A family of regular hypertopes from regular polytopes

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Joint work with Asia Ivić Weiss

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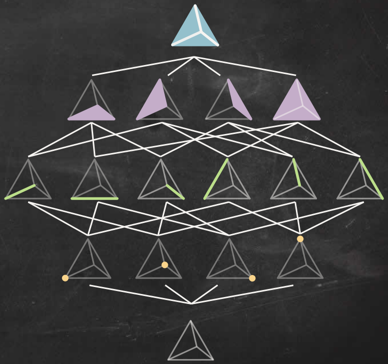

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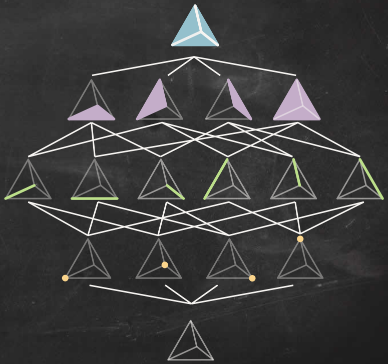

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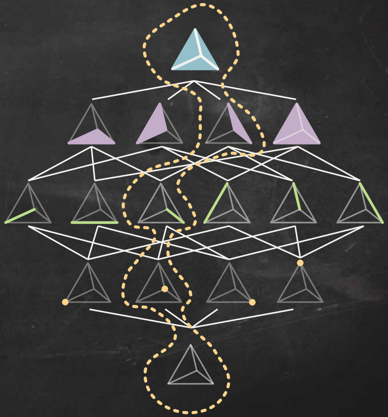

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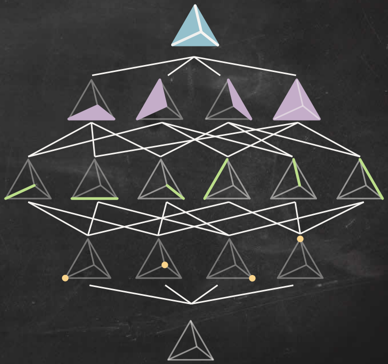

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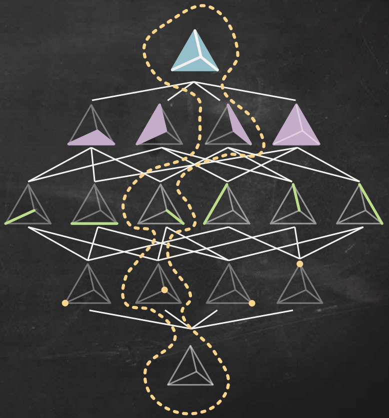

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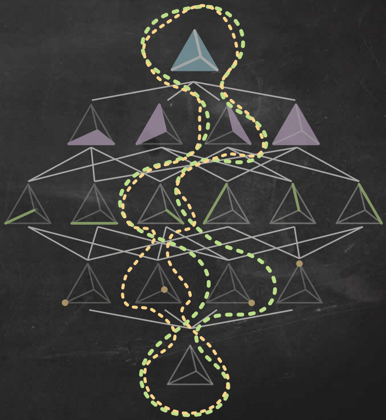

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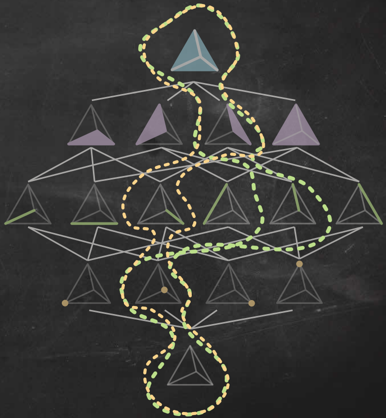

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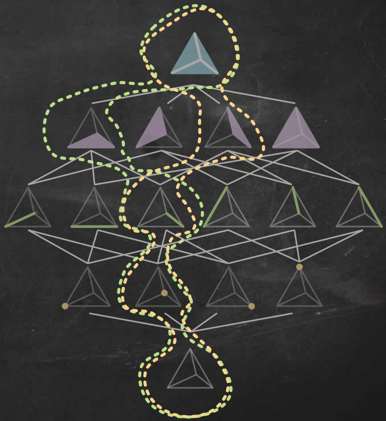

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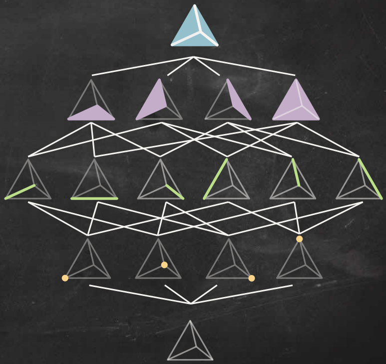
* $\mathcal{P}$ is strongly connected.



