Cubic toroids with few flag-orbits

Antonio Montero
Joint work with José Collins

Centro de Ciencias Matemáticas
National University of México

Seminar of Combinatorics and Group Teory
Faculty of Education
University of Ljubljana
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Maps

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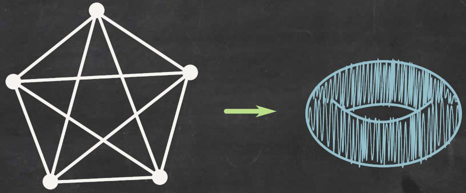

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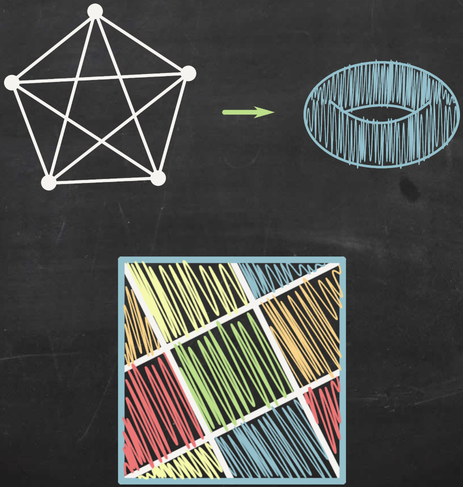

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  - Maniplexes.
* They lose the topological (geometric) spirit of a map...

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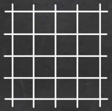

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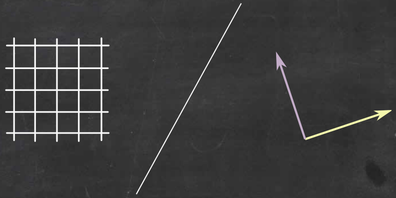

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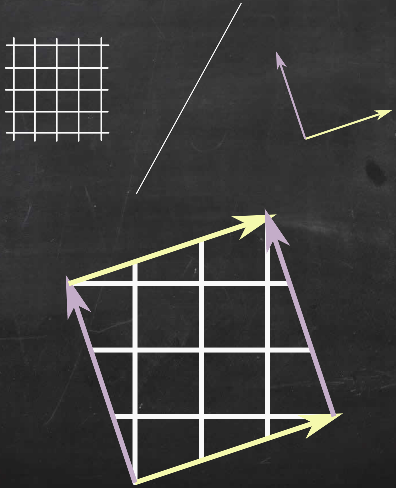

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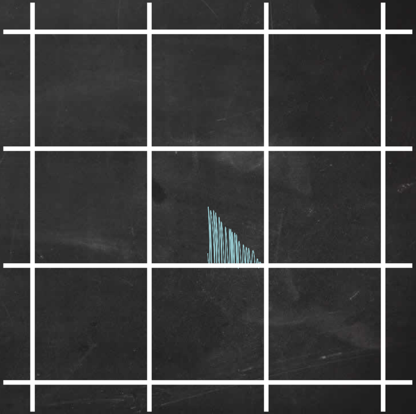

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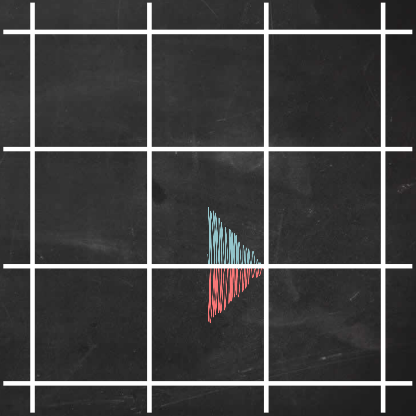

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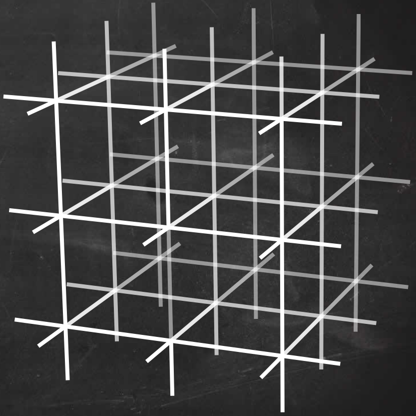

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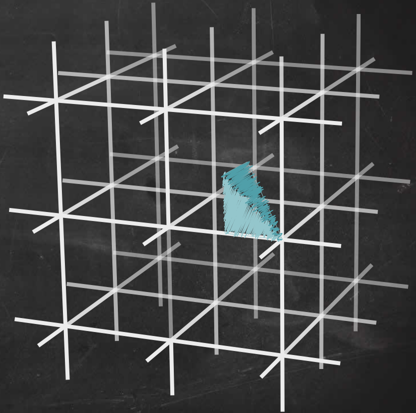

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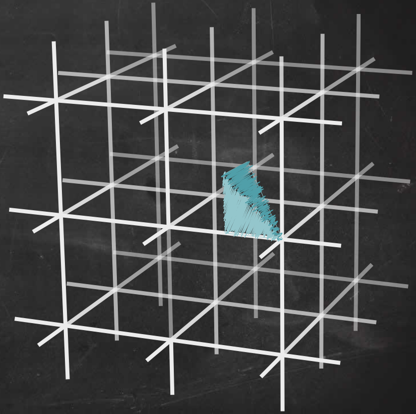

