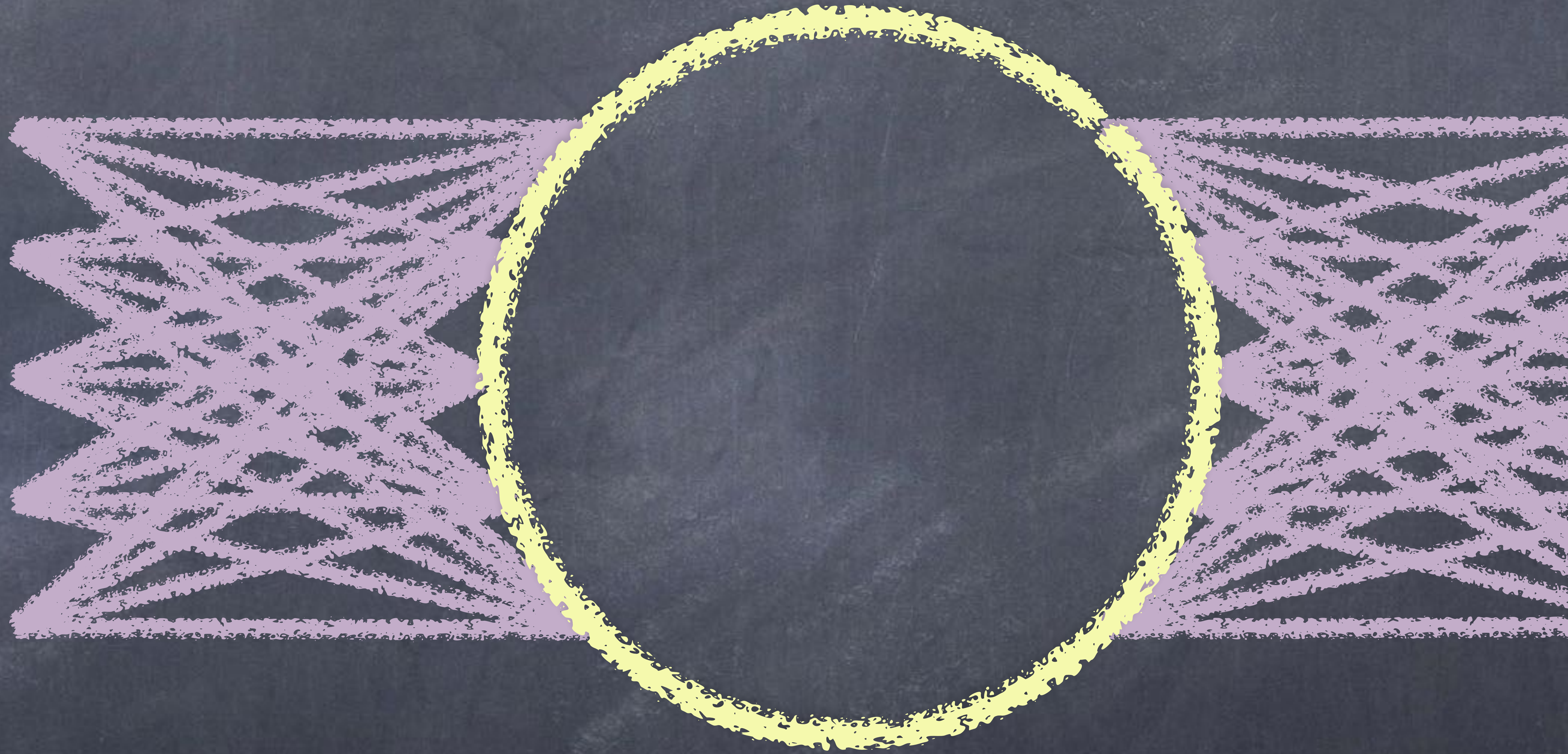

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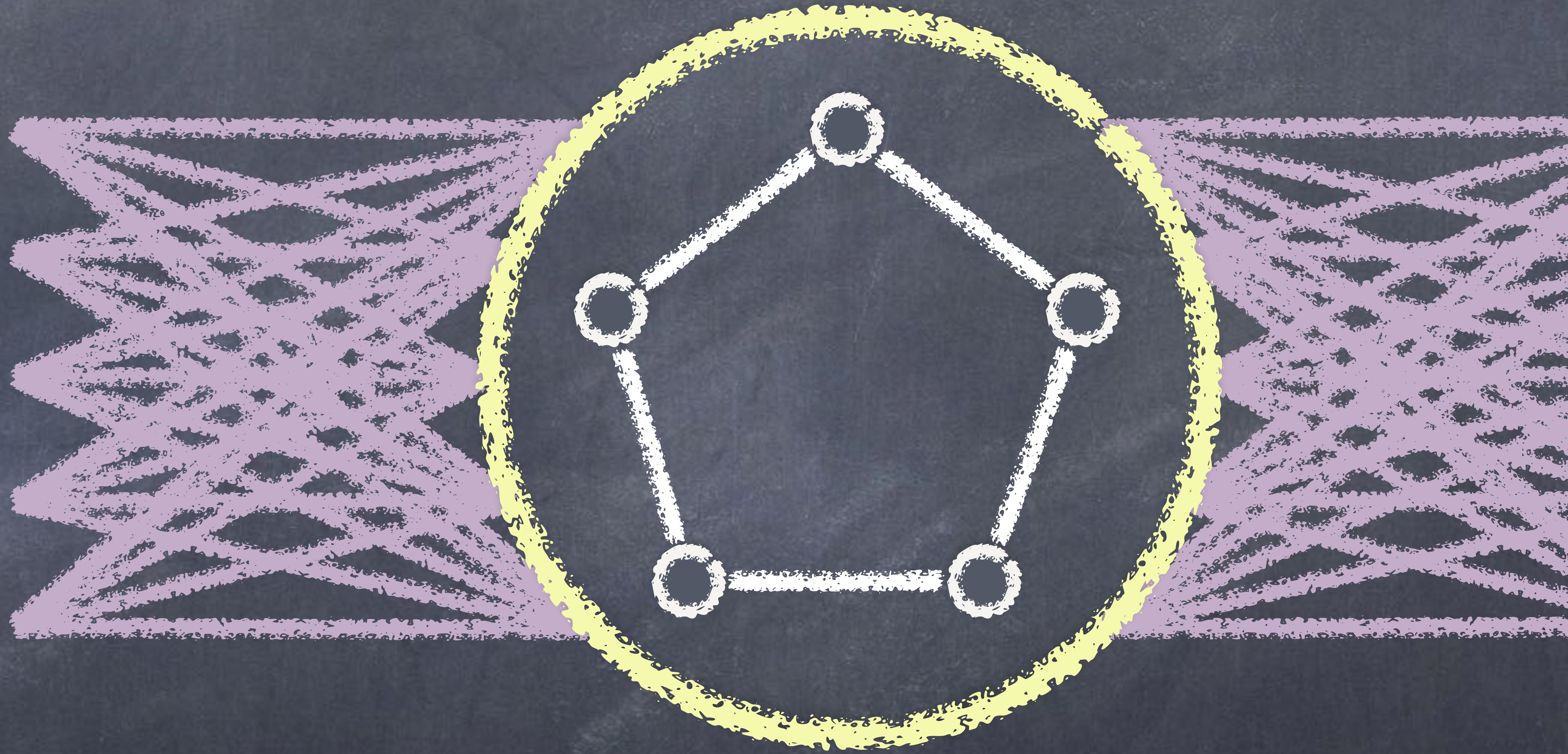