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- An abstract polytope is regular if the action of $\mathrm{Aut}(\mathcal{P})$ on the maximal chains is transitive.

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Given a fixed base maximal chain Φ,

* For every $i \in \{1, \ldots, n-1\}$ there is an automorphism ρ_i s.t.

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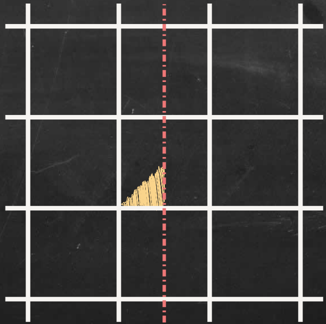

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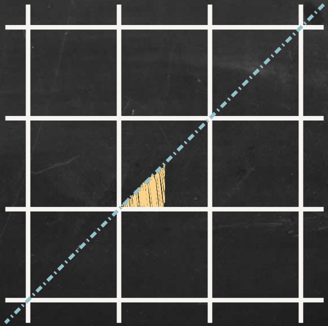

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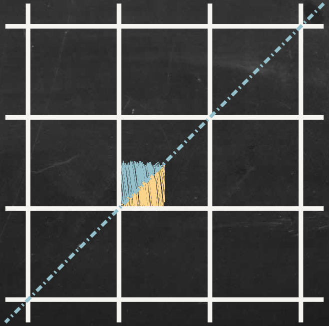

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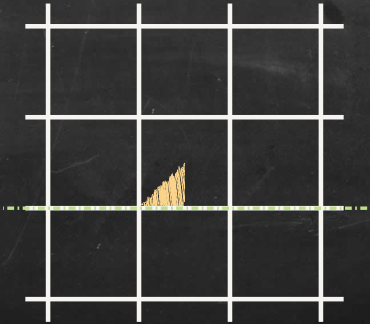

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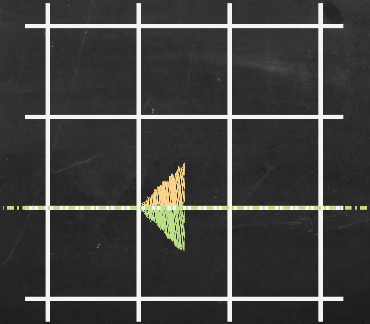

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Theorem (E. Schulte, 1982)

Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be string C-group. Then there exists a regular polytope $\mathcal{P}$ such that $\mathrm{Aut}(\mathcal{P}) = \Gamma$.

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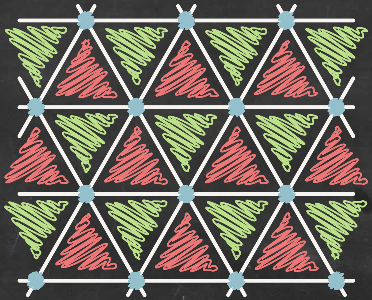

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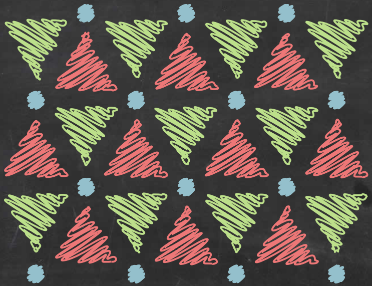

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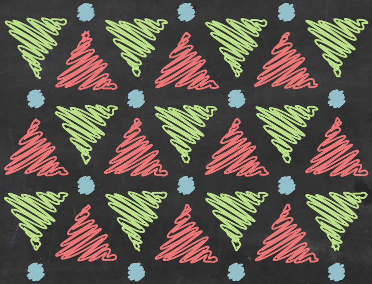

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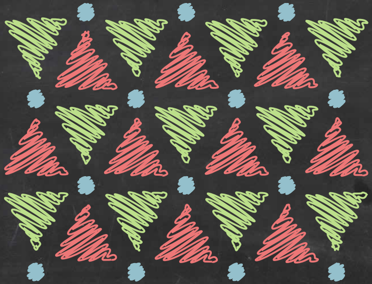

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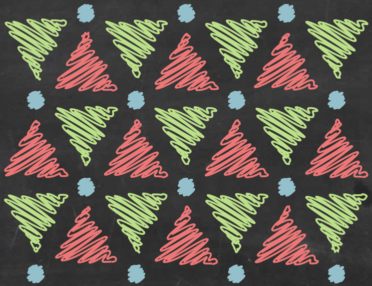

