Chiral extensions of chiral polytopes

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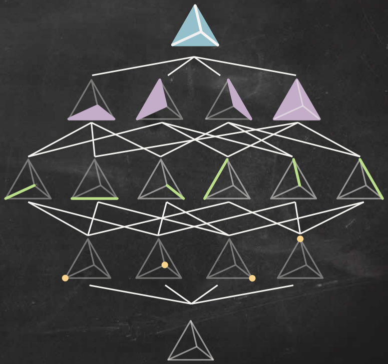

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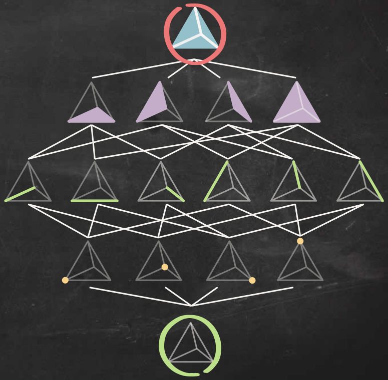

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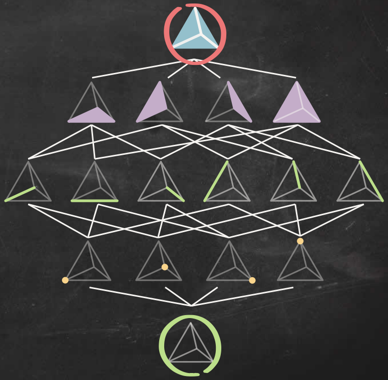

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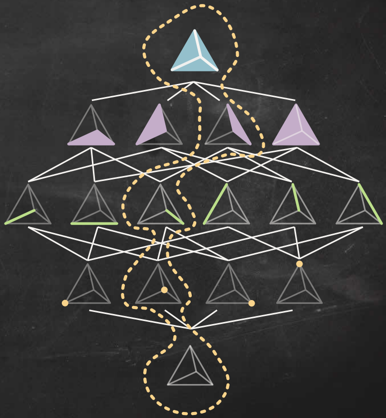

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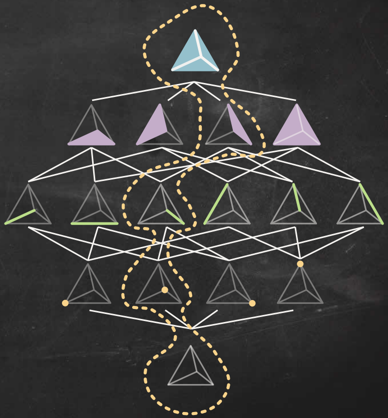

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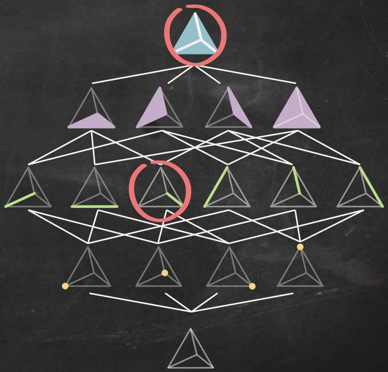

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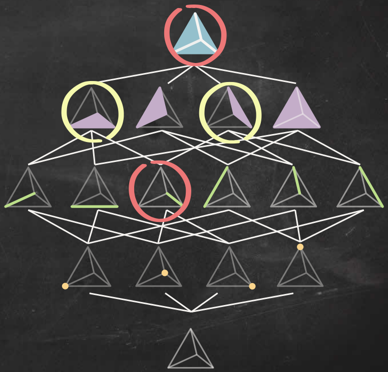

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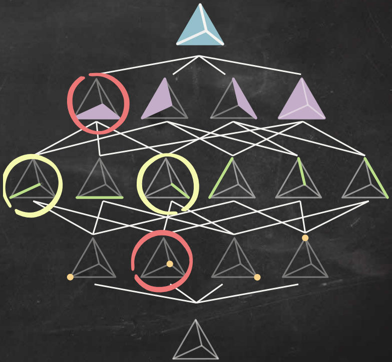

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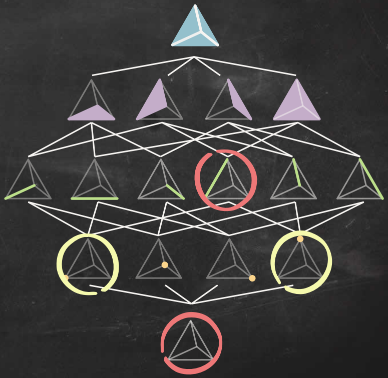

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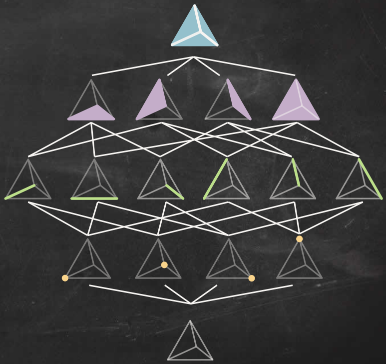
* $\mathcal{P}$ satisfies the diamond condition.
* $\mathcal{P}$ is strongly connected.



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Symmetries

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- An automorphism of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\varphi : \mathcal{P} \to \mathcal{P}$.
- The group $\Gamma(\mathcal{P})$ of automorphisms of $\mathcal{P}$ acts freely on the set of flags.
- An abstract polytope is regular if the action of $\Gamma(\mathcal{P})$ on the flags is transitive.