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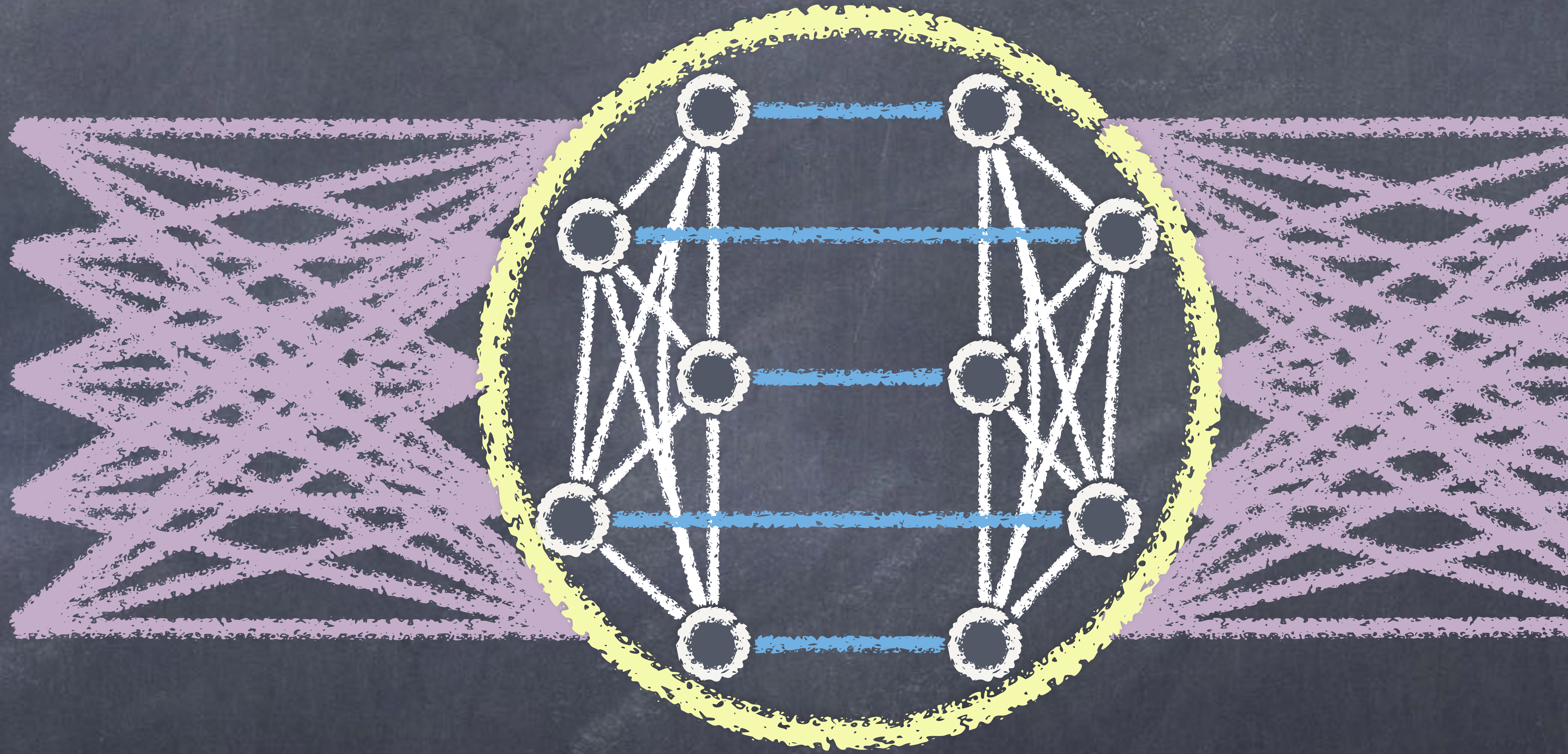

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Theorem (Potočnik, M.)

Let Γ be a vertex-transitive graph with $\mu(\Gamma) = 4$, then

- $\Gamma \cong \mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathcal{P}, m)$