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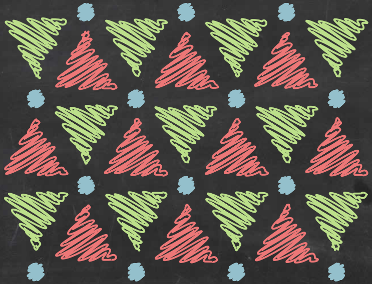

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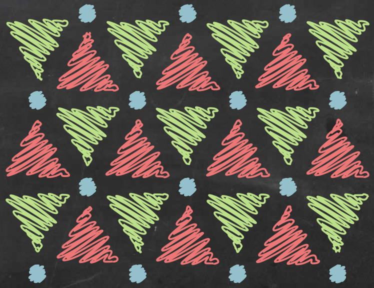

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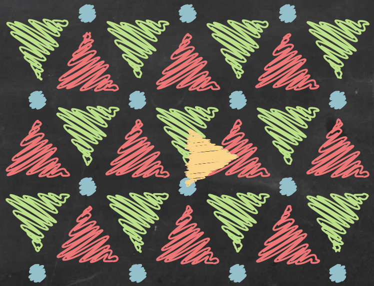

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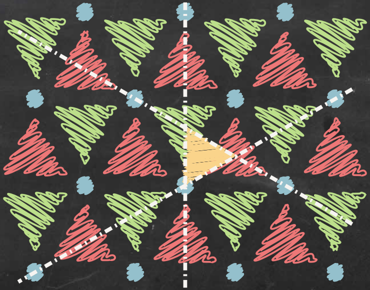

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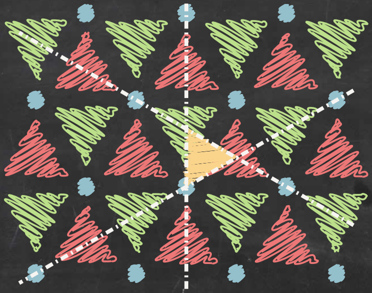

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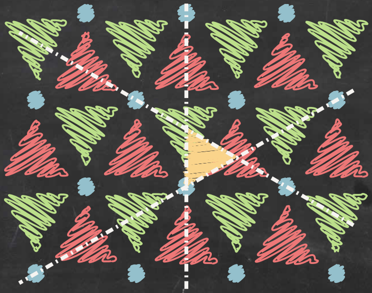
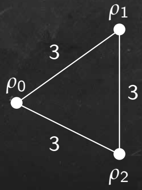


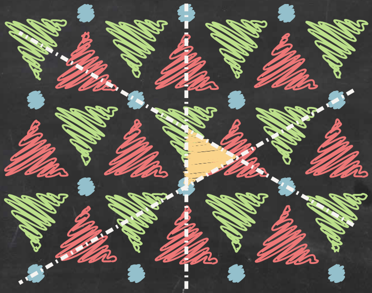


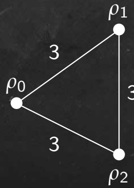
Hypertopes

Theorem (Fernandes-Leemans-Weiss, 2016)

Let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be a C-group. Let $\mathcal{H}$ be the coset geometry associated to Γ. If Γ is flag-transitive on $\mathcal{H}$, then $\mathcal{H}$ is a regular hypertope and $\mathrm{Aut}_I(\mathcal{H}) = \Gamma$.

Hypertopes





The halving operation

Given an abstract n-polytope $\mathcal{P}$ of type $\{p_1, \dots, p_{n-2}, 2s\}$ and automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \varrho_0, \dots, \varrho_{n-1} \rangle$ the halving operation is given by:

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$$\rho_i = \begin{cases} \varrho_i, & \text{if } 0 \leqslant i \leqslant n-2, \\ \varrho_{n-1}\varrho_{n-2}\varrho_{n-1}, & \text{if } i = n-1, \end{cases}$$

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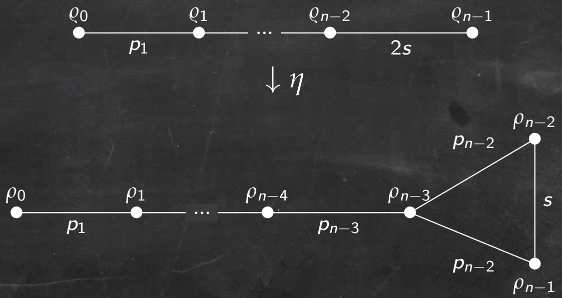