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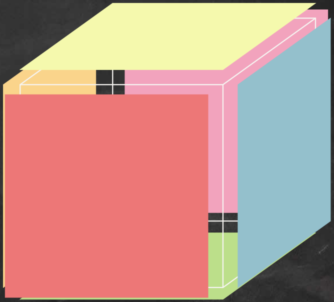

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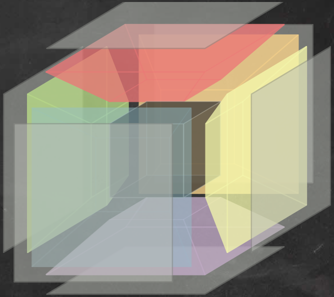

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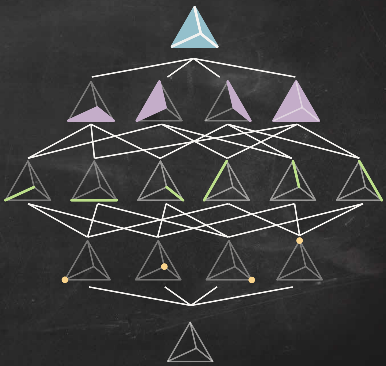

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The extension problem

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In such situation we say that $\mathcal{P}$ is a (regular) extension of $\mathcal{K}$.

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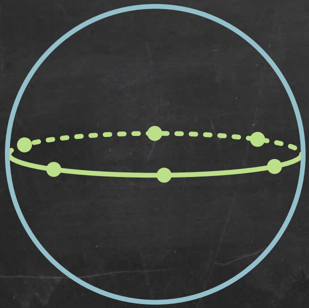

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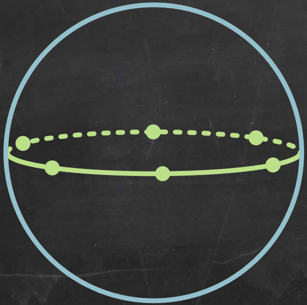

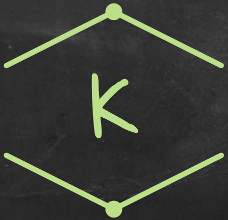

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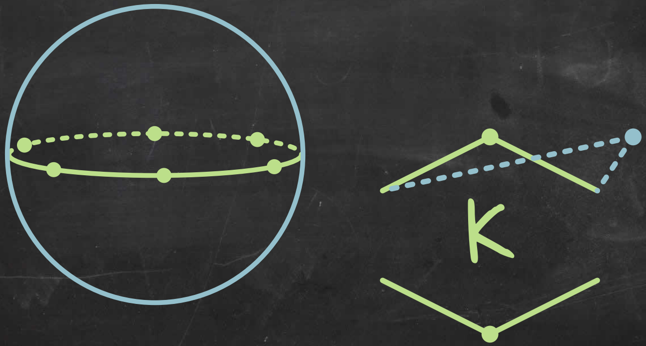

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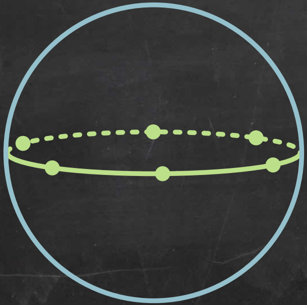

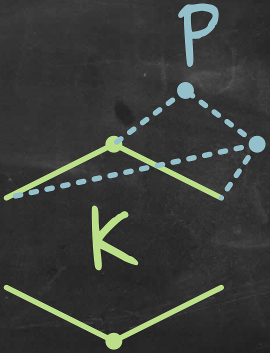

Extensions of type $\{\mathcal{K}, p\}$

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* Given $\mathcal{K}$, determine $q \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{K}$ admits an extension of type $\{\mathcal{K}, q\}$.

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- Harley, 2005: The hemicube $\{4, 3\}/2$ does not have an extension of type $\{\{4, 3\}/2, q\}$ with q odd.

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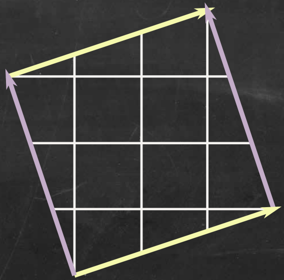

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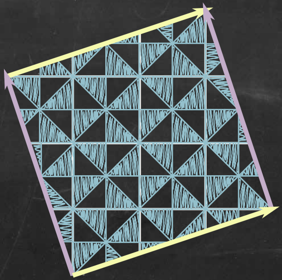

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* Conder, Hubard, O'Reilly and Pellicer, 2017?: Infinitely many chiral n-polytopes with simplicial facets, $n \geqslant 5$.

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* Any construction of chiral extensions cannot be applied recursively.

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* Conder and Zhang, 2017: Abelian covers of chiral polytopes.

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* Let $\mathcal{K}$ be a chiral polytope with regular facets such that $\mathcal{K}$ has a regular quotient with at least two facets. If $\mathcal{K}$ admits a chiral extension of type $\{\mathcal{K}, q\}$, then $\mathcal{K}$ has a chiral extension of type $\{\mathcal{K}, 2n * q\}$ for any $n \in \mathbb{N}$.

Chiral extensions

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* Using our results, given $n \in \mathbb{N}$, we can built extensions of type $\{\{4,4\}_{(b,c)},6n\}$ and for almost any toroidal map $\{4,4\}_{(b,c)}$ and extensions of type $\{\{6,3\}_{(b,c)},6n\}$ for almost any map of type $\{6,3\}_{(b,c)}$.

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    - can we say something about their type?

Thank you for your attention!