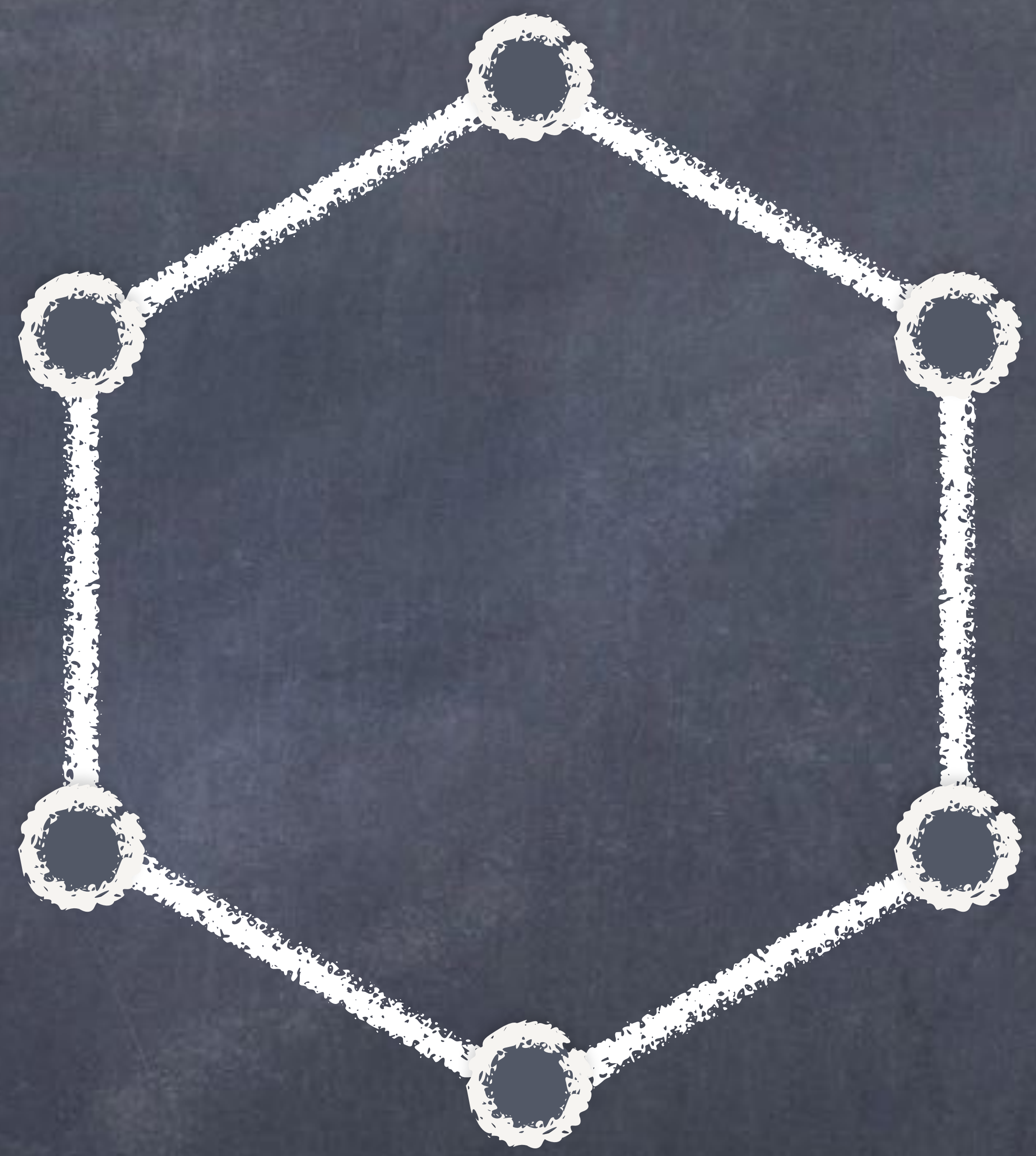
for some Σ, $\mathcal{P}$, λ, κ, and $m \geq 2$ ...