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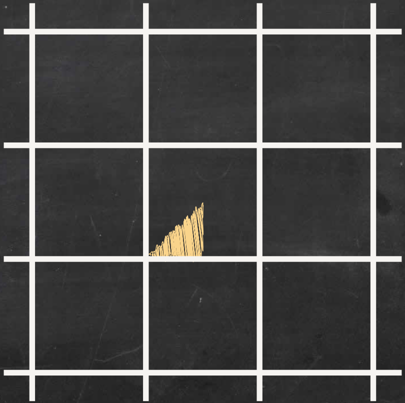
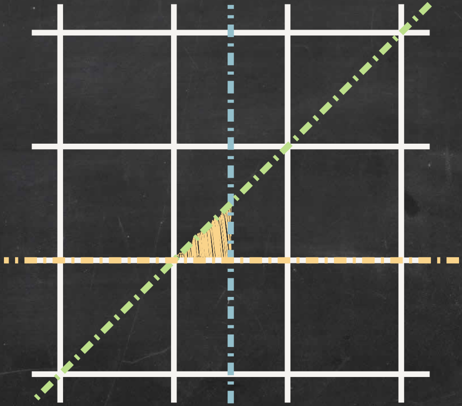


The halving operation



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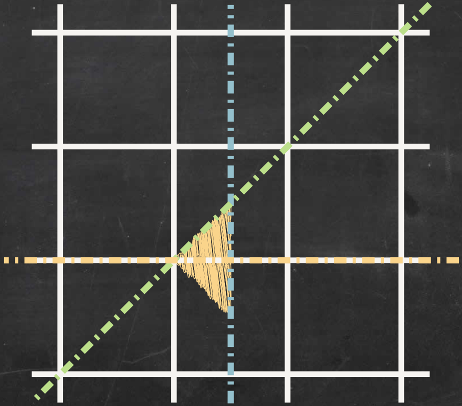

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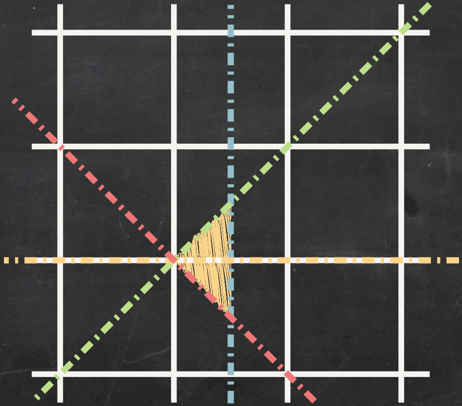

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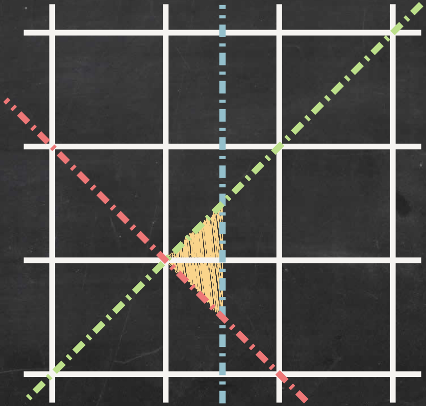

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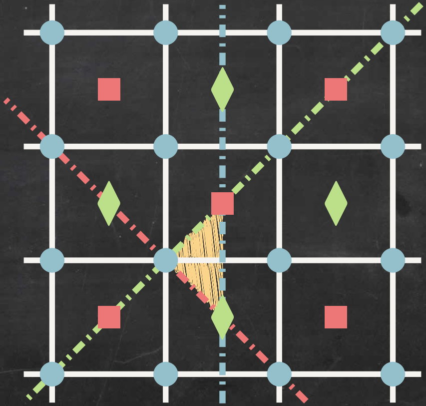

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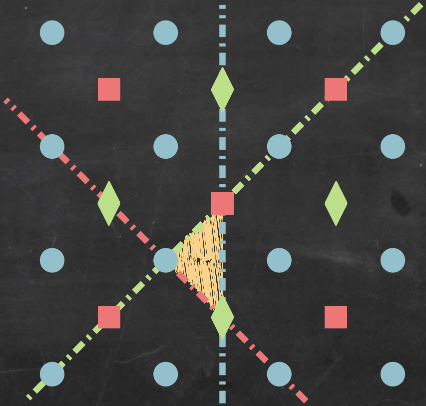