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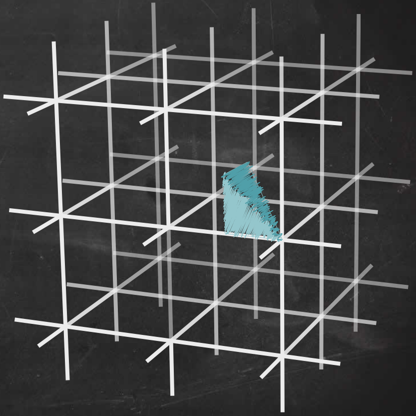

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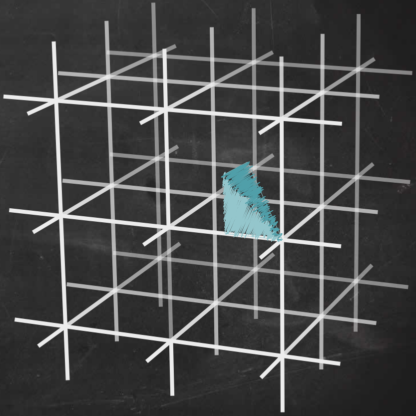

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* It makes sense to define

$$\mathrm{Aut}(\mathcal{U}/\Lambda) = \mathrm{Norm}_{\mathrm{Aut}(\mathcal{U})}(\Lambda)/\Lambda.$$

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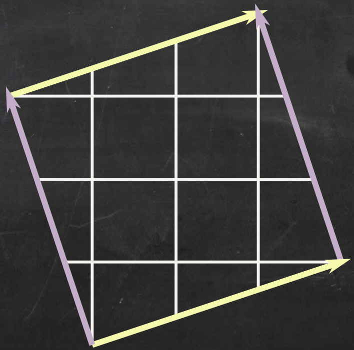

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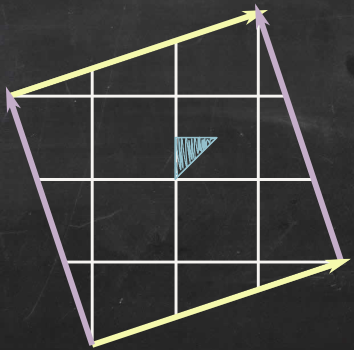

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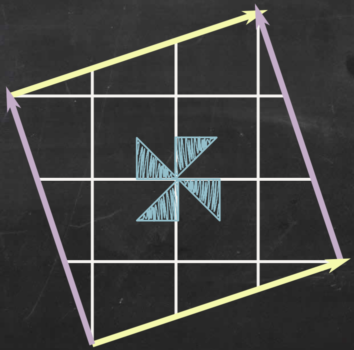

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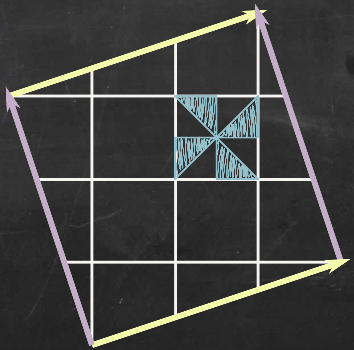

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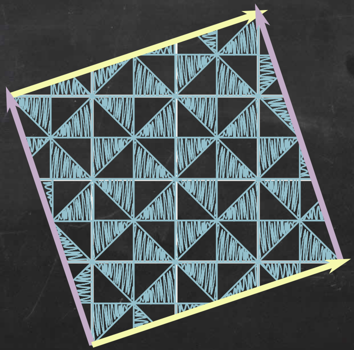

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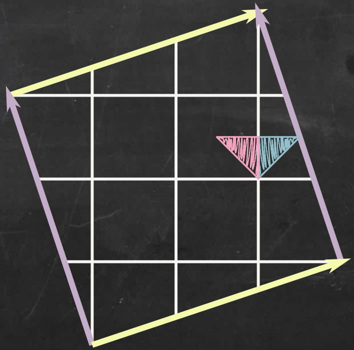

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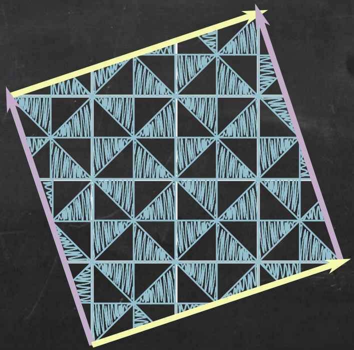

Problem:

Classify (cubic) toroids up to symmetry type.

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* Chiral cubic toroids are classified, they only exist in dimension 2 (chiral maps). (Hartley, McMullen and Schulte, 1999)

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  - Q: Do they even exist if $n > 3$?

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* $\mathcal{U}/\Lambda \cong \mathcal{U}/\Lambda'$ if and only if Λ and Λ are conjugate in $\mathrm{Aut}(\mathcal{U})$.

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  - 4-orbit toroids: none.

Open problems/Future work

* Classify few-orbit toroids induced by non-regular tessellations.

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- Classify few-orbit toroids induced by non-regular tessellations.
- Study few-orbits structures in other Euclidean space forms.
- Achieve a complete classification of (equivelar) toroids on arbitrary dimension.

Hvala!