The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$

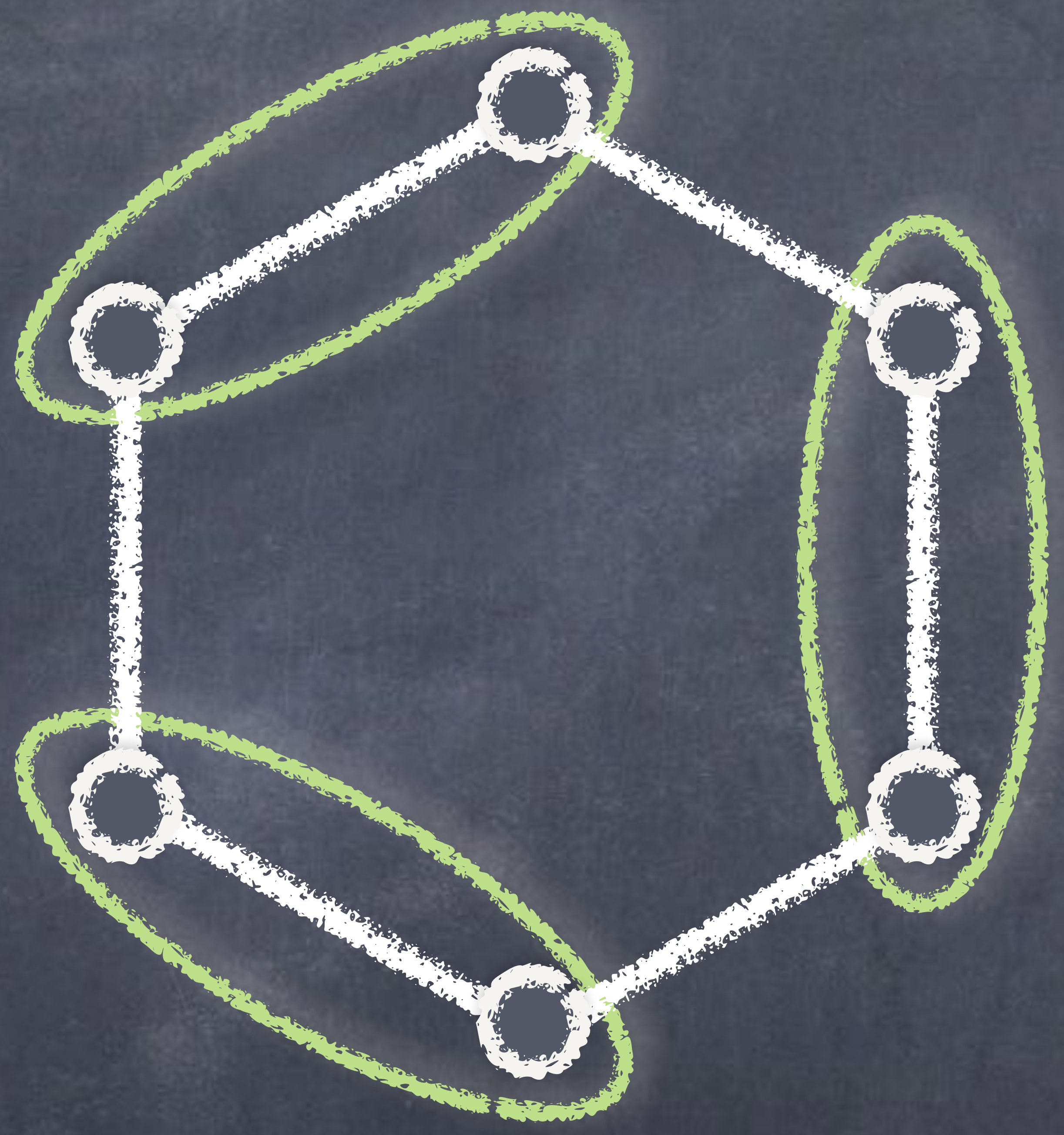
The graphs problem

$$\mathrm{Inf}_\kappa^\lambda(\Sigma, \mathscr{P}, m)$$



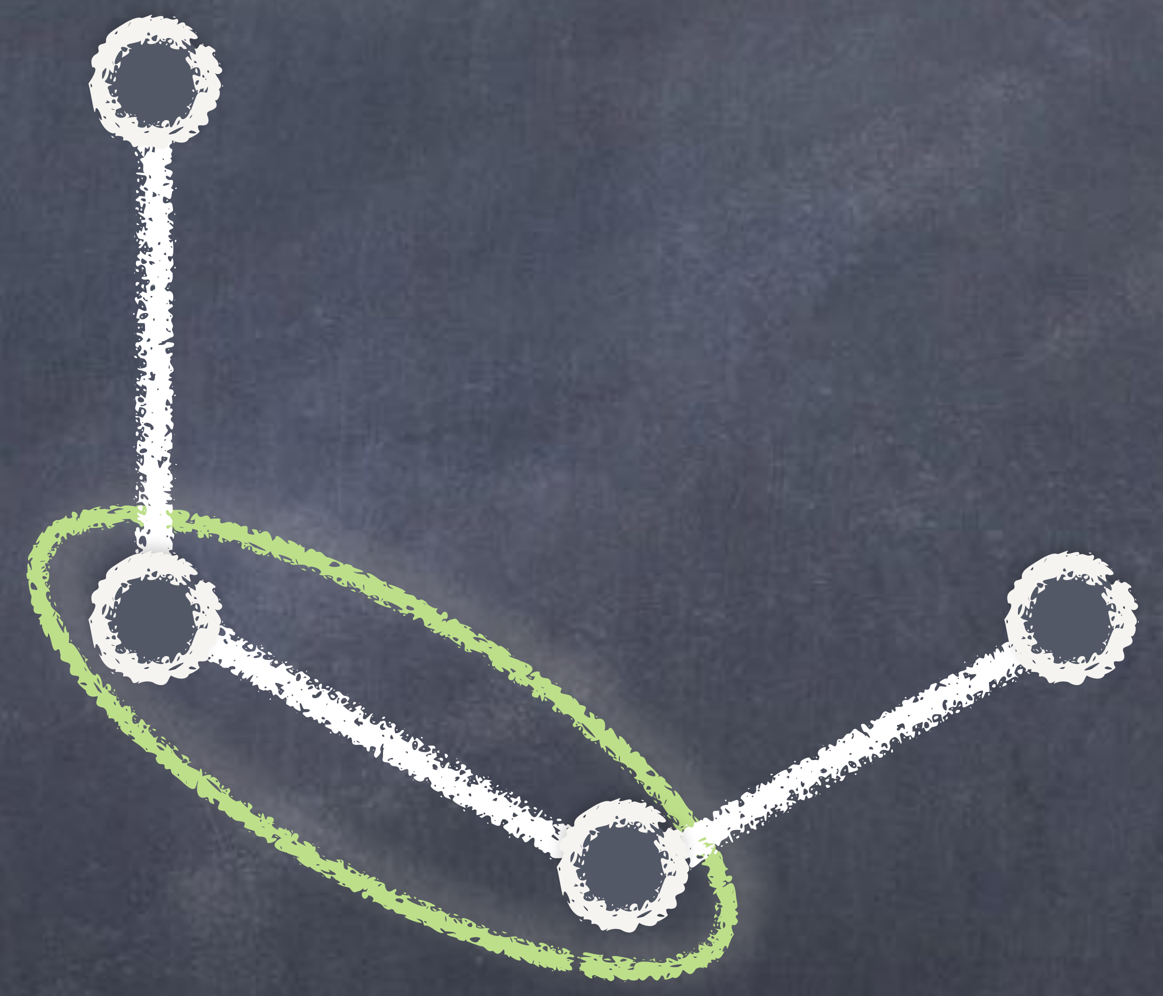
The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$



The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$



The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$

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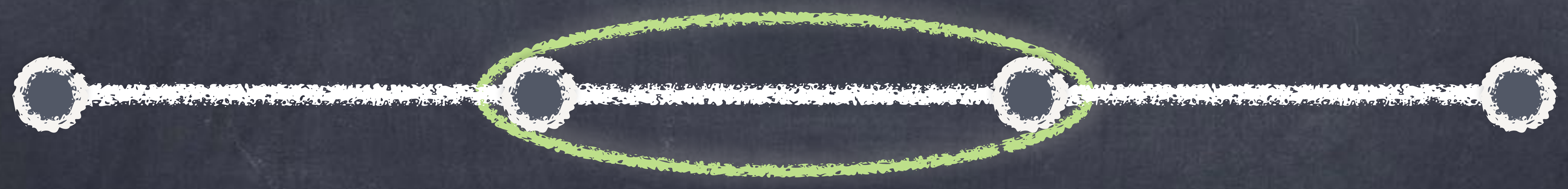
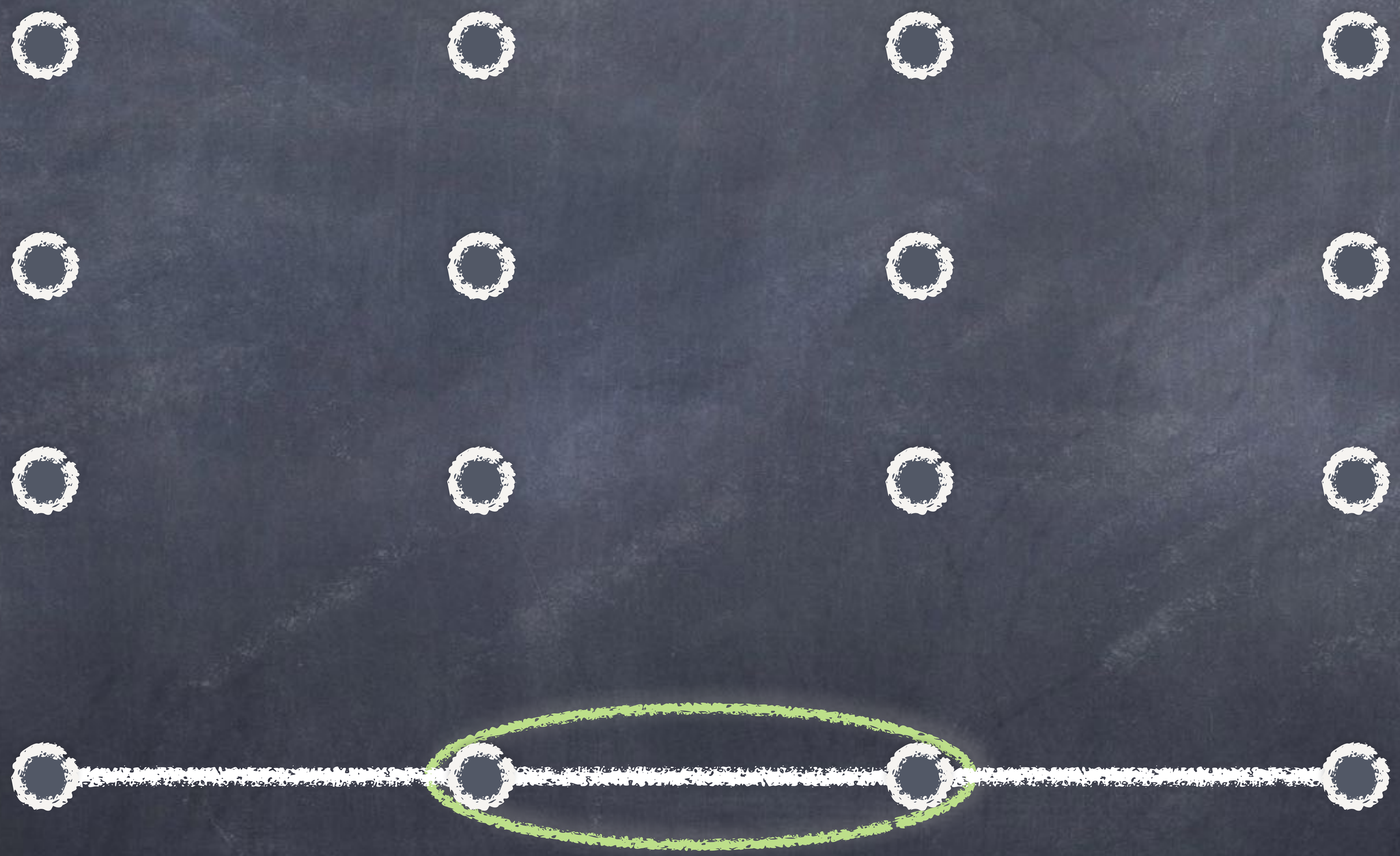
$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$