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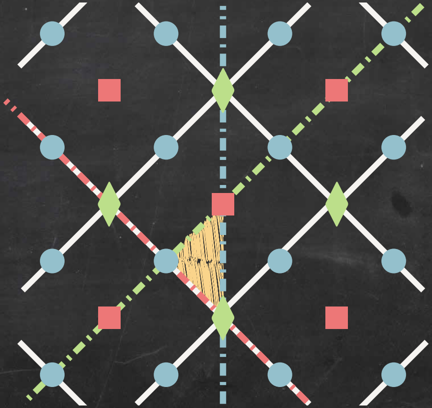

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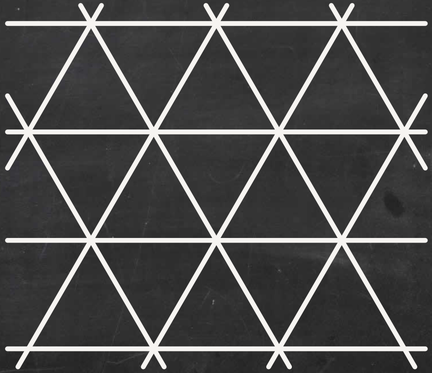

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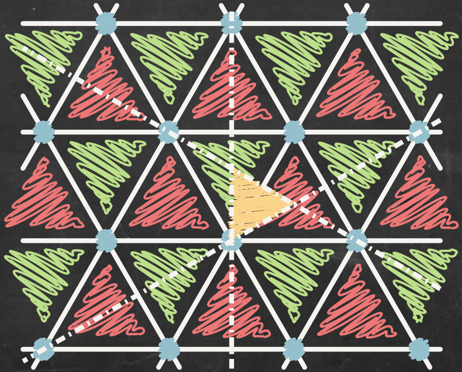

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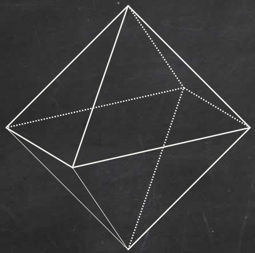

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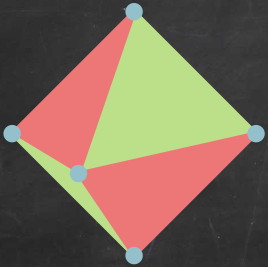

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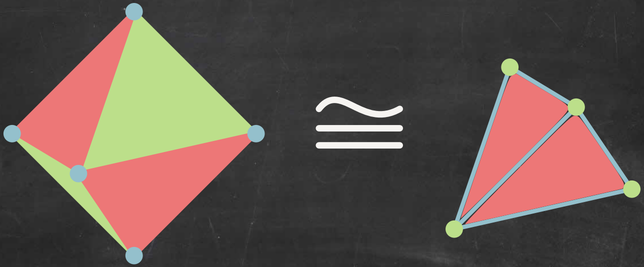

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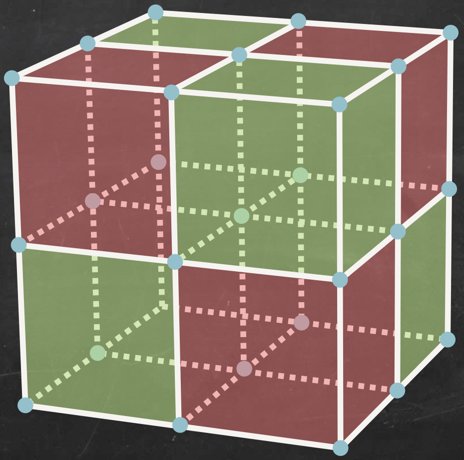

The halving operation

Theorem (M.-Weiss)
Let $\mathcal{P}$ be a non-degenerate, regular n-polytope of type $\{p_1, \ldots, p_{n-2}, 2s\}$. Let $H(\mathcal{P})$ be the group resulting after applying the halving opperation to $\mathrm{Aut}(\mathcal{P})$. Then there exists a regular hypertope $\mathcal{H}(\mathcal{P})$ such that $\mathrm{Aut}_I(\mathcal{H}(\mathcal{P})) = H(\mathcal{P})$.

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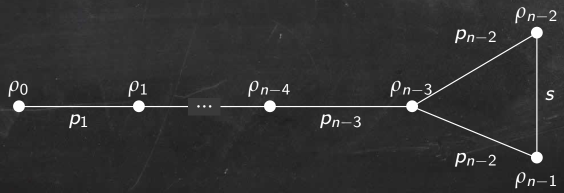

The halving operation

* Finite globally toroidal hypertopes.

The halving operation

* Finite globally toroidal hypertopes.
* Finite locally spherical hypertopes.

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* Regular hypertopes with prescribed labels on the Coxeter diagram

Thank you!