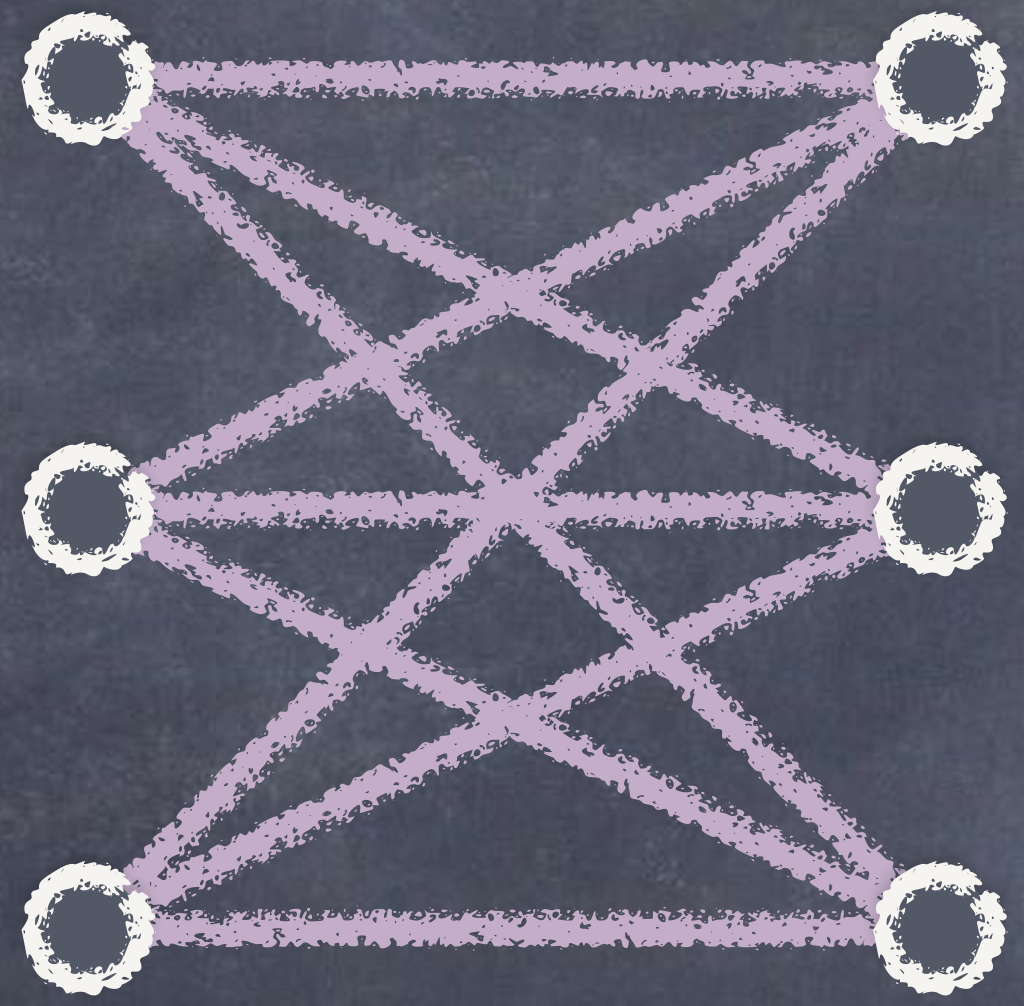
The graphs problem

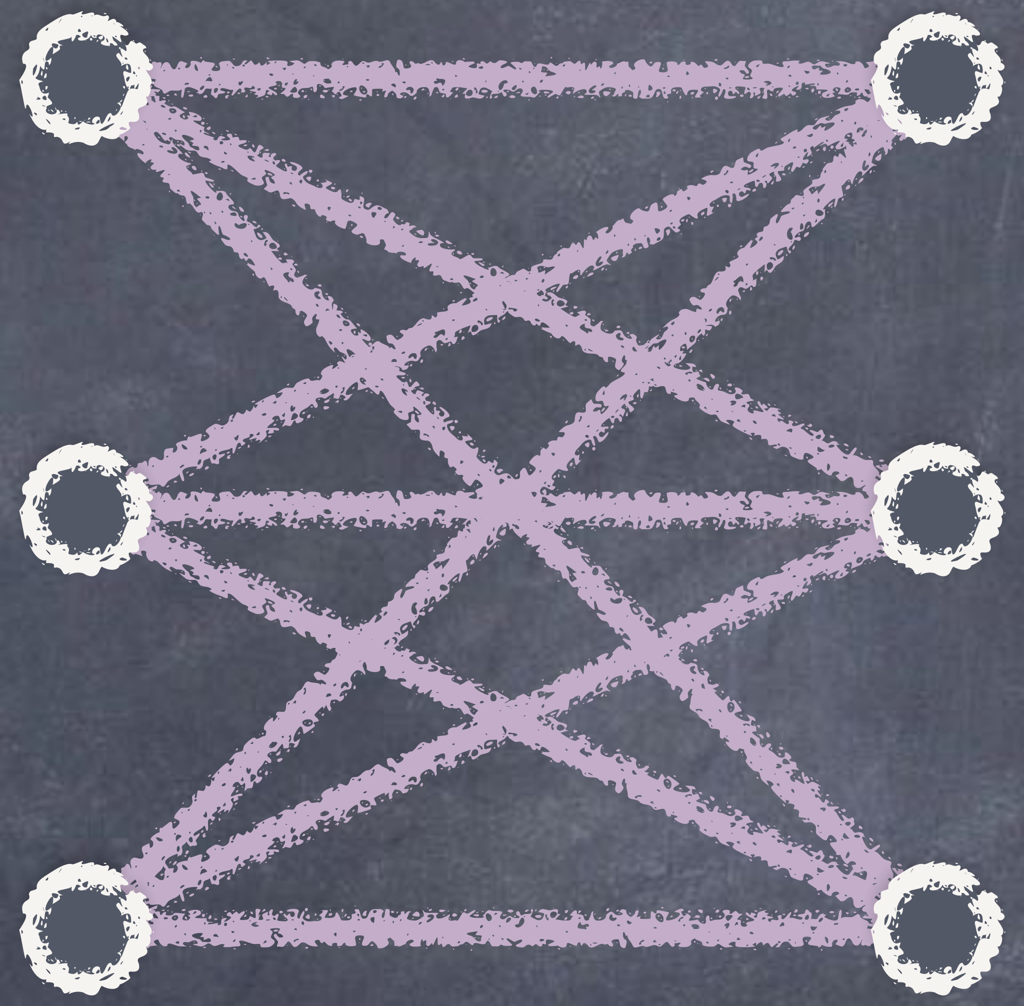
$$\mathrm{Inf}^{\lambda}_{\kappa}(\Sigma, \mathscr{P}, m)$$



The graphs problem

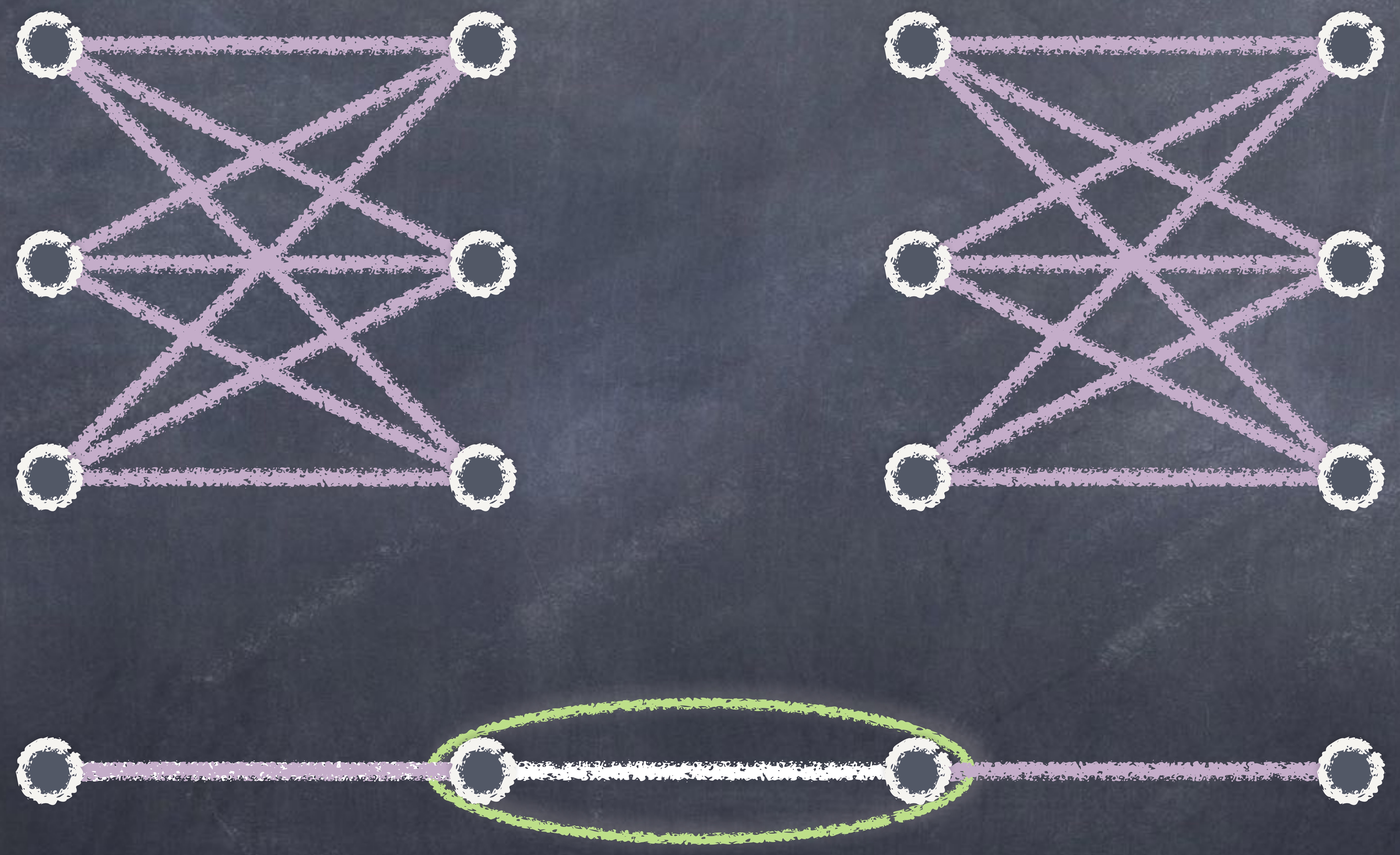
$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$





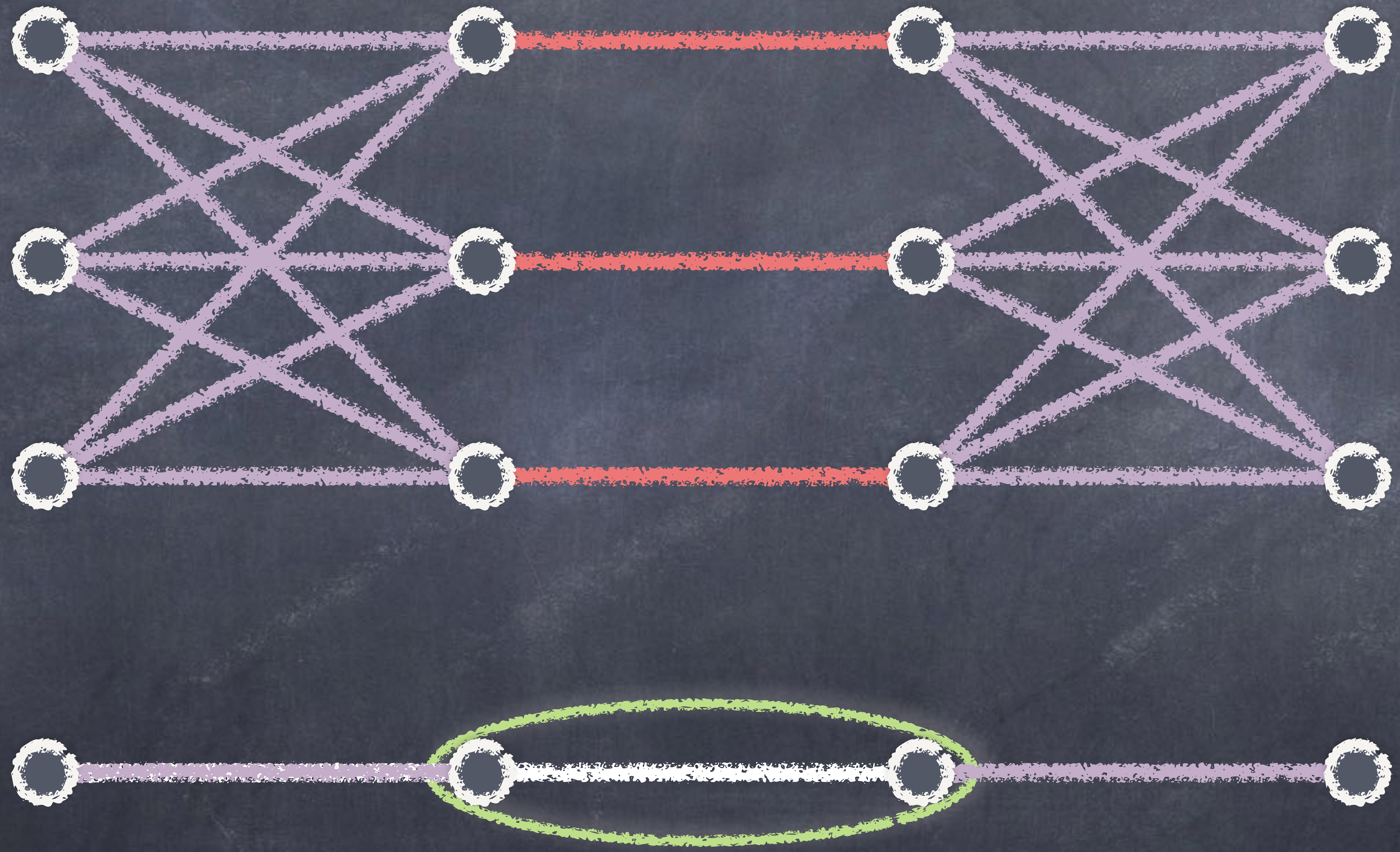
The graphs problem

$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$



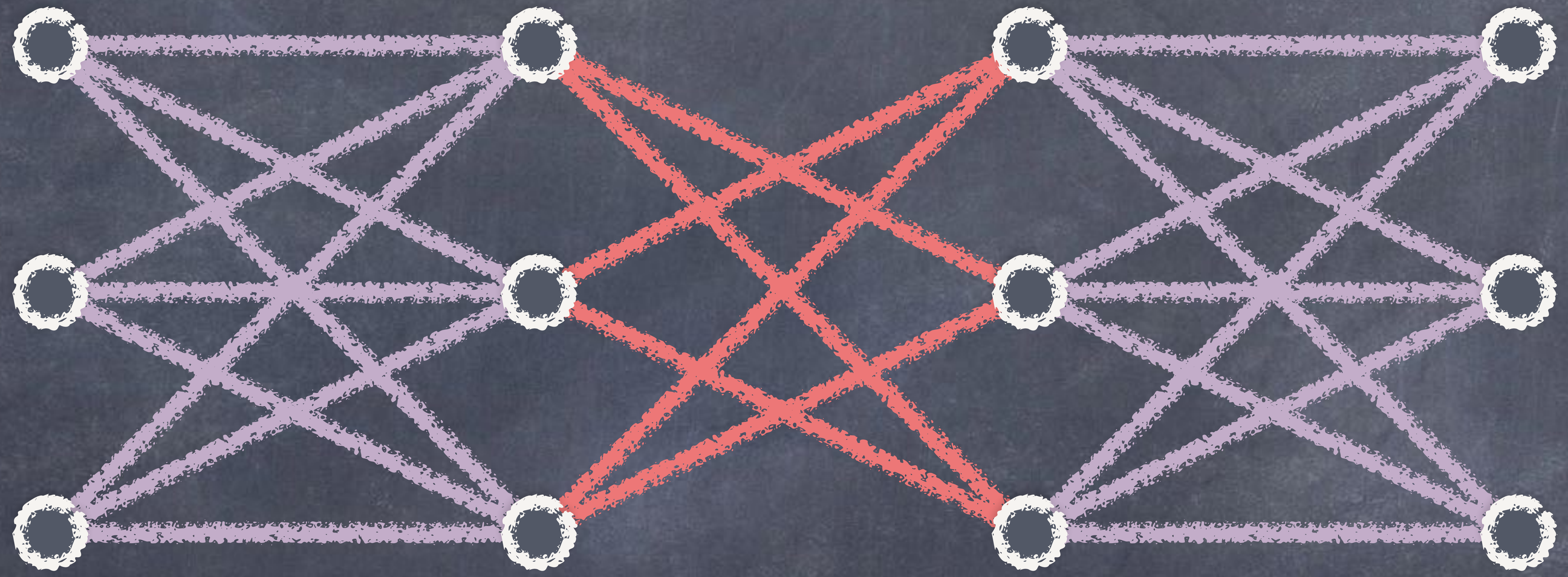
The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$



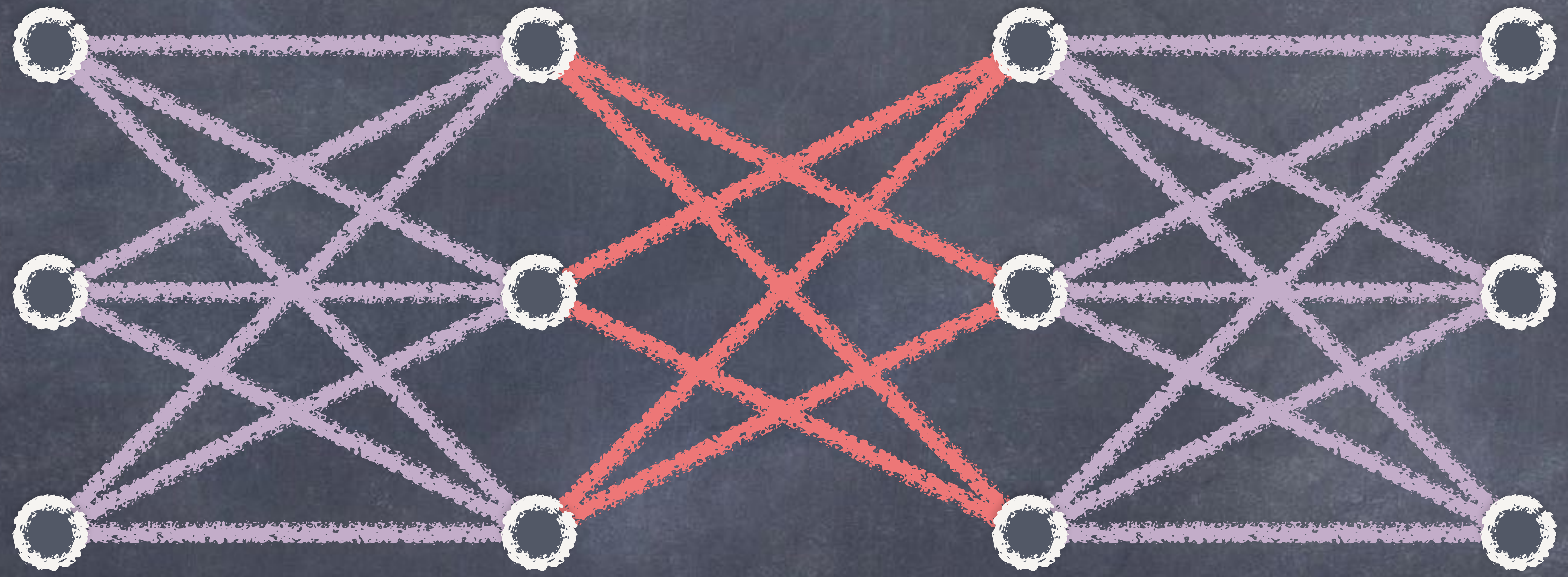
The graphs problem

$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$



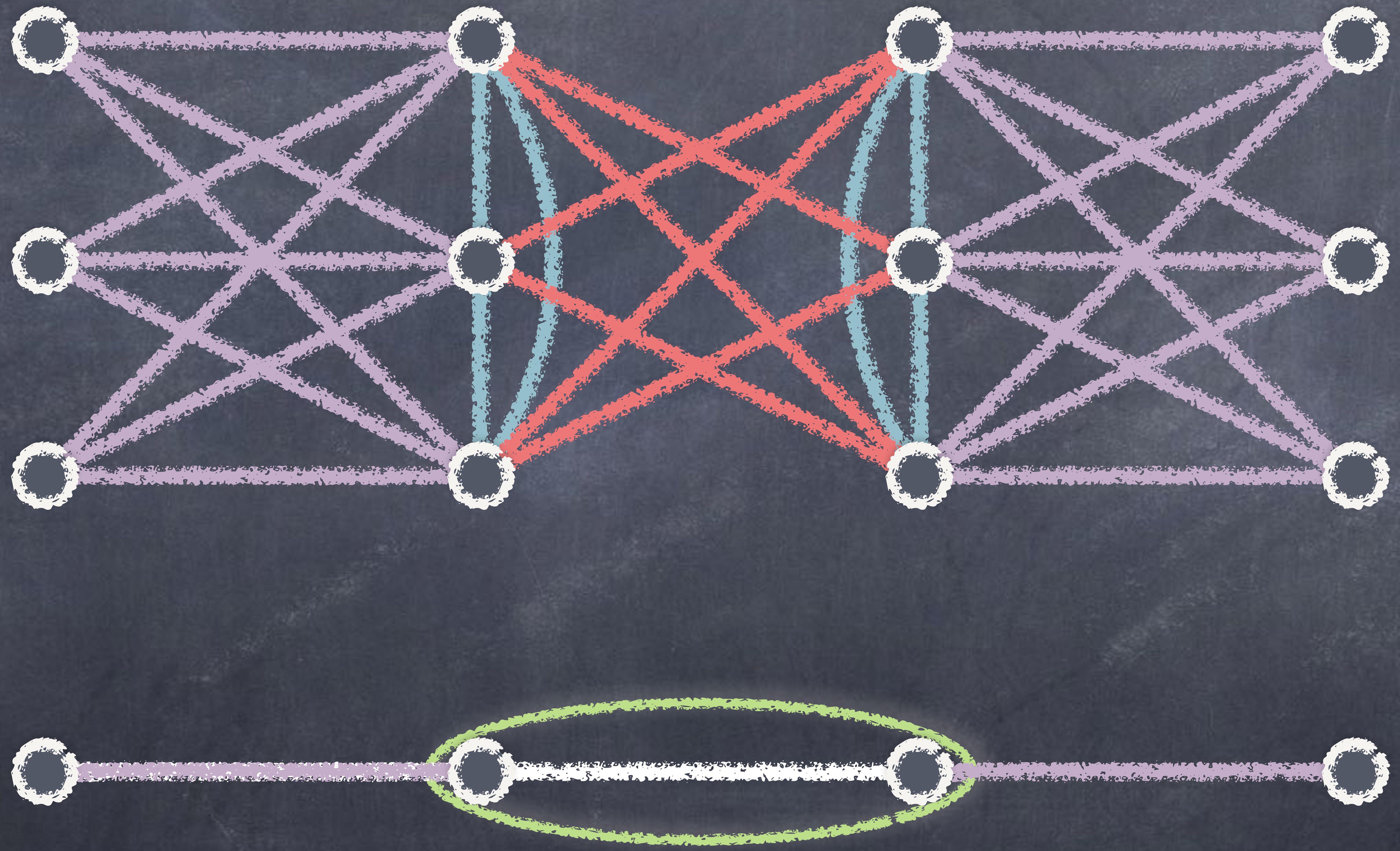
The graphs problem

$$\mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathscr{P}, m)$$



The graphs problem

$$\mathrm{Inf}^{\lambda}_{\kappa}(\Sigma, \mathscr{P}, m)$$



The graphs problem

Theorem (Potocnik, M.)

Let Γ be a vertex-transitive graph with $\mu(\Gamma) = 4$, then

- $\Gamma \cong C_5 \wr \Theta$, Θ a vertex-transitive graph;
- $\Gamma \cong (K_m \square K_2) \wr \Theta$, Θ a vertex-transitive graph, $m \geq 3$,
- $\Gamma \cong (\overline{K_m \square K_2}) \wr \Theta$, Θ a vertex-transitive graph, $m \geq 3$.
- $\Gamma \cong \mathrm{Inf}_{\kappa}^{\lambda}(\Sigma, \mathcal{P}, m)$, for some Σ, $\mathcal{P}$, λ, κ and $m \geq 2$.

Thank you!